Local Pre-Hausdorff Objects in the Category of Interval Spaces

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ABSTRACT. In this paper, we characterize the local T_0, T_1, pre-Hausdorff, and Hausdorff objects in the topological category of interval spaces and interval-preserving maps and investigate the relationships among them. Then, we show that all these objects are hereditary and productive.

1. INTRODUCTION

Separation properties are fundamental in topology because they define various levels of "separability" and are crucial for analyzing and classifying topological spaces based on how well points and sets can be separated by neighborhoods. In 1991, Baran [2] introduced separation properties within the framework of arbitrary set-based topological categories by using initial-final lifts and discrete objects. He initially defined these properties locally, at a specific point p, and later extended them to point-free formulations using the generic element approach from topos theory.

One of these separation properties is the notion of pre-Hausdorff, which was extended to set-based topological categories by Baran [2,6], and has also been characterized in various categories by many authors (see [7,9,10,15]). In 1994, Mielke [11] demonstrated the significance of pre-Hausdorff objects in the general theory of geometric realization, particularly in relation to their intervals and associated homotopic structures. Later, in 1999, Mielke [12] employed pre-Hausdorff objects within topological categories to characterize decidable objects in topos theory. Pre-Hausdorff objects have also been utilized to define various forms of Hausdorff, regular and normal objects [3, 5]. Additionally, Stine [17] showed that there exists a connection between pre-Hausdorff objects and partitions.

Convexity plays a key role across various branches of mathematics. Nevertheless, vector spaces may not offer the most suitable framework for grasping the essential nature of convex sets. To address this limitation, abstract convex structures [19] were developed, finding applications in areas such as topology, graph theory, and lattice theory (see [13,16,18]). These structures can be defined in several ways, including through algebraic closure or hull operators.

In 1971, Calder [8] introduced interval operators –a natural extension of intervals– which offer a common and intuitive method for building convex structures. In 2019, Wang, Li and Xiu [20] investigated the relations between convex spaces and interval spaces from a categorical sense. They showed that there is an adjunction between the category of interval spaces and the category of convex spaces. Moreover, they also proved that the category of arity 2 convex spaces can be embedded in the category of interval spaces as a reflective subcategory.

The main goals of this paper are to characterize the local T_0, T_1, pre-Hausdorff and Hausdorff interval spaces with a categorical approach; to explore the relationships between them; and to determine whether these properties are hereditary and productive.

2. Preliminaries

Definition 2.1 ([19, 20]). Let X be a set. The mapping $\mathcal{I} : X \times X \to P(X)$ that satisfies the following is called an interval operator. For all $a, b \in X$,

- (I1) $a, b \in \mathcal{I}(a, b)$ (Extensive)
- (I2) $\mathcal{I}(a, b) = \mathcal{I}(b, a)$ (Symmetry).

$\mathcal{I}(a, b)$ is the interval between a and b. The pair $(X, \mathcal{I})$ is called an interval space.

Definition 2.2 ([20]). For any non-empty sets $X, \mathcal{Y}$, and any mapping $f : X \to \mathcal{Y}$,

- (1) The mapping $f^{\to} : P(X) \to P(\mathcal{Y})$ is called forward mapping defined by $f^{\to}(A) = \{f(x) \mid x \in A\}$ for any $A \in P(X)$.
- (2) The mapping $f^{\leftarrow} : P(\mathcal{Y}) \to P(X)$ is called backward mapping defined by $f^{\leftarrow}(B) = \{x \mid f(x) \in B\}$ for any $B \in P(\mathcal{Y})$.

Definition 2.3 ([20]). Let $(X, \mathcal{I}_X)$ and $(\mathcal{Y}, \mathcal{I}_\mathcal{Y})$ be interval spaces. Then, the mapping $f : (X, \mathcal{I}_X) \to (\mathcal{Y}, \mathcal{I}_\mathcal{Y})$ is called an interval preserving provided that for all $a, b \in X$,

$$f^{\to}(\mathcal{I}_X(a, b)) \subseteq \mathcal{I}_\mathcal{Y}(f(a), f(b)).$$

All interval spaces and interval preserving mappings form a category, denoted by **IS**.

Example 2.4 ([19, 20]).   (1) Let $\mathbb{R}$ be the set of real numbers and define a mapping $\mathcal{I}_\mathbb{R} : \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ by for all $a, b \in \mathbb{R}$,

$$\mathcal{I}_\mathbb{R}(a, b) = [min\{a, b\}, max\{a, b\}],$$

where $\mathcal{I}_\mathbb{R}$ indicates the interval operator on $\mathbb{R}$.

- (2) Let d be a metric on a set X and define a mapping $\mathcal{I}_d : X \times X \to P(X)$ by for all $a, b \in X$,

$$\mathcal{I}_d(a, b) = \{x \in X \mid d(a, b) = d(a, x) + d(x, y)\},$$

where $\mathcal{I}_d$ indicates the geodesic interval operator on X.

- (3) Let $\mathcal{V}$ be a vector space and define a mapping $\mathcal{I}_\mathcal{V} : \mathcal{V} \times \mathcal{V} \to P(\mathcal{V})$ by for all $a, b \in \mathcal{V}$,

$$\mathcal{I}_\mathcal{V}(a, b) = \{at + (1 - t)b \mid 0 \le t \le 1\},$$

where $\mathcal{I}_\mathcal{V}$ indicates the standard interval operator on $\mathcal{V}$.

- (4) Let $(X, \le)$ be a partially ordered set and define a mapping $\mathcal{I}_\le : X \times X \to P(X)$ by for all $a, b \in X$,

$$\mathcal{I}_\le(a, b) = \begin{cases} \{a, b\}, & \text{if } a, b \text{ are incomparable} \\ \{x \mid a \le x \le b\}, & \text{if } a \le b \end{cases},$$

where $\mathcal{I}_\le$ indicates the ordered interval operator on X.

- (5) Let $\mathcal{S}$ be a semilattice and define a mapping $\mathcal{I}_\mathcal{S} : \mathcal{S} \times \mathcal{S} \to P(\mathcal{S})$ by for all $a, b \in \mathcal{S}$,

$$\mathcal{I}_\mathcal{S}(a, b) = \{x \in \mathcal{S} \mid a \wedge b \le x \le a \text{ or } a \wedge b \le x \le b\},$$

where $\mathcal{I}_\mathcal{S}$ indicates the semilattice interval operator on $\mathcal{S}$.

- (6) Let $\mathcal{L}$ be a lattice and define a mapping $\mathcal{I}_\mathcal{L} : \mathcal{L} \times \mathcal{L} \to P(\mathcal{L})$ by for all $a, b \in \mathcal{L}$,

$$\mathcal{I}_\mathcal{L}(a, b) = \{x \in \mathcal{L} \mid a \wedge b \le x \le a \vee b\} = [a \wedge b, a \vee b],$$

where $\mathcal{I}_\mathcal{L}$ indicates the lattice interval operator on $\mathcal{L}$.

Definition 2.5 ([19]). Let $(X, \mathcal{I})$ be an interval space. Then, a set $\mathcal{Y} \subseteq X$ can be equipped with the relative interval operator, which is defined as follows: For $a, b \in X$,

$$\mathcal{I}_\mathcal{Y}(a, b) = \mathcal{I}(a, b) \cap \mathcal{Y}.$$

The resulting interval space $(\mathcal{Y}, \mathcal{I}_\mathcal{Y})$ is called a subspace of $(X, \mathcal{I})$.

Definition 2.6 ([19]). Let $\{(X_i, \mathcal{I}_i) \mid i \in I\}$ be a collection of interval spaces and $X = \prod_{i \in I} X_i$ be a product set. The product of interval spaces $(X_i, \mathcal{I}_i)$ for $i \in I$ is defined by the following interval operator $\mathcal{I}_*$ on X.

$$\mathcal{I}_*(a, b) = \{ (x_i)_{i \in I} \mid x_i \in \mathcal{I}_i(a_i, b_i) \},$$

where $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$. The pair $(\mathcal{X}, \mathcal{I}_*)$ is called a product space of $(\mathcal{X}_i, \mathcal{I}_i)$ for $i \in I$.

Let $\mathcal{X} \neq \varnothing$ be a set and $\mathcal{A} = \{A_i \mid i \in I\} \overset{dir}{\subseteq} P(\mathcal{X})$ denotes the directed subset of $\mathcal{X}$, which means for any $C, D \in \mathcal{A}$, there exists $E \in \mathcal{A}$ such that $C \subseteq E$ and $D \subseteq E$ [20].

Definition 2.7 ([19, 20]). A convex structure C on the set $\mathcal{X}$ is a subset of $P(\mathcal{X})$ that satisfies the following:

(C1) $\varnothing, \mathcal{X} \in \mathsf{C}$

(C2) $\{A_i \mid i \in I\} \subseteq \mathsf{C}$ implies $\bigcap_{i \in I} A_i \in \mathsf{C}$

(C3) $\{A_j \mid j \in J\} \overset{dir}{\subseteq} \mathsf{C}$ implies $\bigcup_{j \in J} A_j \in \mathsf{C}$.

The pair $(\mathcal{X}, \mathsf{C})$ is called convexity space. The members of C are called convex sets and their complements are called concave sets.

By these axioms, a subset A of a convexity space $(\mathcal{X}, \mathsf{C})$ is included in at least one convex set, namely $\mathcal{X}$. Also, A is included in a smallest convex set

$$co(A) = \bigcap \{C : A \subseteq C \in \mathsf{C}\}.$$

This set is called the convex hull of A. If the set A is finite, then the set $co(A)$ is called polytope [19].

Definition 2.8 ([19, 20]). A convexity space $(\mathcal{X}, \mathsf{C})$ is called arity 2 convexity space if it is satisfies for all $A \in P(\mathcal{X})$ and for all $x, y \in A$,

$$co(\{x, y\}) \subseteq A \text{ implies } A \in \mathsf{C}.$$

Definition 2.9 ([20]). Let $(\mathcal{X}, \mathsf{C}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathsf{C}_{\mathcal{Y}})$ be convexity spaces. Then, the mapping $f : (\mathcal{X}, \mathsf{C}_{\mathcal{X}}) \to (\mathcal{Y}, \mathsf{C}_{\mathcal{Y}})$ is called a convexity preserving provided that

$$B \in \mathsf{C}_{\mathcal{Y}} \text{ implies } f^{\leftarrow}(B) \in \mathsf{C}_{\mathcal{X}}.$$

Remark 2.10. Let **CS** (**CS(2)**) denotes the category of convexity spaces (arity 2 convexity spaces) and convexity preserving mappings. Note that **IS** is the full subcategory of **CS**, and **CS(2)** can be embedded in **IS** as a reflexive subcategory [20].

All preliminary information on convexity and interval spaces and can be found in [19, 20].

Definition 2.11 ([1, 14]). A functor $\mathbb{U} : C \to \mathbf{SET}$ (the category of sets and functions) is called topological if it satisfies the following:

(1) $\mathbb{U}$ is concrete (i.e., amnestic and faithful).

(2) $\mathbb{U}$ consists of small (i.e., set) fibers.

(3) Every $\mathbb{U}$-source has a unique initial lift, i.e., if for every source $\{f_i : X \to (\mathcal{X}_i, \alpha_i) \mid i \in I\}$ there exists a unique structure β on $\mathcal{X}$ such that $g : (\mathcal{Y}, \gamma) \to (\mathcal{X}, \beta)$ is a morphism if and only if for each $i \in I$, $f_i \circ g : (\mathcal{Y}, \gamma) \to (\mathcal{X}_i, \alpha_i)$ is a morphism.

(3′) Each $\mathbb{U}$-sink has a unique final lift. This condition is equivalent to the condition (3).

In addition, if the subterminals have a unique structure, the topological functor $\mathbb{U}$ is called normalized [1, 14].

Remark 2.12. The functor $\mathbb{U} : \mathbf{IS} \to \mathbf{SET}$ is topological [20]. It is also normalized since a unique interval structure exists on the empty set or on a one-point set.

Note that a topological functor $\mathbb{U} : C \to \mathbf{SET}$ has a left adjoint $\mathbb{D} : \mathbf{SET} \to C$, called the discrete functor. An object X in C is called a discrete object if and only if for each object Y in C, every map $\mathbb{U}(X) \to \mathbb{U}(Y)$ lifts to the map $X \to Y$ [1, 14].

Lemma 2.13 ([20]). *Let $\{(\mathcal{X}_i, \mathcal{I}_i) \mid i \in I\}$ be a collection of interval spaces and $\{f_i : (\mathcal{X}, \mathcal{I}') \to (\mathcal{X}_i, \mathcal{I}_i) \mid i \in I\}$ be a source. For any $a, b \in \mathcal{X}$, the initial interval structure on $\mathcal{X}$ is defined as follows:*

$$\mathcal{I}'(a, b) = \bigcap_{i \in I} f_i^{\leftarrow}(\mathcal{I}_i(f_i(a), f_i(b))).$$

Lemma 2.14 ([20])**.** *Let $(X, \mathcal{I})$ be an interval space. Then, the discrete and indiscrete interval structures are given below:*

   (1) *For any distinct $a, b \in X$, $\mathcal{I}_D(a, b) = \{a, b\}$ is the discrete interval structure on X.*
   (2) *For all $a, b \in X$, $\mathcal{I}_I(a, b) = X$ is the indiscrete interval structure on X.*

3. LOCAL T_0, T_1, PRE-HAUSDORFF AND HAUSDORFF INTERVAL SPACES

Let X be a set and p be a point in X. Let $X \vee_p X$ be the wedge product of X at p ([2], p. 334), i.e., two disjoint copies of X identified at p, or in other words, the pushout of $p : 1 \to X$ along itself (where 1 is the terminal object in **SET**).

A point x in $X \vee_p X$ is denoted as x_1 (x_2) if it lies in the first (second) component.

Definition 3.1 ([2])**.** Let X^2 be the cartesian product of X.

   (1) The mapping $\mathbb{A}_p : X \vee_p X \to X^2$ is called principal p-axis map provided that

$$\mathbb{A}_p(x_i) = \begin{cases} (x, p), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

   (2) The mapping $\mathbb{S}_p : X \vee_p X \to X^2$ is called skewed p-axis map provided that

$$\mathbb{S}_p(x_i) = \begin{cases} (x, x), & i = 1 \\ (p, x), & i = 2 \end{cases}$$

   (3) The mapping $\nabla_p : X \vee_p X \to X$ is called fold map at p provided that $\nabla_p(x_i) = x$ for $i = 1, 2$.

Definition 3.2. [2] Let the functor $\mathbb{U} : C \to$ **SET** be topological, $X \in Ob(C)$ with $\mathbb{U}(X) = A$ and $p \in X$.

   (1) X is $\overline{T_0}$ at p provided that the initial lift of the $\mathbb{U}$-source $\{\mathbb{A}_p : X \vee_p X \to \mathbb{U}(X^2) = A^2$ and $\nabla_p : X \vee_p X \to \mathbb{U}\mathbb{D}(A) = A\}$ is discrete, where $\mathbb{D}$ is the discrete functor.
   (2) X is T_1 at p provided that the initial lift of the $\mathbb{U}$-source $\{\mathbb{S}_p : X \vee_p X \to \mathbb{U}(X^2) = A^2$ and $\nabla_p : X \vee_p X \to \mathbb{U}\mathbb{D}(A) = A\}$ is discrete.
   (3) X is $pre\overline{T}_2$ at p provided that the initial lift of the $\mathbb{U}$-source $\{\mathbb{A}_p : X \vee_p X \to \mathbb{U}(X^2) = A^2\}$ and the initial lift of the $\mathbb{U}$-source $\{\mathbb{S}_p : X \vee_p X \to \mathbb{U}(X^2) = A^2\}$ agree.
   (4) X is $\overline{T}_2$ at p provided that X is $\overline{T}_0$ at p and $pre\overline{T}_2$ at p.

Remark 3.3. Note that, for the category **TOP** of topological spaces and continuous functions,

   (1) $\overline{T_0}$ at p (T_1 at p) reduces to the usual local T_0 (local T_1) separation axiom, i.e., for each point x distinct from p, there exists a neighborhood of x missing p or (and) there exists a neighborhood of p missing x [2].
   (2) $pre\overline{T}_2$ at p reduces to the following (called $preT_2$ at p [4]): for each point x distinct from p, if the set $\{x, p\}$ is not indiscrete, then there exist disjoint neighborhoods of x and p [2].

Theorem 3.4. *An interval space $(X, \mathcal{I})$ is $\overline{T_0}$ at p.*

Proof. Let $(X, \mathcal{I})$ be an interval space and $p \in X$. We show that $(X, \mathcal{I})$ is $\overline{T_0}$ at p.

Let $\mathcal{I}'$ be the initial interval structure on $X \vee_p X$ induced by $\mathbb{A}_p : X \vee_p X \to \mathbb{U}(X^2, \mathcal{I}^2) = X^2$ and $\nabla_p : X \vee_p X \to \mathbb{U}(X, \mathcal{I}_D) = X$, where $\mathcal{I}^2$ is the product interval structure on X^2 induced by the projection maps $proj_i : X^2 \to X$ for $i = 1, 2$, and $\mathcal{I}_D$ is the discrete interval structure on X.

Let a and b be any points in the wedge $X \vee_p X$. We consider the following cases.

   (1) If $a = b$, then

$$proj_i\mathbb{A}_p(a) = proj_i\mathbb{A}_p(b), i = 1, 2, \text{ and } \nabla_p(a) = \nabla_p(b).$$

Accordingly, we have

$$proj_i\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(b))) = proj_i\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(a))), \ i = 1, 2,$$

and it follows that

$$a \in proj_i\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(b))) \text{ for } i = 1, 2.$$

Also,

$$\nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(a)))$$
$$= \nabla_p^{\leftarrow}(\{\nabla_p(a)\})$$
$$= \{a\}.$$

By Lemma 2.13, we have

$$\mathcal{I}'(a, b) = \mathcal{I}'(a, a)$$
$$= proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(a))) \cap proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(a))) \cap \nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(a)))$$
$$= \{a\}.$$

Hence, $\mathcal{I}'$ is a discrete interval structure.

(2) If $a \neq b$ and $\nabla_p(a) \neq \nabla_p(b)$, then by Definition 2.1,

$$proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(b) \in \mathcal{I}(proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(b)), \; i = 1, 2,$$

and it follows that

$$a, b \in proj_i\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i\mathbb{A}_p(a), proj_i\mathbb{A}_p(b))) \text{ for } i = 1, 2.$$

Also,

$$\nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \nabla_p^{\leftarrow}(\{\nabla_p(a), \nabla_p(b)\})$$
$$= \{a, b\}.$$

By Lemma 2.13, we have

$$\mathcal{I}'(a, b) = proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) \cap proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) \cap \nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b)))$$
$$= \{a, b\}.$$

Hence, $\mathcal{I}'$ is a discrete interval structure.

(3) If $a \neq b$ and $\nabla_p(a) = \nabla_p(b)$, then $\nabla_p(a) = x = \nabla_p(b)$ for some points $x \in X$ distinct from p. It follows that $a = x_1, b = x_2$ or $a = x_2, b = x_1$ since $a \neq b$.

(a) If $a = x_1$ and $b = x_2$, then

$$proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) = proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{A}_p(x_1), proj_1\mathbb{A}_p(x_2)))$$
$$= proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p)),$$

and

$$proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) = proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{A}_p(x_1), proj_2\mathbb{A}_p(x_2)))$$
$$= proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(p, x)).$$

Since $proj_1\mathbb{A}_p(x_1) = x$, $proj_1\mathbb{A}_p(x_2) = p$ and $x, p \in \mathcal{I}(x, p)$, then $x_1, x_2 \in proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p))$. Similarly, we have $x_1, x_2 \in proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(p, x))$ since $proj_2\mathbb{A}_p(x_2) = x$, $proj_2\mathbb{A}_p(x_1) = p$ and $x, p \in \mathcal{I}(p, x)$.
By Lemma 2.14,

$$\mathcal{I}_D(\nabla_p(a), \nabla_p(b)) = \mathcal{I}_D(\nabla_p(x_1), \nabla_p(x_2))$$
$$= \mathcal{I}_D(x, x)$$
$$= \{x\}$$

and it follows that

$$\nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \nabla_p^{\leftarrow}(\{x\})$$
$$= \{x_1, x_2\}.$$

By Lemma 2.13,

$$\mathcal{I}'(a,b) = proj_1\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) \cap proj_2\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) \cap \nabla_p^\leftarrow(\mathcal{I}_D(\nabla_p(a), \nabla_p(b)))$$
$$= proj_1\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) \cap proj_2\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) \cap \{x_1, x_2\}$$
$$= \{x_1, x_2\}$$
$$= \{a, b\}.$$

(b) Similarly, if $a = x_2$ and $b = x_1$, then we have

$$\mathcal{I}'(a,b) = \{x_1, x_2\}$$
$$= \{a, b\}.$$

Consequently, $\mathcal{I}'$ is a discrete interval structure for all cases and by Definition 3.2, $(\mathcal{X}, \mathcal{I})$ is $\overline{T_0}$ at p.   □

Theorem 3.5. *An interval space $(\mathcal{X}, \mathcal{I})$ is T_1 at p.*

Proof. Let $(\mathcal{X}, \mathcal{I})$ be an interval space and $p \in \mathcal{X}$. Let $\mathcal{I}'$ be the initial interval structure on $\mathcal{X} \vee_p \mathcal{X}$ induced by $\mathbb{S}_p : \mathcal{X} \vee_p \mathcal{X} \to \mathbb{U}(\mathcal{X}^2, \mathcal{I}^2) = \mathcal{X}^2$ and $\nabla_p : \mathcal{X} \vee_p \mathcal{X} \to \mathbb{U}(\mathcal{X}, \mathcal{I}_D) = \mathcal{X}$, where $\mathcal{I}_D$ is the discrete interval structure on $\mathcal{X}$ and $\mathcal{I}^2$ is the product interval structure on $\mathcal{X}^2$ induced by the projection maps $proj_1, proj_2$.

Suppose $a, b \in \mathcal{X} \vee_p \mathcal{X}$ are any points. We consider the following cases.

(1) If $a = b$, then

$$proj_i\mathbb{S}_p(a) = proj_i\mathbb{S}_p(b), i = 1, 2, \text{ and } \nabla_p(a) = \nabla_p(b).$$

It follows that

$$a \in proj_i\mathbb{S}_p^\leftarrow(\mathcal{I}(proj_i\mathbb{S}_p(a), proj_i\mathbb{A}_p(b))), i = 1, 2$$

and

$$\nabla_p^\leftarrow(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \nabla_p^\leftarrow(\{\nabla_p(a)\})$$
$$= \{a\}.$$

By Lemma 2.13, we obtain

$$\mathcal{I}'(a,b) = \mathcal{I}'(a,a)$$
$$= proj_1\mathbb{S}_p^\leftarrow(\mathcal{I}(proj_1\mathbb{S}_p(a), proj_1\mathbb{S}_p(a))) \cap proj_2\mathbb{S}_p^\leftarrow(\mathcal{I}(proj_2\mathbb{S}_p(a), proj_2\mathbb{S}_p(a))) \cap \nabla_p^\leftarrow(\mathcal{I}_D(\nabla_p(a), \nabla_p(a)))$$
$$= \{a\}.$$

Hence, $\mathcal{I}'$ is a discrete interval structure.

(2) Suppose $a \neq b$ and $\nabla_p(a) \neq \nabla_p(b)$. By Definition 2.1, since

$$proj_i\mathbb{S}_p(a), proj_i\mathbb{S}_p(b) \quad \in \quad \mathcal{I}(proj_i\mathbb{S}_p(a), proj_i\mathbb{S}_p(b)), \ i = 1, 2,$$

then

$$a, b \in proj_i\mathbb{S}_p^\leftarrow(\mathcal{I}(proj_i\mathbb{S}_p(a), proj_i\mathbb{A}_p(b))) \text{ for } i = 1, 2.$$

Moreover,

$$\nabla_p^\leftarrow(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \nabla_p^\leftarrow(\{\nabla_p(a), \nabla_p(b)\})$$
$$= \{a, b\}.$$

By Lemma 2.13, we obtain

$$\mathcal{I}'(a,b) = proj_1\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) \cap proj_2\mathbb{A}_p^\leftarrow(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) \cap \nabla_p^\leftarrow(\mathcal{I}_D(\nabla_p(a), \nabla_p(b)))$$
$$= \{a, b\}.$$

Hence, $\mathcal{I}'$ is a discrete interval structure.

(3) Suppose $a \neq b$ and $\nabla_p(a) = \nabla_p(b)$. In that case there exist some points $x \in \mathcal{X}$ distinct from p such that $\nabla_p(a) = x = \nabla_p(b)$. So we must have $a = x_1, b = x_2$ or $a = x_2, b = x_1$.

(a) If $a = x_1$ and $b = x_2$, then

$$proj_1\mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{S}_p(a), proj_1\mathbb{S}_p(b))) = proj_1\mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{S}_p(x_1), proj_1\mathbb{S}_p(x_2)))$$
$$= proj_1\mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, p)),$$
$$proj_2\mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{S}_p(a), proj_2\mathbb{S}_p(b))) = proj_2\mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{S}_p(x_1), proj_2\mathbb{S}_p(x_2)))$$
$$= proj_2\mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, x)).$$

It follows that $x_1, x_2 \in proj_1\mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, p))$ since $x, p \in \mathcal{I}(x, p)$ and $proj_1\mathbb{S}_p(x_1) = x$, $proj_1\mathbb{S}_p(x_2) = p$. Also, we obtain $x_1, x_2 \in proj_2\mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, x))$ since $proj_2\mathbb{S}_p(x_1) = x = proj_2\mathbb{S}_p(x_2)$ and $x \in \mathcal{I}(x, x)$. By Lemma 2.14,

$$\mathcal{I}_D(\nabla_p(a), \nabla_p(b)) = \mathcal{I}_D(x, x)$$
$$= \{x\}$$

and

$$\nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b))) = \{x_1, x_2\}.$$

By Lemma 2.13,

$$\mathcal{I}'(a, b) = proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1\mathbb{A}_p(a), proj_1\mathbb{A}_p(b))) \cap proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2\mathbb{A}_p(a), proj_2\mathbb{A}_p(b))) \cap \nabla_p^{\leftarrow}(\mathcal{I}_D(\nabla_p(a), \nabla_p(b)))$$
$$= \{x_1, x_2\}$$
$$= \{a, b\}.$$

(b) Similarly, if $a = x_2$ and $b = x_1$, then we have

$$\mathcal{I}'(a, b) = \{a, b\}.$$

Consequently, $\mathcal{I}'$ is a discrete interval structure for all cases and by Definition 3.2, $(\mathcal{X}, \mathcal{I})$ is T_1 at p. □

Theorem 3.6. *An interval space $(\mathcal{X}, \mathcal{I})$ is $pre\overline{T_2}$ at p if and only if $\mathcal{X}$ has at most one point, i.e., $\mathcal{X} = \{p\}$.*

Proof. Let $(\mathcal{X}, \mathcal{I})$ be an interval space and $p \in \mathcal{X}$. Suppose that $(\mathcal{X}, \mathcal{I})$ is $pre\overline{T_2}$ at p. Let $\mathcal{I}_A$ and $\mathcal{I}_S$ be the initial interval structures on $\mathcal{X} \vee_p \mathcal{X}$ induced by $\mathbb{A}_p : \mathcal{X} \vee_p \mathcal{X} \to (\mathcal{X}^2, \mathcal{I}^2)$ and $\mathbb{S}_p : \mathcal{X} \vee_p \mathcal{X} \to (\mathcal{X}^2, \mathcal{I}^2)$, respectively, where $proj_i : \mathcal{X}^2 \to \mathcal{X}$ are the projection maps for $i = 1, 2$, and $\mathcal{I}^2$ is the product interval structure on $\mathcal{X}^2$ induced by $proj_1$, $proj_2$. Since $(\mathcal{X}, \mathcal{I})$ is $pre\overline{T_2}$ at p, $\mathcal{I}_{\mathbb{A}}(a, b) = \mathcal{I}_{\mathbb{S}}(a, b)$ for all $a, b \in \mathcal{X} \vee_p \mathcal{X}$.

(1) First, assume that $\mathcal{X}$ has at least two points, i.e., $\mathcal{X} = \{x, p\}$ where $x \neq p$. Then, $\mathcal{I}(x, p) = \{x, p\} = \mathcal{X}$. For $(a, b) = (x_1, x_2)$, by Lemma 2.13, we have

$$\mathcal{I}_{\mathbb{A}}(x_1, x_2) = \bigcap_{i=1}^{2} proj_i\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i\mathbb{A}_p(x_1), proj_i\mathbb{A}_p(x_2)))$$
$$= proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1(x, p), proj_1(p, x))) \cap proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2(x, p), proj_2(p, x)))$$
$$= proj_1\mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p)) \cap proj_2\mathbb{A}_p^{\leftarrow}(\mathcal{I}(p, x))$$
$$= \mathbb{A}_p^{\leftarrow}(proj_1^{\leftarrow}(\{x, p\})) \cap \mathbb{A}_p^{\leftarrow}(proj_2^{\leftarrow}(\{x, p\}))$$
$$= \mathbb{A}_p^{\leftarrow}(\mathcal{X} \times \mathcal{X}) \cap \mathbb{A}_p^{\leftarrow}(\mathcal{X} \times \mathcal{X})$$
$$= \mathbb{A}_p^{\leftarrow}(\mathcal{X} \times \mathcal{X})$$
$$= \mathcal{X} \vee_p \mathcal{X} = \{x_1, x_2, p\},$$

and

$$I_{\mathbb{S}}(x_1, x_2) = \bigcap_{i=1}^{2} proj_i \mathbb{S}_p^{\leftarrow}(I(proj_i \mathbb{S}_p(x_1), proj_i \mathbb{S}_p(x_2)))$$
$$= proj_1 \mathbb{S}_p^{\leftarrow}(I(proj_1(x, x), proj_1(p, x)) \cap proj_2 \mathbb{S}_p^{\leftarrow}(I(proj_2(x, x), proj_2(p, x)))$$
$$= proj_1 \mathbb{S}_p^{\leftarrow}(I(x, p)) \cap proj_2 \mathbb{S}_p^{\leftarrow}(I(x, x))$$
$$= \mathbb{S}_p^{\leftarrow}(proj_1^{\leftarrow}(\{x, p\})) \cap \mathbb{S}_p^{\leftarrow}(proj_2^{\leftarrow}(\{x\}))$$
$$= \mathbb{S}_p^{\leftarrow}(X \times X) \cap \mathbb{S}_p^{\leftarrow}(X \times \{x\})$$
$$= \mathbb{S}_p^{\leftarrow}(X \times \{x\})$$
$$= \{x_1, x_2\}.$$

Since $x \neq p$, we obtain $\{x_1, x_2, p\} \neq \{x_1, x_2\}$ and hence $I_{\mathbb{A}}(x_1, x_2) \neq I_{\mathbb{S}}(x_1, x_2)$, a contradiction. Therefore, X cannot have two elements.

(2) Suppose $X = \{x, y, p\}$, where $x \neq p \neq y$ and $x \neq y$, then either $p \in I(x, y)$ or $p \notin I(x, y)$.

(a) Let $p \in I(x, y)$. Then, $I(x, y) = \{x, y, p\} = X$. For $(a, b) = (x_1, y_1)$, by Lemma 2.13, we have

$$I_{\mathbb{A}}(x_1, y_1) = \bigcap_{i=1}^{2} proj_i \mathbb{A}_p^{\leftarrow}(I(proj_i \mathbb{A}_p(x_1), proj_i \mathbb{A}_p(y_1)))$$
$$= proj_1 \mathbb{A}_p^{\leftarrow}(I(proj_1(x, p), proj_1(y, p))) \cap proj_2 \mathbb{A}_p^{\leftarrow}(I(proj_2(x, p), proj_2(y, p)))$$
$$= proj_1 \mathbb{A}_p^{\leftarrow}(I(x, y)) \cap proj_2 \mathbb{A}_p^{\leftarrow}(I(p, p))$$
$$= \mathbb{A}_p^{\leftarrow}(proj_1^{\leftarrow}(I(x, y))) \cap \mathbb{A}_p^{\leftarrow}(proj_2^{\leftarrow}(\{p\}))$$
$$= \mathbb{A}_p^{\leftarrow}(X \times X) \cap \mathbb{A}_p^{\leftarrow}(X \times \{p\})$$
$$= \mathbb{A}_p^{\leftarrow}(X \times \{p\})$$
$$= \{x_1, y_1, p\},$$

and

$$I_{\mathbb{S}}(x_1, y_1) = \bigcap_{i=1}^{2} proj_i \mathbb{S}_p^{\leftarrow}(I(proj_i \mathbb{S}_p(x_1), proj_i \mathbb{S}_p(y_1)))$$
$$= proj_1 \mathbb{S}_p^{\leftarrow}(I(proj_1(x, x), proj_1(y, y))) \cap proj_2 \mathbb{S}_p^{\leftarrow}(I(proj_2(x, x), proj_2(y, y)))$$
$$= proj_1 \mathbb{S}_p^{\leftarrow}(I(x, y)) \cap proj_2 \mathbb{S}_p^{\leftarrow}(I(x, y))$$
$$= \mathbb{S}_p^{\leftarrow}(proj_1^{\leftarrow}(X)) \cap \mathbb{S}_p^{\leftarrow}(proj_2^{\leftarrow}(X))$$
$$= \mathbb{S}_p^{\leftarrow}(X \times X) \cap \mathbb{S}_p^{\leftarrow}(X \times X)$$
$$= \mathbb{S}_p^{\leftarrow}(X \times X)$$
$$= X \vee_p X = \{x_1, y_1, x_2, y_2, p\}.$$

Since $x \neq p \neq y$ and $x \neq y$, we obtain $\{x_1, y_1, p\} \neq \{x_1, y_1, x_2, y_2, p\}$ and $I_{\mathbb{A}}(x_1, y_1) \neq I_{\mathbb{S}}(x_1, y_1)$, a contradiction. Hence, X cannot have three elements.

(b) Let $p \notin I(x, y)$. Then, $I(x, y) = \{x, y\}$. For $(a, b) = (x_1, y_2)$, by Lemma 2.13, we have

$$\mathcal{I}_{\mathbb{A}}(x_1, y_2) = \bigcap_{i=1}^{2} proj_i \mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_i \mathbb{A}_p(x_1), proj_i \mathbb{A}_p(y_2)))$$

$$= proj_1 \mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_1(x, p), proj_1(p, y))) \cap proj_2 \mathbb{A}_p^{\leftarrow}(\mathcal{I}(proj_2(x, p), proj_2(p, y)))$$

$$= proj_1 \mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p)) \cap proj_2 \mathbb{A}_p^{\leftarrow}(\mathcal{I}(p, y))$$

$$= \mathbb{A}_p^{\leftarrow}(proj_1^{\leftarrow}(\mathcal{I}(x, p))) \cap \mathbb{A}_p^{\leftarrow}(proj_2^{\leftarrow}(\mathcal{I}(p, y)))$$

$$= \mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p) \times \mathcal{X}) \cap \mathbb{A}_p^{\leftarrow}(\mathcal{X} \times \mathcal{I}(p, y))$$

$$= \mathbb{A}_p^{\leftarrow}(\mathcal{I}(x, p) \times \mathcal{I}(p, y))$$

$$= \{z_1, t_2, p \mid p \neq z \in \mathcal{I}(x, p), p \neq t \in \mathcal{I}(p, y)\},$$

and

$$\mathcal{I}_{\mathbb{S}}(x_1, y_2) = \bigcap_{i=1}^{2} proj_i \mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_i \mathbb{S}_p(x_1), proj_i \mathbb{S}_p(y_2)))$$

$$= proj_1 \mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_1(x, x), proj_1(p, y))) \cap proj_2 \mathbb{S}_p^{\leftarrow}(\mathcal{I}(proj_2(x, x), proj_2(p, y)))$$

$$= proj_1 \mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, p)) \cap proj_2 \mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, y))$$

$$= \mathbb{S}_p^{\leftarrow}(proj_1^{\leftarrow}(\mathcal{I}(x, p))) \cap \mathbb{S}_p^{\leftarrow}(proj_2^{\leftarrow}(\{x, y\}))$$

$$= \mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, p) \times \mathcal{X}) \cap \mathbb{S}_p^{\leftarrow}(\mathcal{X} \times \{x, y\})$$

$$= \mathbb{S}_p^{\leftarrow}(\mathcal{I}(x, p) \times \{x, y\})$$

$$= \{z_1, t_2 \mid p \neq z \in \mathcal{I}(x, p), t \in \{x, y\}\}.$$

Since $x \neq p \neq y$ and $x \neq y$, we have $\{z_1, t_2, p \mid p \neq z \in \mathcal{I}(x, p), p \neq t \in \mathcal{I}(p, y)\} \neq \{z_1, t_2 \mid p \neq z \in \mathcal{I}(x, p), t \in \{x, y\}\}$ and hence $\mathcal{I}_{\mathbb{A}}(x_1, y_2) \neq \mathcal{I}_{\mathbb{S}}(x_1, y_2)$, a contradiction. Here $p \in \mathcal{I}_{\mathbb{A}}(x_1, y_2)$, but $p \notin \mathcal{I}_{\mathbb{S}}(x_1, y_2)$. Therefore, $\mathcal{X}$ cannot have three elements.

(3) Suppose $Card\mathcal{X} = n > 3$. Similar to the case $Card\mathcal{X} = 3$, for any distinct elements $z, t \in \mathcal{X}$, either $p \in \mathcal{I}(z, t)$ or $p \notin \mathcal{I}(z, t)$.

  (a) Let $p \in \mathcal{I}(z, t)$. For $a = z_1, b = t_1 \in \mathcal{X} \vee_p \mathcal{X}$, by Lemma 2.13, we have

$$\mathcal{I}_{\mathbb{A}}(z_1, t_1) = \{\alpha_1 \mid \alpha \in \mathcal{I}(z, t)\}$$

and

$$\mathcal{I}_{\mathbb{S}}(z_1, t_1) = \{\alpha_1, \alpha_2 \mid \alpha \in \mathcal{I}(z, t)\}.$$

$\mathcal{I}_{\mathbb{A}}(z_1, t_1) \neq \mathcal{I}_{\mathbb{S}}(z_1, t_1)$ is obtained, a contradiction.

  (b) Let $p \notin \mathcal{I}(z, t)$. For $a = z_1, b = t_2 \in \mathcal{X} \vee_p \mathcal{X}$, by Lemma 2.13, we have

$$\mathcal{I}_{\mathbb{A}}(z_1, t_2) = \{\alpha_1, \beta_2, p \mid p \neq \alpha \in \mathcal{I}(z, p), p \neq \beta \in \mathcal{I}(p, t)\}$$

and

$$\mathcal{I}_{\mathbb{S}}(z_1, t_2) = \{\alpha_1, \beta_2 \mid p \neq \alpha \in \mathcal{I}(z, p), \beta \in \mathcal{I}(z, t)\}.$$

$\mathcal{I}_{\mathbb{A}}(z_1, t_2) \neq \mathcal{I}_{\mathbb{S}}(z_1, t_2)$ is obtained, a contradiction.

Hence, $\mathcal{X}$ cannot have more than three elements.

We concluded that $\mathcal{X}$ has at most one point.

Conversely, let $(\mathcal{X}, \mathcal{I})$ be an interval space and $p \in \mathcal{X}$. Let $\mathcal{X} = \{p\}$. Then, $\mathcal{I}(p, p) = \{p\}$. By the Definition 3.2 and Lemma 2.13, we show that for all $a, b \in \mathcal{X} \vee_p \mathcal{X}$, $\mathcal{I}_{\mathbb{A}}(a, b) = \mathcal{I}_{\mathbb{S}}(a, b)$. Since $\mathcal{X}$ has only one element, then $a = b = p$. It follows that

$$\mathcal{I}_{\mathbb{A}}(a, b) = \mathcal{I}_{\mathbb{A}}(p, p) = \{p\} = \mathcal{I}_{\mathbb{S}}(p, p) = \mathcal{I}_{\mathbb{S}}(a, b).$$

Consequently, we obtained that the interval space $(\mathcal{X}, \mathcal{I})$ is $pre\overline{T_2}$ at p.

□

Theorem 3.7. *An interval space $(\mathcal{X}, \mathcal{I})$ is $\overline{T_2}$ at p if and only if $\mathcal{X} = \{p\}$.*

Proof. It follows from Theorems 3.4 and 3.6. □

Theorem 3.8. *Let $(X, \mathcal{I})$ be an interval space, $\mathcal{Y} \subseteq X$ and $p \in \mathcal{Y}$.*

(1) *A subspace $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ is $\overline{T}_0$ at p and T_1 at p, i.e., in **IS**, local $\overline{T}_0$ and local T_1 interval spaces are hereditary.*
(2) *If $(X, \mathcal{I})$ is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p, then the subspace $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p, i.e., in **IS**, local $pre\overline{T}_2$ and local $\overline{T}_2$ interval spaces are hereditary.*

Proof. (1) By Theorems 3.4 and 3.5, it is obvious.

(2) Suppose that $p \in \mathcal{Y}$ and $(X, \mathcal{I})$ is $pre\overline{T}_2$ at p (or $\overline{T}_2$ at p). Then, $X = \{p\}$ is one-point set by Theorem 3.6 (or 3.7). Since $p \in \mathcal{Y} \subseteq X = \{p\}$, we have $\mathcal{Y} = \{p\}$. Consequently, by Definition 2.5 and Theorem 3.6 (or 3.7), it follows that the subspace $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ is $pre\overline{T}_2$ at p (or $\overline{T}_2$ at p). □

Theorem 3.9. *Let $\{(X_i, \mathcal{I}_i) \mid i \in I\}$ be a collection of interval spaces, $(X, \mathcal{I}_*)$ be the product space, where $X = \prod_{i \in I} X_i$ and $\mathcal{I}_*(a, b) = \{ (x_i)_{i \in I} \mid x_i \in \mathcal{I}_i(a_i, b_i) \}$. Let $p = (p_i)_{i \in I}$ be a point in X. Then,*

(1) *A product space $(X, \mathcal{I}_*)$ is $\overline{T}_0$ at p and T_1 at p.*
(2) *The product space $(X, \mathcal{I}_*)$ is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p if and only if for each $i \in I$, the interval space $(X_i, \mathcal{I}_i)$ is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p.*

Proof. (1) By Theorems 3.4 and 3.5, it is obvious.

(2) Suppose that the product space $(X, \mathcal{I}_*)$ is $pre\overline{T}_2$ at p (or $\overline{T}_2$ at p). Then, $X = \{p = (p_i)_{i \in I}\}$ is one-point set by Theorem 3.6 (or 3.7). Let $proj_i : (X, \mathcal{I}_*) \to (X_i, \mathcal{I}_i)$ be the projection maps defined as $proj_i(p) = p_i$ for $i \in I$ with $p \in X$. For each $i \in I$, $X_i = proj_i(X) = \{p_i\}$ is one-point set since $X = \{p\}$. Hence, by Theorem 3.6 (or 3.7), it follows that the interval space $(X_i, \mathcal{I}_i)$ for each $i \in I$, is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p.

Conversely, for each $i \in I$, let the interval space $(X_i, \mathcal{I}_i)$ be $pre\overline{T}_2$ at p (or $\overline{T}_2$ at p). Then, by Theorem 3.6 (or 3.7), $X_i = \{p_i\}$ is one-point set for each $i \in I$. It follows that $X = \{p = (p_i)_{i \in I}\}$ is one-point set, since $X = \prod_{i \in I} X_i$. Consequently, the product space $(X, \mathcal{I}_*)$ is $pre\overline{T}_2$ at p (or $\overline{T}_2$ at p) by Theorem 3.6 (or 3.7). □

Remark 3.10. (1) For the category **IS** of interval spaces and interval preserving mappings, $(X, \mathcal{I})$ is $pre\overline{T}_2$ at p or $\overline{T}_2$ at p, then $(X, \mathcal{I})$ is $\overline{T}_0$ at p or T_1 at p, but in general, the converse is not true.

$$pre\overline{T}_2 \text{ at } p = \overline{T}_2 \text{ at } p \implies T_1 \text{ at } p = \overline{T}_0 \text{ at } p.$$

(2) In **IS**, if $X = \{p\}$, then the interval space $(X, \mathcal{I})$ is $\overline{T}_0$ at p, T_1 at p, $pre\overline{T}_2$ at p or $\overline{T}_2$ at p.

Corollary 3.11. *Let $\mathbf{T_i}(C)$ be the full subcategory of the category C consisting of all local $\mathbf{T_i}$ $(i = 0, 1)$ objects. By Theorems 3.4 and 3.5, the following categories are isomorphic.*

(1) **IS**
(2) $\mathbf{T_0}(\mathbf{IS})$
(3) $\mathbf{T_1}(\mathbf{IS})$.

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