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A CHAIN RULE FOR REDUCED FUNCTIONAL DIFFERENTIAL INCLUSIONS AND STABILITY THEOREMS

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Abstract: In order to represent real-world problems, modeling and stability concepts of a system are two essential steps, and functional differential inclusions become favorable among other methods because of their flexibility and robustness to handle those problems. Thus, functional differential inclusions (FDIs) provide a solid foundation for engineering problems, and the calculation of their derivatives becomes an important issue in checking the stability of them. Especially, to check the Lyapunov stability, various chain rules for FDIs are defined in the literature. In this work, a new chain rule is introduced in terms of the reduction procedure, a comparison with another one is represented, and the stability theorems in terms of Lyapunov are extended to the reduced functional differential inclusions.

Keywords: Functional differential inclusions, Set-valued analysis, Convex analysis, Stability

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1. Introduction

Functional differential inclusions (FDIs), a sophisticated branch of mathematical analysis, have emerged as a versatile framework for modeling and analyzing complex dynamical systems that exhibit intricate interactions between time-dependent variables and their histories. These powerful mathematical tools, which generalize the classic concept of differential equations, have proven invaluable in capturing the nuanced behaviors of systems that are influenced both by their current state and past trajectories. Functional differential inclusions are particularly valuable when modeling systems where the rate of change depends not only on the current state but also on the history of the state variables. These inclusions generalize functional differential equations so that they allow multi-valued right-hand sides. This allows the derivative to belong to a set of possible values rather than being uniquely defined. This property makes FDIs particularly useful for modeling systems with uncertainty, non-smooth dynamics, or control constraints. They provide benefits from different perspectives such as flexibility (FDIs handle non-uniqueness in solutions, making them suitable for systems with inherent ambiguity or multiple possible trajectories) or robustness (they model systems under uncertainty effectively, providing a framework for robust analysis and synthesis). They play a vital role in several domains such as control systems and optimization, modeling uncertainty, non-smooth and hybrid systems, population dynamics and biological models, material Science and mechanics, economics and finance in economic modeling. FDIs are widely applied in control theory, particularly in optimal control and differential games where the system's dynamics are influenced by control actions that may vary within a range of admissible values. This is common in situations with bang-bang control or state-dependent constraints (Aubin and Cellina, 1984). FDIs are used to model systems with uncertainty in the dynamics, such as when the exact form of the derivative is unknown but is known to lie within a specified set. This is critical in engineering and economics, where systems are influenced by external disturbances or imprecise measurements (Filippov, 1988). Hybrid systems that involve both continuous dynamics and discrete transitions often exhibit non-smooth behavior. FDIs can describe such systems' evolution by accommodating jumps, switches, and other discontinuities (Clarke et al., 1998). In population dynamics and epidemiology, FDIs may be used to account for time delays in reproduction or infection processes and to model systems with uncertain growth rates or carrying capacities. In economic modeling, FDIs may be used to represent markets where decisions depend on historical trends or delayed responses to changes in economic indicators. These applications align with the foundational works by Hale (1977) and Kolmanovskii and Myshkis (1992) in functional differential equations.

Apart from the previously mentioned works, recent studies on functional differential inclusions have advanced the theory and applications significantly. In (Aitalioubrahim and Raghib, 2023), explored existence results for inclusions driven by maximal monotone operators with nonconvex perturbations are explored, broadening classical frameworks. In (Bokalo et al., 2024), strong nonlinear functional-differential variational inequalities without initial conditions are focused on, addressing dynamic processes beyond fixed starts. Research on optimal control of hereditary differential inclusions has provided insights into systems with memory effects (Mahmudov and Mastaliyeva, 2024). Additionally, the topological properties of solution sets have been studied to better understand stability and robustness (Haddad, 1981b). These studies reflect a dynamic research front, bridging advanced mathematics with applications in areas such as engineering and the sciences.

In this work, a new chain rule for functional differential inclusions is obtained by the reduction procedure defined in (Kamalapurkar et al., 2020), a comparison with the chain rule in (Liu et al., 2015) is done and the stability theorems for the reduced functional differential inclusion are obtained. The paper's structure can be summarized as the following. In Section 2, preliminaries are mentioned and the necessary definitions and theorems in order to be utilized in Sections 5.1, 5.2 and 5.3 are given. In sections 3 and 4 a new chain rule is defined and a comparison between two chain rules is considered. In Sections 5.1, 5.2 and 5.3 stability theorems for autonomous and nonautonomous systems and Razumikhin type theorems are given, respectively.

2. Functional Differential Inclusions

Observe the functional (delay) differential inclusion;

$$\dot{x} \in \mathcal{F}(t, x_t) \tag{1}$$

where $\mathcal{F}: R_{\geq 0} \times D \rightrightarrows R^n$ is a functional that is set valued and moreover, it is bounded on closed bounded subsets of D. The solutions of equation (1) are assumed to satisfy basic assumptions in (Liu et al., 2016, p. 3216) and the property (Liu et al., 2016, Theorem 1). The definitions for solution, precompact solution, maximal solution can be found in (Surkov, 2007; Kamalapurkar et al., 2020). The definitions of weakly invariant set and strongly forward invariant set can be found in (LaSalle, 1976; Surkov, 2007; Kamalapurkar et al., 2020). The set-valued functional F satisfies the, $\mathcal{F}$ is upper semi-continuous and $\mathcal{F}(\phi)$ is nonempty, compact, and convex for each $\phi \in D$ where $D \subset C_r$ is an open set containing the origin and C_r is the collection of continuous functions from $[-r, 0]$ to $\mathcal{R}^n$, with the norm $\|\phi\|_r = max_{-r\leq s\leq 0}|\phi(s)|$ and $\phi \in C([-r, 0]; R^n)$. With the given basic assumptions there exists at least a solution for (1) on the interval$[t_0 - r, T]$ for some $T > 0$ (Liu et al., 2016, Haddad, 1981a). The chain rule for delay differential inclusion (1) is given by as the following in (Liu et al., 2016). A locally absolutely continuous function x is called a solution for 2.1 if it is a solution of 2.1 and verifies $x_{t_0} = \phi$. In this case, the solution will be represented by $x(t, t_0, \phi)$ where t_0 is the initial time and $\phi \in C_r$ is the initial value. The definitions of weakly forward invariant set, strongly forward invariant set, precompactness of solutions, the conditions that admits local solutions can be found in (Liu et al., 2015).

3. Set-Valued Derivatives

Proposition 3.1. Observe the functional (delay) differential inclusion (1). Let $V: R_{\geq 0} \times R \times \mathcal{C}_r \to R_{\geq 0}$ is a functional that is locally Lipschitz and moreover the following conditions hold (Liu et al., 2016):

(1) Functional V is composite locally absolutely continuous;

(2) For each point$(\alpha, \beta) \in (R_{\geq 0} \times R) \times \mathcal{C}_r$ it is possible to find the invariant directional derivative;

(3) $V'(\alpha, \beta, z) = V^o(\alpha, \beta, z)$ for each $(\alpha, \beta) \in (R_{\geq 0} \times R) \times \mathcal{C}_r$ and $z \in R_{\geq 0} \times R$

Then for any solution $x: [t_0 - r, T]$ of (1), it holds that for almost all $t \in [t_0, T)$

$$\dot{V}(t,x(t),x_t) \in \dot{\bar{V}}(t,x(t),x_t)$$

$$\textit{where } \dot{\bar{V}}(t,x(t),x_t) = \cap_{\xi\in\partial V(\alpha,x_t)}\xi \begin{bmatrix} 1 \\ \mathcal{F}(t,x_t) \\ 1 \end{bmatrix} \textit{ and } \alpha = (t,x(t)). \tag{2}$$

Since the chain rule in (Liu et al., 2016) represents an analogy of the chain rule defined in (Paden and Sastry, 1987) for the functional differential inclusions, then it is possible to define a chain rule analogous to the one defined in (Bacciotti, and Ceargioli, 1999).

Definition 3.2. For a locally Lipschitz functional $V: R_{\geq 0} \times R \times \mathcal{C}_r \to R_{\geq 0}$ that satisfies the conditions of (Liu et al., 2016, Proposition 1) (given in proposition 3.1), the set valued derivative of V with respect to 2.1 is defined as

$$\dot{\tilde{V}}(t,x(t),x_t) = \left\{a \in \mathbb{R} : \exists \psi \in \mathcal{F}(t,x_t) \text{ such that } \xi^T \begin{bmatrix} 1 \\ \psi \\ 1 \end{bmatrix} = a, \forall \xi \in \partial V(t,x(t),x_t)\right\}. \tag{3}$$

Proposition 3.3. Suppose that the set valued functional $\mathcal{F}: R_{\geq 0} \times D$ is upper semicontinuous, with compact, nonempty, and convex values. If V is a locally Lipschitz functional which satisfies properties of (Liu et al., 2016, Proposition 1) (given in proposition 3.1) and max $\dot{\bar{V}} \leq 0$ or max $\dot{\tilde{V}} \leq 0$, then the trivial solution of (1) is stable.

Proof. It is straightforward to verify the proof using results of (Shevitz and Paden, 1994; Bacciotti and Ceragioli, 1999).

4. Construction of Reduced Functional Differential Inclusions and Generalized Time Derivatives

Definition 4.1. For any regular, locally Lipschitz function $U: R_{\geq 0} \times R \times C_r \to R$ and the set-valued functional $H: R_{\geq 0} \times D \rightrightarrows R^n$, the reduction $R_U^H: R_{\geq 0} \times D \rightrightarrows R^n$ is define as.

$$R_U^H(t,x(t),x_t) = \{\psi \in \mathcal{F}(t,x_t) | \xi^T \begin{bmatrix} 1 \\ \psi \\ 1 \end{bmatrix} = 0, \forall \xi \in \partial V(t,x(t),x_t)\}. \quad (4)$$

By using 4.1, the chain rule 3.2 can be written as follows.

$$\max \dot{\overline{V}}(t,x(t),x_t) = \min_{p \in \partial V(t,x(t),x_t)} \max_{q \in R_V^{\mathcal{F}}(t,x_t)} \xi^T[1;\psi;1]. \quad (5)$$

Proposition 4.2. Suppose that $\mathcal{F}: R_{\geq 0} \times D \rightrightarrows R^n$ is a functional which is upper semi-contiuous and takes nonempty, convex and compact values. Let $V: R_{\geq 0} \times R \times C_r \to R$ be a locally Lipschitz functional with the conditions c1, c2 and c3 of (Liu et al., 2016, Proposition 1) and V : R≥0 × R × Cr → R be a regular functional. If then the trivial solution of 2.1 is stable.

$$\min_{\xi \in \partial V(t,x(t),x_t)} \max_{\psi \in R_U^{\mathcal{F}}(t,x_t)} \xi^T[1;\psi;1] \leq 0, \quad (6)$$

Proof. Using theorem 4.4 of this paper, one may prove the stability and global asymptotic stability of equation 5.1. Then the statement of the proposition is concluded. □

Definition 4.3. Let $\mathcal{U} = \{U_i\}_{i=1}^{\infty} where \{U_i\}_{i=1}^{\infty}$ where $\{U_i\}_{i=1}^{\infty}$ is a collection of real-valued locally Lipschitz regular functions. $\mathcal{F}: R \times \mathcal{C}_r$ is defined as

$$\mathcal{F}_U(t,x_t) = \mathcal{F}(t,x_t) \cap (\cap_{i=1}^{\infty} R_{U_i}^{\mathcal{F}}(t,x_t)). \quad (7)$$

Theorem 4.4. *Let U be defined as in Definition 4.3 and $x(t): \mathcal{I}_{\mathcal{D}} \to R$ is a solution of 2.1. Then, $\dot{x}(t) \in \mathcal{F}_{\mathcal{U}}(t,x_t)$ for almost all $t \in \mathcal{I}_{\mathcal{D}}$.*

Proof. By using the fact that x(t) is locally absolutely continuous and $U(t,x(t),\beta)$ is composite locally absolutely continuous, $U_i(t,x(t),x_t)$ is locally absolutely continuous. If E_0 is the set of measure zero such that x(t) and $U_i(t,x(t),x_t)$ are not differentiable. By following the steps in (Shevitz and Paden, 1994; Liu et al., 2016, Kamalapurkar et al., 2020) it can be verified that the right derivative of $U_i(t,x(t),x_t)$ is equal to

$$\dot{U}_i(t,x(t),x_t) = \max\{\xi \begin{bmatrix} 1 \\ \mathcal{F}(t,x_t) \\ 1 \end{bmatrix} | \xi \in \partial U_i(w,x_t)\}. \quad (8)$$

With a similar approach, the left derivative of $U_i(t,x(t),x_t)$ is equal to

$$\dot{U}_i(t,x(t),x_t) = \min\{\xi \begin{bmatrix} 1 \\ \mathcal{F}(t,x_t) \\ 1 \end{bmatrix} | \xi \in \partial U_i(w,x_t)\}. \quad (9)$$

The remaining steps follow from (Kamalapurkar et al., 2020, Theorem 1) and therefore $\dot{x}(t) \in \mathcal{F}_{\mathcal{U}}$

Generalized time derivative definitions are given as follows.

Definition 4.5. The U-generalized time derivative of V, $\dot{\overline{V}}_{\mathcal{U}}$, whenever V is regular, is equal to

$$\dot{\overline{V}}_{\mathcal{U}} := \min_{\xi \in \partial V(t,x(t),x_t)} \max_{\psi \in \tilde{\mathcal{F}}_{\mathcal{U}}(t,x_t)} \xi^T[1;\psi;1]. \quad (10)$$

The U-generalized time derivative of V, $\dot{\overline{V}}_{\mathcal{U}}$, whenever V is not regular, is equal to

$$\dot{\overline{V}}_{\mathcal{U}} := \max_{\xi \in \partial V(t,x(t),x_t)} \max_{\psi \in \tilde{\mathcal{F}}_{\mathcal{U}}(t,x_t)} \xi^T[1;\psi;1]. \quad (11)$$

Definition 4.6. If V is locally Lipschitz, positive definite and if $\dot{\overline{V}}_{\mathcal{U}} \leq 0$, then V is called a U-generalized Lyapunov function for 2.1.

Theorem 4.7. *If $V \in Lip(C_r, R)$, then $\forall x(\cdot) \in \mathcal{S}(\Omega)$,*

$$\dot{V}(t,x(t),x_t) \in (\partial V(t,x(t),x_t))^T \begin{bmatrix} 1 \\ \mathcal{F}_{\mathcal{U}}(t,x_t) \\ 1 \end{bmatrix} \quad (12)$$

for almost all $t \in \mathcal{I}_{\mathcal{D}}$. Moreover, if there exists a function $W: \Omega \to R$ such that $\dot{\overline{V}}_{\mathcal{U}}(t,x(t),x_t) \leq W(t,x(t),x_t)$, $\forall (t,x(t),x_t) \in \Omega$ then $\dot{V}_{\mathcal{U}}(t,x(t),x_t) \leq W(t,x(t),x_t)$, for almost all $t \in \mathcal{I}_{\mathcal{D}}$.

Proof. Investigate two cases: when V is regular or not. If regular, the result follows from (Bacciotti, and Ceargioli, 1999). If not regular, then the result is obtained from (Shevitz and Paden, 1994). □

5. Stability Theorems for Reduced Functional Differential Inclusions

5.1. Stability of Autonomous Systems

Consider the functional differential inclusion;

$$\dot{x} \in \mathcal{F}(x_t) \tag{13}$$

Theorem 5.1. *Let V be a functional such that $V: \mathcal{C} \rightarrow R$ and $\dot{\overline{V}}_{\mathcal{U}} \leq 0$ for all $\phi \in G$ and $f \in F(\phi)$. Moreover, assume that x(t) is precompact and x_t remains in G for all t ≥ 0. Then, for some c, x_t approaches $M_v \cap V^{-1}(c)$.*

Proof. Two cases are investigated. If V is regular then the result follows from (Liu et al., 2015, Theorem 1) and from very well-known Arzela-Ascoli theorem, omega limit set is compact. If V is not regular, then the result follows from (Kamalapurkar et al., 2020, Theorem 19) and (Moreau and Valadier, 1987).

5.2. Stability of Nonautonomous Systems

Definition 5.2. The differential inclusion (1) is said to be uniformly stable at x = 0 if $\forall \epsilon > 0$ $\exists \delta > 0$ such that if $x(\cdot) \in \overline{\mathcal{B}}(0,\delta) \times \mathcal{R}_{\geq 0}$, then $x(\cdot)$ is complete and $x(t) \in \overline{\mathrm{B}}(0,\epsilon)$, $\forall t \geq t_0$.

Theorem 5.3. Consider (1). Let $V: \mathrm{R}_{\geq 0} \times \mathcal{C}_r \rightarrow \mathrm{R}_{\geq 0}$ be a locally Lipschitz functional. If there exist positive definite functions u, v , w that is positive for s > 0, and u(0) = v(0) = 0. If the following condition is satisfied

$$u(|\phi(0)|) \leq V(t,\phi) \leq v(|\phi|) \tag{14}$$

$$\dot{\overline{V}}_{\mathcal{U}}(t,\phi) \leq -w(|\phi(0)|) \tag{15}$$

then the solution $x = 0$ of (2.1) is uniformly stable.

Proof. Investigation of two cases is enough to prove the theorem. If $\dot{\overline{V}}_{\mathcal{U}}$ is regular follow (Liu et al., 2016, Theorem 3). If $\dot{\overline{V}}_{\mathcal{U}}$ is not regular follow similar steps to (Kamalapurkar et al., 2020, Theorem 19) and (Moreau and Valadier, 1987).

5.3. Razumikhin Theorem

The chain rule will be defined by reduced Lyapunov functions in this case. More specifically, the chain rule will be defined as

$$\dot{\overline{V}}(t,x(t)) = \{a \in \mathbb{R} : \exists \psi \in \mathcal{F}(t,x_t) \text{ such that } \xi^T \begin{bmatrix} 1 \\ \psi \end{bmatrix} = a, \forall \xi \in \partial V(t,x(t))\}. \tag{16}$$

Theorem 5.4. Observe the equation (1). $u, v, w: \mathrm{R}^+ \rightarrow \mathrm{R}^+$ such that u, v, w are continuous functions and they are non-decreasing, u(s), v(s) positive whenever s > 0 and u(0) = v(0) = 0. Moreover, v is strictly increasing. Assume that a continuous function $V: \mathrm{R} \times \mathrm{R}^n \rightarrow \mathrm{R}$, with the following conditions

$$u(|x|) \leq V(t,x) \leq v(|x|), \quad t \in \mathbb{R},\, x \in \mathbb{R}^n, \tag{17}$$

and

$$\dot{\overline{V}}_{\mathcal{U}}(t,\phi(0)) \leq -w(|\phi(0)|) \;\; if \;\; V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)), \tag{18}$$

exist for $\theta \in [-r,0]$. Then the trivial solution, $x = 0$, of (2.1) is uniformly stable.

Proof. Start by choosing some variable θ_0 from [−r,0]. Check the cases where $\theta_0 < 0$ and θ_0. Then use of theorem 5.3 for non-autonomous case of this paper. This finishes the proof of uniform stability.

6. Conclusion

In summary, the chain rule and stability theorems constitute fundamental analytical tools for functional differential inclusions, offering a robust framework for the investigation of intricate system behaviors and the verification of solution dependability. The chain rule, particularly in its generalized forms applicable to set-valued mappings and non-smooth analysis, facilitates the computation of derivatives along solution trajectories, thereby enabling the characterization of system sensitivity to perturbations and parameter variations. Stability theorems, encompassing Lyapunov-like approaches and fixed-point methodologies, furnish criteria for ascertaining the qualitative properties of solutions, including boundedness, convergence, and robustness in the face of uncertainties. These theoretical underpinnings are not merely abstract mathematical constructs but have tangible implications across various scientific and engineering disciplines. Functional differential inclusions, which inherently incorporate memory effects and hereditary characteristics, demand sophisticated mathematical techniques for their analysis, where the chain rule allows for the effective propagation of derivative information through the functional arguments of the inclusion, while stability theorems provide a rigorous basis for assessing the long-term behavior of solutions, ensuring that the system's response remains within acceptable bounds.

Future investigations may extend the reduction-based chain rule to stochastic and more general classes of functional differential inclusions, broadening its theoretical scope. At the same time, integrating these analytical advances with computational techniques could facilitate the stability assessment of complex systems, reinforcing the connection between abstract analysis and applied practice.

Author Contributions

The percentages of the author' contributions are presented below. The author reviewed and approved the final version of the manuscript.

	N.G.
C	100
D	100
S	100
DCP	100
DAI	100
L	100
W	100
CR	100
SR	100

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision.

Conflict of Interest

The author declared that there is no conflict of interest.

Ethical Consideration

Authors of this manuscript follow all ethical guidelines including authorship, citation, data reporting, and publishing original research.

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