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Regular Perturbation Approach to Bratu Equations with Fractional Exponent

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Keywords

Non-linear Differential Equation, Regular Perturbation Method, Bratu's Type Equation, Fractional Exponent, Approximate Solution.

Abstract

In this paper, the regular perturbation method is employed to obtain approximate solution of Bratu differential equations with fractional exponent. Thus, comparison of numerical results is done using different values of the perturbation parameter ε. It is evident that the perturbation method is an alternative approach that should be taken into consideration while resolving a variety of real-life problems in differential equations. To show the recommended approach, three test problems were considered. The maple 18 program was used to perform calculations.

1. Introduction

Consider the following nonlinear Bratu differential equation:

$$\frac{d^2\varphi}{d\tau^2} + \gamma e^{\frac{1}{\alpha}\varphi(\tau)} = 0; \quad 0 < \tau < 1; \quad \gamma \in \Re,\ \alpha \in \mathbb{N}, \alpha > 1 \tag{1}$$

$$\text{with initial conditions} \quad \varphi(0) = \varphi'(0) = 0 \tag{2}$$

Simplifying the solid fuel ignition model allowed for the development of the fundamental thermal combustion theory, which explains the significance of Eqn (1). This problem is known as a one-dimensional Bratu-type problem. Many investigations into Bratu-type differential equations have been carried out in order to achieve this. Safaa Ali Salema, Thair Younis Thanoona [1] used the perturbation method to solve the problem of the Bratu type. M. Zarebnia and M. Hoshyar [2] employed spline technique for the solution of Bratu's type problem. A.M. Wazwaz [3] used method of successive differentiation for solving Bratu's type equations. M. Saravi, M. Hermann and D. Kaiser [4] used He's variational Iteration Method for the solution of Bratu's equation. For the numerical solution of second order initial value problems of Bratu-type equations, H.B. Fenta and G.A. Derese [5] used the sixth order Runge-Kutta seven stages technique. Bratu-type problems were addressed by A. Ezekiel [6] using the New Improved Variational Homotopy Perturbation Technique. Bratu's problem was approximated analytically by H.N. Hassan and M. S. Semary [7] using the optimal homotopy analysis technique. When solving the Bratu problem numerically, Y. Aregbesola [8] employed the weighted residual technique. For the handling of equations of the Bratu's type, A.M. Wazwaz [9] employed Adomians' decomposition technique. For the resolution of Bratu's problem, Y. Changqing and H. Jianhua [10] used the Chebyshev wavelets method. Furthermore, Otaide I.J and Ugbene I.J [11] used Taylor's series method to obtain closed form solution to Bratu

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equations and also derived approximate solutions to Bratu problems [12] using variational iteration method via Shifted Legendre polynomials. In this study, the regular perturbation method is applied to Bratu differential equations with fractional exponent. This is accomplished by expressing each solution in a power series with regard to achieving an approximation solution and then comparing the influence of the non-linear part with a small coefficient ε known as the perturbation coefficient. The results so far are encouraging and reliable.

2. The Essential Concept of Regular Perturbation Method

Let the perturbed Bratu differential equation (1) be written in the form

$$\frac{d^2\varphi}{d\tau^2} + \gamma e^{\varepsilon\frac{1}{\alpha}\varphi(\tau)} = 0 \tag{3}$$

Let the nonlinear term in (3) be a small perturbation such that the solution (3) can be expressed as a power series solution for ε sufficiently small.
In the sequel we have the following:

$$\varphi(\tau) = \varphi_0(\tau) + \varepsilon\varphi_1(\tau) + \varepsilon^2\varphi_2(\tau) + \cdots \tag{4}$$

Where epsilon ε represents the perturbation parameters and lies in the range $0 < \varepsilon \leq 1$.
We now substitute (4) into (3) to obtain:

$$\left(\varphi''_0(\tau) + \varepsilon\varphi''_1(\tau) + \varepsilon^2\varphi''_2(\tau) + \cdots\right) + \gamma\left(1 + \frac{\varepsilon\varphi_0}{\alpha} + \frac{\varepsilon^2{\varphi_0}^2}{2\alpha^2} + \cdots\right) = 0$$

Comparing like powers of ε^0, ε^1 and ε^2 respectively, we have that:

$$\varphi''_0(\tau) = \gamma, \varphi''_1(\tau) = \frac{\varphi_0}{\alpha}\gamma \text{ and } \varphi''_2(\tau) = \frac{{\varphi_0}^2}{2\alpha^2}\gamma.$$

Finally, by recursive integration $\varphi_0(\tau)$, $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are derived.

3. Mathematical Applications

In this section, three Bratu differential equations were solved using the suggested methodology.

Example 1:

Consider the Bratu equation (1), by choosing $\gamma = -\frac{2}{3}$ and $\alpha = 2$ to obtain:

$$\frac{d^2\varphi}{d\tau^2} - \frac{2}{3}\, e^{\frac{1}{2}\varphi(\tau)} = 0, \quad 0 < \tau < 1 \tag{5}$$

$$\varphi(0) = \varphi'(0) = 0$$

The exact solution of the system (5), is given as:

$$\varphi(\tau) = \frac{\tau^2}{3} + \frac{\tau^4}{108} + \frac{\tau^6}{2430} + \cdots$$

Solution:

Let the perturbed Bratu differential equation (5) be written in the form

$$\frac{d^2\varphi}{d\tau^2} - \frac{2}{3}\, e^{\varepsilon\frac{1}{2}\varphi(\tau)} = 0$$

Substituting (4) into (5), we have that:

$$\left(\varphi''_0(\tau) + \varepsilon\varphi''_1(\tau) + \varepsilon^2\varphi''_2(\tau) + \cdots\right) - \frac{2}{3}\left(1 + \frac{\varepsilon\varphi_0}{2} + \frac{\varepsilon^2{\varphi_0}^2}{8} + \cdots\right) = 0$$

Comparing like powers of ε^0, ε^1 and ε^2 respectively, gives:

$$\varphi''_0(\tau) = \frac{2}{3},\ \varphi''_1(\tau) = \frac{\varphi_0}{3},\ \text{and}\ \varphi''_2(\tau) = \frac{{\varphi_0}^2}{12}.$$

By recursive integration, we obtain the following:

$$\varphi_0(\tau) = \frac{\tau^2}{3},\ \varphi_1(\tau) = \frac{\tau^4}{108}\ \text{and}\ \varphi_2(\tau) = \frac{\tau^6}{3240}$$

Hence the approximate solution is given as

$$\varphi(\tau) = \frac{\tau^2}{3} + \varepsilon\frac{\tau^4}{108} + \varepsilon^2\frac{\tau^6}{3240}$$

Table 1. Comparison of numerical results for $\varepsilon = 1$, $\varepsilon = 0.5$ and $\varepsilon = 0.1$ for example 1.

τ	Exact solution	Absolute error based on the suggested method for $\varepsilon = 1$	Absolute error based on the suggested method for $\varepsilon = 0.5$	Absolute error based on the suggested method for $\varepsilon = 0.1$
0.0	0.0000000000e-0	0.00000e-00	0.00000e-00	0.00000e-00
0.1	0.3334259671e-2	1.03000e-10	4.63298e-07	8.33742e-07
0.2	0.1334817448e-1	6.59000e-09	7.42880e-06	1.33595e-05
0.3	0.3007530000e-1	7.50000e-08	3.77438e-05	6.77978e-05
0.4	0.5357205597e-1	4.21000e-07	1.19888e-04	2.15006e-04
0.5	0.8391846707e-1	1.60751e-06	2.94576e-04	5.27215e-04
0.6	0.1212192000e-0	4.80000e-06	6.15600e-04	1.09906e-03
0.7	0.1656048966e-0	1.21038e-05	1.15091e-03	2.04889e-03
0.8	0.2172338041e-0	2.69696e-05	1.98395e-03	3.52040e-03
0.9	0.2762937000e-0	5.46750e-05	3.21519e-03	5.68456e-03
1.0	0.3430041152e-0	1.02881e-04	4.96399e-03	8.74177e-03

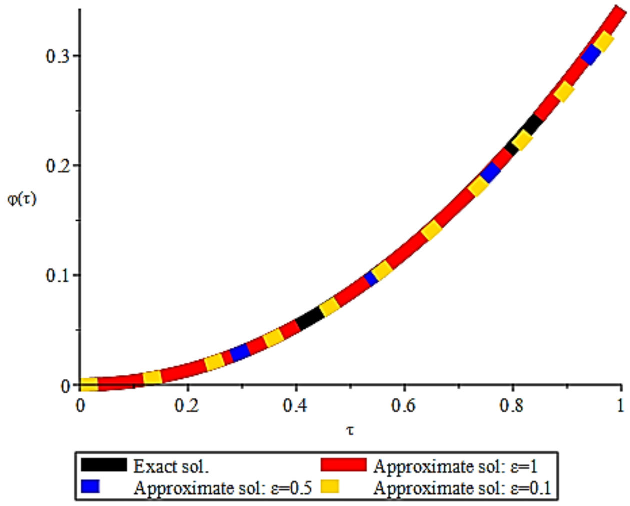

Figure 1. Graphical comparison of exact and approximate solutions for example 1.

Example 2:

Consider the Bratu equation (1), by choosing $\gamma = -\frac{1}{2}$ and $\alpha = 3$ to obtain:

$$\frac{d^2\varphi}{d\tau^2} - \frac{1}{2}\, e^{\frac{1}{3}\varphi(\tau)} = 0, \quad 0 < \tau < 1, \qquad (6)$$
$$\varphi(0) = \varphi'(0) = 0$$

The exact solution of the system (6), is given as:

$$\varphi(\tau) = \frac{\tau^2}{4} + \frac{\tau^4}{288} + \frac{\tau^6}{12960} + \cdots$$

Solution:

Let the perturbed Bratu differential equation (6) be written in the form

$$\frac{d^2\varphi}{d\tau^2} - \frac{1}{2}\, e^{\varepsilon\frac{1}{3}\varphi(\tau)} = 0$$

Substituting (4) into (6), we have that:

$$\left(\varphi''_0(\tau) + \varepsilon\varphi''_1(\tau) + \varepsilon^2\varphi''_2(\tau) + \cdots\right) - \frac{1}{2}\left(1 + \frac{\varepsilon\varphi_0}{3} + \frac{\varepsilon^2{\varphi_0}^2}{18} + \cdots\right) = 0$$

Comparing like powers of ε^0, ε^1 and ε^2 respectively, gives:

$$\varphi''_0(\tau) = \frac{1}{2},\ \varphi''_1(\tau) = \frac{\varphi_0}{6}, \text{ and } \varphi''_2(\tau) = \frac{{\varphi_0}^2}{36}.$$

By recursive integration, we obtain the following:

$$\varphi_0(\tau) = \frac{\tau^2}{4},\ \varphi_1(\tau) = \frac{\tau^4}{288} \text{ and } \varphi_2(\tau) = \frac{\tau^6}{17280}$$

Hence the approximate solution is given as

$$\varphi(\tau) = \frac{\tau^2}{4} + \varepsilon\frac{\tau^4}{288} + \varepsilon^2\frac{\tau^6}{17280}$$

Table 2. Comparison of numerical results for $\varepsilon = 1$, $\varepsilon = 0.5$ and $\varepsilon = 0.1$ for example 2

τ	Exact solution	Absolute error based on the suggested method for $\varepsilon = 1$	Absolute error based on the suggested method for $\varepsilon = 0.5$	Absolute error based on the suggested method for $\varepsilon = 0.1$
0.0	0.0000000000e-0	0.00000e-00	0.00000e-00	0.00000e-00
0.1	0.2500347299e-2	1.90000e-11	1.73674e-07	3.12576e-07
0.2	0.1000556050e-1	1.24000e-09	2.78179e-06	5.00490e-06
0.3	0.2252818125e-1	1.40600e-08	1.41082e-05	2.53683e-05
0.4	0.4008920494e-1	7.90100e-08	4.47012e-05	8.03137e-05
0.5	0.6271821952e-1	3.01410e-07	1.09487e-04	1.96509e-04
0.6	0.9045360000e-1	9.00000e-07	2.27925e-04	4.08573e-04
0.7	0.1233427585e-0	2.26950e-06	4.24216e-04	7.59322e-04
0.8	0.1614424494e-0	5.05680e-06	7.27546e-04	1.30008e-03
0.9	0.2048191312e-0	1.02515e-05	1.17238e-03	2.09101e-03
1.0	0.2535493827e-0	1.92901e-05	1.79880e-03	3.20158e-03

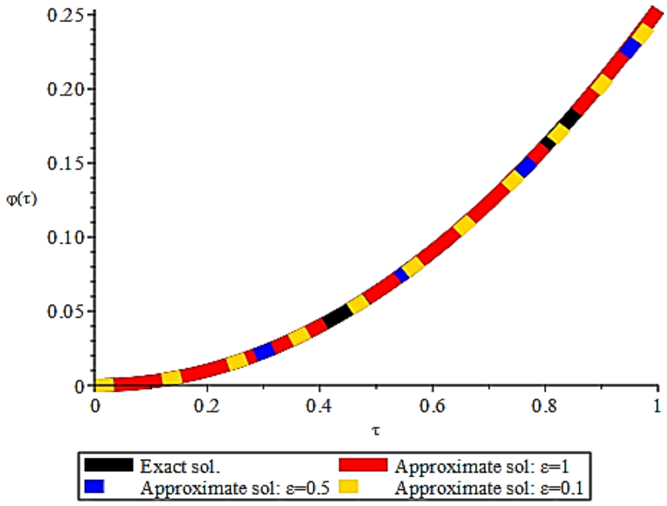

Figure 2. Graphical comparison of exact and approximate solutions for example 2

Example 3:

Consider the Bratu equation (1), by choosing $\gamma = -\frac{1}{5}$ and $\alpha = 4$ to obtain:

$$\frac{d^2\varphi}{d\tau^2} - \frac{1}{5}\, e^{\frac{1}{4}\varphi(\tau)} = 0, \quad 0 < \tau < 1, \tag{7}$$

$$\varphi(0) = \varphi'(0) = 0$$

The exact solution of the system (7), is given as:

$$\varphi(\tau) = \frac{\tau^2}{10} + \frac{\tau^4}{2400} + \frac{\tau^6}{360000} + \cdots$$

Solution:

Let the perturbed Bratu differential equation (7) be written in the form

$$\frac{d^2\varphi}{d\tau^2} - \frac{1}{5}\, e^{\varepsilon\frac{1}{4}\varphi(\tau)} = 0$$

Substituting (4) into (7), we have that:

$$\left(\varphi''_0(\tau) + \varepsilon\varphi''_1(\tau) + \varepsilon^2\varphi''_2(\tau) + \cdots\right) - \frac{1}{5}\left(1 + \frac{\varepsilon\varphi_0}{4} + \frac{\varepsilon^2{\varphi_0}^2}{32} + \cdots\right) = 0$$

Comparing like powers of ε^0, ε^1 and ε^2 respectively, gives:

$$\varphi''_0(\tau) = \frac{1}{5},\ \varphi''_1(\tau) = \frac{\varphi_0}{20},\ \text{and}\ \varphi''_2(\tau) = \frac{{\varphi_0}^2}{160}.$$

By recursive integration, we obtain the following:

$$\varphi_0(\tau) = \frac{\tau^2}{10},\ \varphi_1(\tau) = \frac{\tau^4}{2400}\ \text{and}\ \varphi_2(\tau) = \frac{\tau^6}{48000}$$

Hence the approximate solution is given as

$$\varphi(\tau) = \frac{\tau^2}{10} + \varepsilon\frac{\tau^4}{2400} + \varepsilon^2\frac{\tau^6}{48000}$$

Table 3. Comparison of numerical results for $\varepsilon = 1$, $\varepsilon = 0.5$ and $\varepsilon = 0.1$ for example 3

τ	Exact solution	Absolute error based on the suggested method for $\varepsilon = 1$	Absolute error based on the suggested method for $\varepsilon = 0.5$	Absolute error based on the suggested method for $\varepsilon = 0.1$
0.0	0.0000000000e-0	0.00000e-00	0.00000e-00	0.00000e-00
0.1	0.1000041670e-2	1.80000e-11	2.08320e-08	3.75030e-08
0.2	0.4000666845e-2	1.15500e-09	3.33179e-07	6.00165e-07
0.3	0.9003377025e-2	1.31630e-08	1.68573e-06	3.03937e-06
0.4	0.1601067805e-1	7.39500e-08	5.32339e-06	9.61053e-06
0.5	0.2502608507e-1	2.82120e-07	1.29829e-05	2.34776e-05
0.6	0.3605412960e-1	8.42400e-07	2.68866e-05	4.87199e-05
0.7	0.4910036847e-1	2.12422e-06	4.97349e-05	9.03398e-05
0.8	0.6417139485e-1	4.73315e-06	8.46962e-05	1.54274e-04
0.9	0.8127485122e-1	9.59547e-06	1.35396e-04	2.47403e-04
1.0	0.1004194445e-0	1.80555e-05	2.05903e-04	3.77570e-04

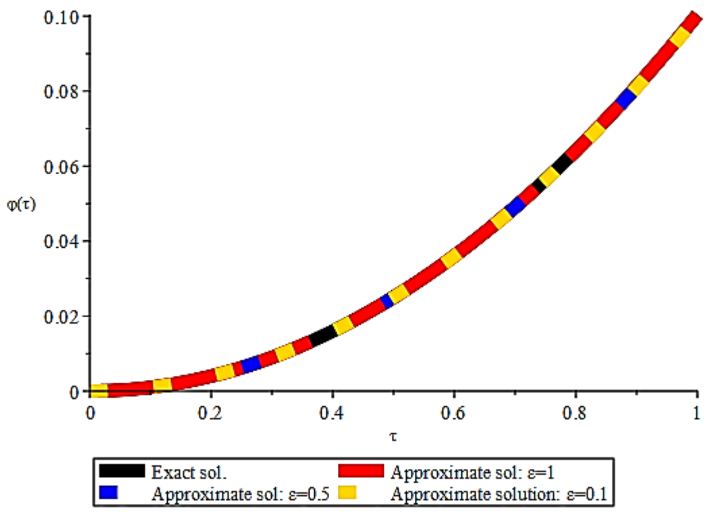

Figure 3. Graphical comparison of exact and approximate solutions for example 3.

4. Conclusion

This study has effectively produced numerical solutions for nonlinear Bratu differential equations with fractional exponents using the regular perturbation method. The approximate solutions for different values of ε has been successfully arrived at. Thus, from figures 1, 2 and 3, we can see that the results are close and accurate using different values for the perturbated parameter ε. From Tables 1, 2 and 3 it is observed that the suggested scheme converges better when $\varepsilon = 1$. Beyond recollection, the numerical findings demonstrated that the current method is a mathematical strategy that works well for the class of problems being studied.

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Declaration of Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authorship Contribution Statement

Ikechukwu Otaide: Supervision, Conceptualization, Methodology, Reviewing and Editing.
Egborge Usu Oghenerukevwe: Visualization, Investigation, Typing and Software.

References

[1] S. A. Salem, T. Y. Thanoon, "On solving Bratu's type equation by perturbation Method," Int. J. Nonlinear Anal. Appl., vol.13, no.1, pp. 2755-2763, Sep. 2021, doi:10.22075/ijnaa.2022.6000.

[2] M. Zarebnia and M. Hoshyar, "Solution of Bratu-type equation via spline method," Acta Univ. Apulensis, vol. 37, pp. 61–72, 2014.

[3] A.M. Wazwaz, "The successive differentiation method for solving Bratu equation and Bratu-type equations," Rom. J. Phys. Vol. 61, no.5-6, pp. 774–783, Sept. 2016.

[4] M. Saravi, M. Hermann and D. Kaiser, "Solution of Bratu's Equation by He's variational Iteration Method," Amer. J. Comput. Appl. Math., vol.3, no. 1, pp. 46-48, Mar. 2013, doi:10.5923/j.ajcam.20130301.08.

[5] H.B. Fenta and G.A. Derese, "Numerical solution of second order initial value problems of Bratu-type equations using sixth order Runge-Kutta seven stages method," Int. J. Comput. Sci. Appl. Math., vol. 5, no.1, Feb.2019, doi: 10.12962/j24775401.v5i1.3806.

[6] A. Ezekiel, "New Improved Variational Homotopy Perturbation Method for Bratu-Type Problems," Amer. J. Comput. Math., vol. 3, no. 2, pp. 110-113, Jan. 2013, doi: 10.4236/ajcm.2013.32018.

[7] H. N. Hassan and M. S. Semary, "Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method," Commun. Numerical Anal. vol. 2013, pp.1-14, Feb.2013, doi: 10.5899/2013/can-00139.

[8] Y. Aregbesola, "Numerical solution of Bratu problem using the method of weighted residual," Elect. J. South African Math. Soc., vol. 3, no. 1, pp.1-7, 2003.

[9] A.M. Wazwaz, "Adomians decomposition method for a reliable treatment of the Bratu-type equations," Appl. Math. Comput., vol.166, no. 3, pp. 652-663, Jul. 2005, doi: 10.1016/j.amc.2004.06.059.

[10] Y. Changqing and H. Jianhua, "Chebyshev wavelets method for solving Bratu's problem," Bound. Value Prob., vol 1, 142, Jun. 2013, doi: 10.1186/1687-2770-2013-142.

[11] I. J. Otaide and I. J. Ugbene, "Application of the Taylor Series Technique to the solution of Bratu Problems," FNAS-JMNS, vol.2, no.1, pp. 19-26, Mar. 2025.

[12] I.J. Otaide and I.J. Ugbene, "Step-By-Step Application of Shifted Legendre Polynomials on Numerical Assessment of Non-Linear Bratu Differential Equations," Zamfara International Journal of Education (ZIJE), vol.4, no.5, pp. 415-420, Dec. 2024, doi: 10.5281/zenodo.15258640.