SYMMETRY REDUCTIONS AND EXACT SOLUTIONS TO THE
SEVENTH-ORDER KDV TYPES OF EQUATION

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Abstract. In present paper, the seventh-order KdV types of equation is con-
sidered by the Lie symmetry analysis. All of the geometric vector fields of the
KdV equation are obtained, then the symmetry reductions and exact solutions
to the KdV equation are investigated by the dynamical system and the power
series method.

1. Introduction

Recently, mathematics and physics field have devoted considerable effort to the
study of solutions to ordinary and partial differential equations (ODEs and PDEs).
Among many powerful methods for solving the equation, Lie symmetry analysis pro-
vides an effective procedure for integrability, conservation laws, reducing equations
and exact solutions of a wide and general class of differential systems representing
real physical problems [12, 15]. Sinkala et al [14] have performed the group classifi-
cation of a bond-pricing PDE of mathematical finance to discover the combinations
of arbitrary parameters that allow the PDE to admit a nontrivial symmetry Lie
algebra, and computed the admitted Lie point symmetries, identify the correspond-
ing symmetry Lie algebra and solve the PDE. Under the condition of the symmetry
group of the PDE is nontrivial, it contains a standard integral transform of the
fundamental solution for PDEs, and fundamental solution can be reduced to in-
vverting a Laplace transform or some other classical transform in [1]. In [7], by the
direct construction method, all of the first-order multipliers of the the generalized
nonlinear second-order equation are obtained, and the corresponding complete con-
servation laws of such equations are provided. Furthermore, Lie symmetry analysis
helps to study their group theoretical properties, and effectively assists to derive
several mathematical characteristics related with their complete integrability [10].
Also, Lie symmetry analysis and dynamical system method is a feasible approach
dealing with exact explicit solutions to nonlinear PDEs and systems, (see, e.g.,
Liu et al. have derived the symmetries, bifurcations and exact explicit solutions to the KdV equation by using Lie symmetry analysis and the dynamical system method [5, 6]. The KdV equation models the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn’s F-ring [4], etc. In present paper, we will investigate the vector fields, symmetry reductions and exact solutions to the KdV equation with power law nonlinearity and linear damping with dispersion

\[ u_t + u^2 u_x + uu_{4x} + 2uu_{3x} + u_{xx}^2 + uu_7 = 0, \]

where \( u = u(x,t) \) is the unknown functions, \( x \) is the spatial coordinate in the propagation direction and \( t \) is the temporal coordinates, which occur in different contexts in mathematical physics.

The rest of this paper is organized as follows: in Section 2, the vector fields of Eqs. (1.1) are presented by using Lie symmetry analysis method. Based on the optimal system method, all the similarity reductions to the Eqs. (1.1) are obtained. In Section 3, the exact analytic solutions to the equations are investigated by means of the power series method. Finally, the conclusions will be given in Section 4.

2. Lie symmetry analysis and similarity reductions

Recall that the geometric vector field of a PDE equation is as follows:

\[ V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \]

where the coefficient functions \( \xi(x,t,u), \tau(x,t,u), \eta(x,t,u) \) of the vector field are to be determined later.

If the vector field (2.1) generates a symmetry of the equation (1.1), then \( V \) must satisfy the Lie symmetry condition

\[ \text{Pr}V(\Delta)|_{\Delta=0} = 0, \]

where \( \text{Pr}V \) denotes the 7-th prolongation of \( V \), and \( \Delta = u_t + u^2 u_x + uu_{4x} + 2uu_{3x} + u_{xx}^2 + uu_7 \). Moreover, the prolongation \( \text{Pr}V \) depends on the equation

\[ \text{Pr}V = \eta \partial_u + \eta^x \partial_u + \eta \partial_u + \eta^{2x} \partial_u + \eta^{3x} \partial_u + \eta^{4x} \partial_u + \eta^{7x} \partial_u, \]

where the coefficient functions \( \eta^kx (k=1,2,3,4,7) \) are given as

\[ \eta^kx = D_x^k(\eta - \tau u_x - \xi u) + \tau u_{kxt} + \xi u_{(k+1)x}, \quad k = 1,2,3,4,7, \]

here symbol \( D_x \) denotes the total differentiation operator and is defined as

\[ D_x = \partial_x + u_x \partial_u + u_{xx} \partial_u + \ldots. \]

Then, in terms of the Lie symmetry analysis method, we obtain that all of the geometric vector fields of Eq. (1.1) are as follows:

\[ V_1 = x\partial_x + 7t\partial_t - 3u\partial_u, \quad V_2 = \partial_x, \quad V_3 = \partial_t. \]

Moreover, it is necessary to show that the vector fields of Eq. (1.1) are closed under the Lie bracket, we have

\[ [V_i, V_i] = 0, \quad i = 1,2,3, \]

\[ [V_1, V_2] = -[V_2, V_1] = V_2, \quad [V_1, V_3] = -[V_3, V_1] = 7V_3, \quad [V_2, V_3] = -[V_3, V_2] = 0. \]

In the preceding section, we obtained the vector fields and the optimal systems of Eq. (1.1). Now, we deal with the symmetry reductions and exact solutions to the equations. We will consider the following similarity reductions and group-invariant
solutions based on the optimal system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

For the generator $V_1$, we have

\begin{equation}
(2.2) \quad u = t^{-\frac{3}{7}} f(z),
\end{equation}

where $z = x t^{-\frac{1}{7}}$. Substituting (2.2) into Eq. (1.1), we reduce it into the following ODE

\begin{equation}
(2.3) \quad f^{(7)} + f^2 f^{(4)} + 2 f^{(3)} + f'' + \frac{1}{7}zf' - \frac{3}{7}f = 0,
\end{equation}

where $f' = \frac{df}{dz}$.

For the generator $V_2$, we get the trivial solution to Eq. (1.1) is $u(x,t) = c$, where $c$ is an arbitrary constant.

For the generator $V_3$, we have

\begin{equation}
(2.4) \quad u = f(z),
\end{equation}

where $z = x$. Substituting (2.4) into Eq. (1.1), we reduce it into the following ODE

\begin{equation}
(2.5) \quad f^{(7)} + f^2 f^{(4)} + 2 f^{(3)} + f'' = 0,
\end{equation}

where $f' = \frac{df}{dz}$.

For the generator $V_3 + \nu V_2$, $\nu$ is an arbitrary constant, we have

\begin{equation}
(2.6) \quad u = f(z),
\end{equation}

where $z = x - \nu t$. Substituting (2.6) into Eq. (1.1), we reduce it into the following ODE

\begin{equation}
(2.7) \quad f^{(7)} + f^2 f^{(4)} + 2 f^{(3)} + f'' - \nu f' = 0,
\end{equation}

where $f' = \frac{df}{dz}$.

3. The exact power series solutions

By seeking for exact solutions of the PDEs, we mean that those can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE. In terms of this definition, the exact solutions to Eq. (1.1) are obtained actually in both of the preceding Sections 2. In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. But this is not always the case, even for simple semilinear PDEs. However, we know that the power series can be used to solve differential equations, including many complicated differential equations [9, 13]. In this section, we will consider the exact analytic solutions to the reduced equations by using the power series method. Once we get the exact analytic solutions of the reduced ODEs, the exact power series solutions to the original PDEs are obtained, now we consider the solutions of ODEs (2.3), (2.5) and (2.7).

3.1 Exact analytic solutions to Eq. (2.3)
In view of (2.3), we seek a solution in a power series of the form

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n. \]  

(3.1)

Substituting (3.1) into (2.3), and comparing coefficients, then we obtain the following recursion formula:

\[
    c_{n+7} = -\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \times \left( \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_ic_{k-i}c_{n-k+1} 
    + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4)c_kc_{n-k+4} 
    + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1)c_{k+1}c_{n-k+3} 
    + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2)c_{k+2}c_{n-k+2} 
    - \frac{1}{7} nc_n - \frac{3}{7} c_n \right),
\]  

(3.2)

for all \( n = 0, 1, 2, \ldots \).

Thus, for arbitrarily chosen constants \( c_i \) \((i = 0, 1, \ldots, 6)\), we obtain

\[
    c_7 = -\frac{1}{5040} \left( c_0^2c_1 + 24c_0c_4 + 12c_1c_3 + 4c_2^2 - \frac{3}{7} c_1 \right). 
\]  

(3.3)

Furthermore, from (3.2), it yield

\[
    c_8 = -\frac{1}{20160} \left( c_0c_1^2 + c_0^2c_2 + 60c_0c_5 + 36c_1c_4 + 24c_2c_3 - \frac{2}{7} c_1 \right), 
\]  

(3.4)

\[
    c_9 = -\frac{1}{181440} \left( 3c_0^2c_3 + c_1^3 + 6c_0c_1c_2 + 360c_0c_6 + 240c_1c_5 + 144c_2c_4 
    + 72c_3^2 + 24c_0c_4 - \frac{5}{7} c_2 \right),
\]

and so on.

Thus, for arbitrary chosen constant numbers \( c_i \) \((i = 0, 1, \ldots)\), the other terms of the sequence \( \{c_n\}_{n=0}^{\infty} \) can be determined successively from (3.3) and (3.4) in a unique manner. This implies that for Eq. (2.3), there exists a power series solution (3.1) with the coefficients given by (3.3) and (3.4). Furthermore, it is easy to prove the convergence of the power series (3.1) with the coefficients given by (3.3) and (3.4). Therefore, this power series solution (3.1) to Eq. (2.3) is an exact analytic solution.
Thus, the exact power series solution of Eq. (2.3) can be written as
\[ f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + \sum_{n=1}^{\infty} c_{n+7} z^{n+7} \]
\[ = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 \]
\[ - \frac{1}{5040} (c_6^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1) z^7 \]
\[ - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \]
\[ \times \left( \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_i c_{k-i} c_{n-k+1} \right) \]
\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4)c_k c_{n-k+4} \]
\[ + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1)c_{k+1} c_{n-k+3} \]
\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2)c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n - \frac{3}{7} c_n \] \( z^{n+7}. \)

Thus, the exact power series solution of Eq. (1.1) is
\[ u(x, t) = c_0 t^{-\frac{3}{7}} + c_1 x t^{-\frac{4}{7}} + c_2 x^2 t^{-\frac{5}{7}} + c_3 x^3 t^{-\frac{6}{7}} + c_4 x^4 t^{-\frac{7}{7}} + c_5 x^5 t^{-\frac{8}{7}} \]
\[ + c_6 x^6 t^{-\frac{9}{7}} - \frac{1}{5040} (c_6^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1) x^7 t^{-\frac{10}{7}} \]
\[ - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \]
\[ \times \left( \sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1)c_i c_{k-i} c_{n-k+1} \right) \]
\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4)c_k c_{n-k+4} \]
\[ + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1)c_{k+1} c_{n-k+3} \]
\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2)c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n \]
\[ - \frac{3}{7} c_n \] \( x^{n+7} t^{-\frac{n+10}{7}}. \)

3.2 Exact analytic solutions to Eq. (2.5)

In view of (2.5), we have
\[ \frac{1}{3} f^3 + f f'' + f' f''(6) + c = 0, \]
where $c$ is an integration constant.

We seek a solution of Eq. (3.6) in a power series of the form (3.1). Substituting (3.1) into (3.6), and comparing coefficients, we obtain

$$c_{n+6} = \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right) \left(1 \right) + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)c_{k+1} c_{n-k+2},$$

for all $n = 1, 2, \ldots$

Thus, for arbitrarily chosen constants $c_i$ ($i = 0, 1, \ldots, 5$), we have

$$c_6 = -\frac{1}{720} \left( \frac{1}{3} c_3^3 + 6c_0 c_3 + 2c_1 c_2 + c \right).$$

Furthermore, from (3.7), we have

$$c_7 = -\frac{1}{5040} \left( c_0^2 c_1 + 24c_0 c_4 + 12c_1 c_3 + 4c_1 c_2 \right),$$

$$c_8 = -\frac{1}{20160} \left( c_0^2 c_2 + c_0 c_1^2 + 60c_0 c_5 + 36c_1 c_4 + 24c_2 c_3 \right),$$

and so on.

Hence, the power series solution of Eq. (2.5) can be written as

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + \sum_{n=1}^{\infty} c_{n+6} z^{n+6}$$

$$= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 - \frac{1}{720} \left( \frac{1}{3} c_3^3 + 6c_0 c_3 + 2c_1 c_2 + c \right) z^6$$

$$- \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right) z^{n+6} + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)c_{k+1} c_{n-k+2} z^{n+6}.$$
Thus, the exact power series solution of Eq. (1.1) is

\[ u(x, t) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 - \frac{1}{720} \left( \frac{1}{6} c_0^3 + 6c_0 c_3 + 2c_1 c_2 + c \right) x^6 - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right. \\
\left. + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
+ \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} \right) x^{n+6}. \]

(3.8)

3.3 Exact analytic solutions to Eq. (2.7)

In view of (2.7), we have

\[ \frac{1}{3} f^3 + f f''' + f' f''(6) - \nu f + c = 0, \]

(3.9)

where \( c \) is an integration constant.

Similarly, we seek a solution of Eq. (3.9) in a power series of the form (3.1). Substituting (3.1) into (3.9), and comparing coefficients, we obtain

\[ c_{n+6} = - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right. \\
\left. + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
+ \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} - \nu c_n \right), \]

(3.10)

for all \( n = 1, 2, \ldots \).

Thus, for arbitrarily chosen constants \( c_i \) (\( i = 0, 1, \ldots, 5 \)), we can get

\[ c_6 = -\frac{1}{720} \left( \frac{1}{3} c_0^3 + 6c_0 c_3 + 2c_1 c_2 - \nu c_0 + c \right). \]

Furthermore, from (3.10), we have

\[ c_7 = -\frac{1}{5040} \left( c_0^2 c_1 + 24c_0 c_4 + 12c_1 c_3 + 4c_1 c_2 - \nu c_1 \right), \]

\[ c_8 = -\frac{1}{20160} \left( c_0^2 c_2 + c_0 c_1^2 + 60c_0 c_5 + 36c_1 c_4 + 24c_2 c_3 - \nu c_2 \right), \]

and so on.
Hence, the power series solution of Eq. (2.7) can be written as

\[ f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + \sum_{n=1}^{\infty} c_{n+6} z^{n+6} \]

\[ = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 - \frac{1}{720} \left( \frac{1}{3} c_0^3 + 6 c_0 c_3 + 2 c_1 c_2 - \nu c_0 \right) z^6 \]

\[ - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right) \]

\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \]

\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} - \nu c_n \] \( z^{n+6} \).

Thus, we obtain the traveling wave solution to Eq. (1.1) as follows

\[ u(x,t) = c_0 + c_1 (x - \nu t) + c_2 (x - \nu t)^2 + c_3 (x - \nu t)^3 + c_4 (x - \nu t)^4 \]

\[ + c_5 (x - \nu t)^5 - \frac{1}{720} \left( \frac{1}{3} c_0^3 + 6 c_0 c_3 + 2 c_1 c_2 \right) (x - \nu t)^6 \]

\[ - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \left( \frac{1}{3} \sum_{k=0}^{n} \sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \right) \]

\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \]

\[ + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} - \nu c_n \] \( (x - \nu t)^{n+6} \).

**Remark 3.1.** We would like to reiterate that the power series solutions which have been obtained in this section are exact analytic solutions to the equations. Moreover, we can see that these power series solutions converge for the given chosen constants \( c_i \) (\( i = 0, 1, \ldots, 6 \)) of (3.5), \( c_i \) (\( i = 0, 1, \ldots, 5 \)) of (3.8) and (3.11), respectively, it is actual value for mathematical and physical applications.

4. **Summary and Discussion**

In this paper, we have obtained the symmetries and similarity reductions of the seventh-order KdV types of equations by using Lie symmetry analysis method. All the group-invariant solutions to the equations are considered based on the optimal system method. Then the exact analytic solutions are investigated by using the power series method, and we can see that these power series solutions converge.
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