Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 131–139 (2015) ©KJM

# SOME GENERATING RELATIONS INVOLVING 2-VARIABLE LAGUERRE AND EXTENDED SRIVASTAVA POLYNOMIALS

Ahmed Ali Al Gonah [1]

ABSTRACT: In this paper, we derive families of bilateral and mixed multilateral generating relations involving 2-variable Laguerre and extended Srivastava polynomials. Further, several bilateral and trilateral generating functions involving 2-variable Laguerre polynomials and other classical polynomials are obtained as applications of main results.

## 1. INTRODUCTION

Srivastava [9] introduced the Srivastava polynomials (SP) $S_n^N(w)$ by the following series definition:

$$S_n^N(w) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n,k}\ w^k \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$ and $(\lambda)_\nu$, $(\lambda)_0 \equiv 1$, denotes the Pochhammer symbol defined by [10]

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}, \tag{1.2}$$

in terms of familiar Gamma function.

Afterward, González *et al*. [3] extended the SP $S_n^N(w)$ as follows:

$$S_{n,q}^N(w) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n+q,k}\ w^k \ (q, n \in \mathbb{N}_0; N \in \mathbb{N}), \tag{1.3}$$

which were investigated rather extensively in [3] and more recently in [6]. The polynomials $S_{n,q}^N(w)$ called as extended Srivastava polynomials (ESP), since $S_{n,0}^N(w) = S_n^N(w)$.

It is important that, appropriate choices of the double sequence $\{A_{n,k}\}$ in equation (1.3) give many well known polynomials such as Laguerre, Jacobi and Bessel polynomials (see [3]). Here, we will recall them and add further new particular cases as the following remarks:

---

[1]2010 *Mathematics Subject Classification*: 33B10, 33C45, 33E30.
*Keywords*: Generating relations, Laguerre polynomials, Extended Srivastava polynomials.

*Remark* 1.1. ([3; p.147] see also [6]) Choosing $A_{q,n} = (-\alpha - q)_n$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{-1}{w}\right) = \frac{n!}{(-w)^n} L_n^{(\alpha+q)}(w), \tag{1.4}$$

where $L_n^{(\alpha)}(w)$ denotes the associated Laguerre polynomials defined by [10; p.42]

$$L_n^{(\alpha)}(w) = \frac{(-w)^n}{n!}\,{}_2F_0\left(-n, -\alpha - n; -; \frac{-1}{w}\right) \tag{1.5}$$

and ${}_pF_q$ is the generalized hypergeometric function defined by [10; p.42]:

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.6}$$

where $p, q \in \mathbb{N}_0$ and for $p = q = 1$ reduces to the confluent hypergeometric function ${}_1F_1$.

*Remark* 1.2. ([3; p.146]) Choosing $A_{q,n} = \frac{(\alpha+\beta+1)_{2q}(-\beta-q)_n}{(\alpha+\beta+1)_q(-\alpha-\beta-2q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{2}{1+w}\right) = n!(\alpha + \beta + q + n + 1)_q \left(\frac{2}{1+w}\right)^n P_n^{(\alpha+q,\beta+q)}(w), \tag{1.7}$$

where $P_n^{(\alpha,\beta)}(w)$ denotes the classical Jacobi polynomials defined by [8; p.255]

$$P_n^{(\alpha,\beta)}(w) = \binom{\alpha+\beta+2n}{n}\left(\frac{1+w}{2}\right)^n {}_2F_1\left(-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{1+w}\right). \tag{1.8}$$

*Remark* 1.3. ([3; p.148]) Choosing $A_{q,n} = (-\alpha - q)_n$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{-w}{\beta}\right) = y_n(w, 1 - \alpha - q - 2n, \beta)\ (\beta \neq 0), \tag{1.9}$$

where $y_n(w, \alpha, \beta)$ denotes the Bessel polynomials defined by [10; p.75]

$$y_n(w, \alpha, \beta) = {}_2F_0\left(-n, \alpha + n - 1; -; \frac{-w}{\beta}\right). \tag{1.10}$$

Now, we add the following new particular cases as remarks:

*Remark* 1.4. Choosing $A_{q,n} = \frac{2^q(\nu)_q(\frac{1}{2}-\nu-q)_n}{(1-2\nu-2q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}\left(\frac{2}{1-w}\right) = \frac{n!2^q(\nu)_q}{(w-1)^n} C_n^{\nu+q}(w), \tag{1.11}$$

where $C_n^{\nu}(w)$ denotes the classical Gegenbauer polynomials defined by [8; p.279]

$$C_n^{\nu}(w) = \frac{2^{2n}(\nu)_n}{n!}\left(\frac{w-1}{2}\right)^n {}_2F_1\left(-n, \frac{1}{2} - \nu - n; 1 - 2\nu - 2n; \frac{2}{1-w}\right). \tag{1.12}$$

*Remark* 1.5. Choosing $A_{q,n} = \frac{n!(p+1)_q}{(-p-q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S^1_{n,q}(w) = n!(p+1)_q\ g_n^{(p+q)}(w), \tag{1.13}$$

where $g_n^{(p)}(w)$ denotes the Cesaro polynomials defined by [10; p.449]

$$g_n^{(p)}(w) = \binom{p+n}{n} {}_2F_1(-n, 1; -p-n; w). \tag{1.14}$$

*Remark* 1.6. Choosing $A_{q,n} = \frac{(a)_{2q}}{(a)_q (a+q)_n}$ $(q, n \in \mathbb{N}_0)$ in Eq. (1.3), we get

$$S_{n,q}^1(w) = n!(a+q+2n)_q \; R_n(a+q, w), \tag{1.15}$$

where $R_n(a, w)$ denotes the Shively's pseudo Laguerre polynomials defined by [8; p.298]

$$R_n(a, w) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1(-n; a+n; w). \tag{1.16}$$

Next, we recall that the 2-variable Laguerre polynomials (2VLP) $L_n(x, y)$ are defined by the series definition (see[1,2])

$$L_n(x, y) \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k \; x^k \; y^{n-k}}{(r!)^2 \; (n-k)!} \tag{1.17}$$

and specified by the following generating functions:

$$\exp(yt)\, C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}, \tag{1.18}$$

or, equivalently

$$\frac{1}{(1-yt)} \exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x, y) t^n \; \big(|yt| < 1\big), \tag{1.19}$$

where $C_0(x)$ denotes the $0^{th}$ order Tricomi function. The $n^{th}$ order Tricomi functions $C_n(x)$ are defined by [10]

$$C_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k \; x^k}{k! \; (n+k)!}. \tag{1.20}$$

Also, we note that the 2VLP $L_n(x, y)$ satisfy the following generating function:

$$\frac{1}{(1-yt)^a} \; {}_1F_1\Big(a; 1; \frac{-xt}{1-yt}\Big) = \sum_{n=0}^{\infty} (a)_n \; L_n(x, y) \frac{t^n}{n!} \; \big(|yt| < 1\big), \tag{1.21}$$

which for $a = 1$ reduces to Eq. (1.19).

The 2VLP $L_n(x, y)$ are linked to the classical Laguerre polynomials $L_n(x)$ by the relations

$$L_n(x, y) = y^n \; L_n\left(\frac{x}{y}\right), \tag{1.22}$$

$$L_n(x, 1) = L_n(x), \tag{1.23}$$

where $L_n(x)$ are defined by [8]

$$L_n(x) \;=\; \sum_{k=0}^{n} \frac{(-1)^k \, n! \, x^k}{(k!)^2 \, (n-k)!}. \tag{1.24}$$

The aim of this paper is to derive some families of bilateral and mixed multilateral generating relations involving the 2VLP $L_n(x, y)$ and the ESP $S_{n,q}^N(w)$ by using series rearrangement techniques. Also, the above mentioned remarks will be used to obtain some illustrative bilateral and trilateral generating functions involving the 2VLP $L_n(x,y)$ and many classical polynomials in terms of the confluent hypergeometric function.

## 2. BILATERAL GENERATING RELATIONS

We prove the following results:

**Theorem 2.1.** *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the ESP $S^N_{n,q}(w)$ holds true:*

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\frac{t^q}{q!}\frac{u^n}{n!} = \sum_{q,n=0}^{\infty} L_{q+Nn}(x,y)\ A_{q+Nn,n}\frac{(t+u)^q}{q!}\frac{\left(w(-u)^N\right)^n}{n!}. \tag{2.1}$$

**Proof.** Denoting the l.h.s. of Eq. (2.1) by $\Delta_1$ and using definition (1.3), we have

$$\Delta_1 = \sum_{q,n=0}^{\infty} L_{q+n}(x,y)\sum_{k=0}^{[\frac{n}{N}]}\frac{(-1)^{Nk}}{k!(n-Nk)!}A_{q+n,k}\ w^k\frac{t^q}{q!}\ u^n. \tag{2.2}$$

Replacing $n$ by $n+Nk$ in the above equation and using the lemma [10; p.101]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk), \tag{2.3}$$

in the resultant equation, we find

$$\Delta_1 = \sum_{q,n,k=0}^{\infty} L_{q+n+Nk}(x,y)\frac{(-1)^{Nk}}{k!}A_{q+n+Nk,k}\ w^k\frac{t^q}{q!}\frac{u^{n+Nk}}{n!}. \tag{2.4}$$

Again, replacing $q$ by $q-n$ in the r.h.s. of Eq. (2.4) and using the lemma [10; p.100]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{2.5}$$

in the resultant equation, we get

$$\Delta_1 = \sum_{q,k=0}^{\infty} L_{q+Nk}(x,y)A_{q+Nk,k}\ \frac{t^q}{q!}\frac{\left(w(-u)^N\right)^k}{k!}\sum_{n=0}^{q}\frac{(-q)_n}{n!}\left(\frac{-u}{t}\right)^n, \tag{2.6}$$

which on using the binomial expansion [10]

$$(1-x)^{-\lambda} = \sum_{n=0}^{\infty}(\lambda)_n\frac{x^n}{n!}, \tag{2.7}$$

in the r.h.s., yields the r.h.s. of Eq. (2.1), then the proof of Theorem (2.1) is completed.

*Remark* 2.1. Taking $u = -t$ in assertion (2.1) of Theorem 2.1, we deduce the following consequence of Theorem 2.1.

**Corollary 2.1.** *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the ESP $S^N_{n,q}(w)$ holds true:*

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\ \frac{t^q}{q!}\frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} L_{Nn}(x,y)\ A_{Nn,n}\ \frac{\left(wt^N\right)^n}{n!}. \tag{2.8}$$

*Remark* 2.2. Taking $t = 0$ in assertion (2.1) of Theorem 2.1 and using the relation $S^N_{n,0}(w) = S^N_n(w)$, we deduce the following consequence of Theorem 2.1.

**Corollary 2.2.** *The following family of bilateral generating relation involving the 2VLP $L_n(x,y)$ and the SP $S^N_n(w)$ holds true:*

$$\sum_{n=0}^{\infty} L_n(x,y)\ S^N_n(w)\frac{u^n}{n!} = \sum_{q,n=0}^{\infty} L_{q+Nn}(x,y)\ A_{q+Nn,n}\ \frac{u^q}{q!}\frac{\left(w(-u)^N\right)^n}{n!}. \tag{2.9}$$

In the next section, Corollaries 2.1 and 2.2 will be exploited to get families of mixed multilateral generating relations involving the 2VLP $L_n(x,y)$, ESP $S^N_{n,q}(w)$ and SP $S^N_n(w)$ with the help of the method considered in [10,5,7].

## 3. MULTILATERAL GENERATING RELATIONS

First, we prove the following theorem by using Corollary 2.1:

**Theorem 3.1.** *Corresponding to an identically non-vanishing function $\Omega_\mu(\xi_1,\ldots,\xi_l)$ of complex variables $\xi_1,\ldots,\xi_l$ $(l \in \mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta) := \sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\eta^k,\ \ (a_k \neq 0, \psi \in \mathbb{C}). \tag{3.1}$$

*Then we have, for $n, p \in \mathbb{N}$,*

$$\sum_{q,n=0}^{\infty}\sum_{k=0}^{[\frac{q}{p}]} a_k\ L_{q+n-pk}(x,y)S^N_{n,q-pk}(w)\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\ \eta^k\ \frac{(-1)^n t^{n+q-pk}}{(q-pk)!n!}$$

$$= \Lambda_{\mu,\psi}(\xi_1,\ldots,\xi_l;\eta)\sum_{n=0}^{\infty} L_{Nn}(x,y)\ A_{Nn,n}\ \frac{\left(wt^N\right)^n}{n!}. \tag{3.2}$$

*provided that each member of assertion (3.2) exists.*

**Proof.** Denoting the l.h.s. of Eq. (3.2) by $\Delta_2$ and using relation (2.3) , we find

$$\Delta_2 = \sum_{k=0}^{\infty} a_k\ \Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_l)\eta^k \sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ S^N_{n,q}(w)\ \frac{t^q}{q!}\frac{(-t)^n}{n!}. \tag{3.3}$$

Using Eqs. (3.1) and (2.8) in the r.h.s. of Eq. (3.3), we get the r.h.s. of Eq. (3.2), then the proof of Theorem 3.1 is completed.

Next, proceeding on the same lines of proof of Theorem 3.1 and using Corollary 2.2, we get the following result:

**Theorem 3.2.** *Corresponding to an identically non-vanishing function $\Omega_\mu(\xi_1,\dots,\xi_l)$ of complex variables $\xi_1,\dots,\xi_l$ $(l\in\mathbb{N})$ and of complex order $\mu$, let*

$$\Lambda_{\mu,\psi}(\xi_1,\dots,\xi_l;\eta):=\sum_{k=0}^{\infty}a_k\ \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_l)\eta^k,\ (a_k\neq 0,\psi\in\mathbb{C}),$$

$$\Theta_{n,p}^{\mu,\psi}(x,y,z,w;\xi_1,\dots,\xi_l;\tau)=\sum_{k=0}^{[\frac{n}{p}]}a_k\ L_{n-pk}(x,y)\ S_{n-pk}^{N}(w)\ \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_l)\frac{\tau^k}{(n-pk)!}, \tag{3.4}$$

*where $n,p\in\mathbb{N}$. Then, we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y,z,w;\xi_1,\dots,\xi_l;\frac{\eta}{t^p}\right)\ t^n$$
$$=\Lambda_{\mu,\psi}(\xi_1,\dots,\xi_l;\eta)\sum_{q,n=0}^{\infty}L_{q+Nn}(x,y)\ A_{q+Nn,n}\ \frac{t^q}{q!}\frac{\left(w(-t)^N\right)^n}{n!}. \tag{3.5}$$

*provided that each member of assertion (3.5) exists.*

Notice that, for every suitable choice of the coefficients $a_k$ $(k\in\mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_l)$, $(l\in\mathbb{N})$, is expressed in terms of simpler function of one and more variables, the assertions of Theorems 3.1 and 3.2 can be applied in order to derive various families of multilateral generating relations involving the 2VLP $L_n(x,y)$ and the ESP $S_{n,q}^N(w)$.

For example, if we set $l=1$, $\xi_1=v$, $\psi=1$, $\Omega_{\mu+k}(v)=y_j(v,\mu+k,\beta)$ and $a_k=\binom{\mu+j+k-2}{k}$, $(k,j\in\mathbb{N}_0,\mu\in\mathbb{C})$ in assertion (3.2) of Theorem 3.1 and making use of the following generating relation [4; p.270]:

$$\sum_{n=0}^{\infty}\binom{\mu+j+n-2}{k}\ y_j(x,\mu+n,\beta)\ t^n=(1-t)^{1-\mu-j}\ y_j\left(\frac{x}{1-t},\mu,\beta\right), \tag{3.6}$$

we readily obtain the following mixed trilateral generating function:

$$\sum_{q,n=0}^{\infty}\sum_{k=0}^{[\frac{q}{p}]}\binom{\mu+j+k-2}{k}\ L_{q+n-pk}(x,y)S_{n,q-pk}^{N}(w)\ y_j(v,\mu+k,\beta)\frac{(-1)^n\eta^k}{(q-pk)!}\frac{t^{n+q-pk}}{n!}$$
$$=(1-\eta)^{1-\mu-j}\ y_j\left(\frac{v}{1-\eta},\mu,\beta\right)\sum_{n=0}^{\infty}L_{Nn}(x,y)A_{Nn,n}\frac{\left(wt^N\right)^n}{n!}. \tag{3.7}$$

In the next section, we derive some bilateral and trilateral generating functions for the 2VLP $L_n(x,y)$ in terms of the confluent hypergeometric function as applications of the results derived in Sections 2 and 3 with the help of generating function (1.21) and the remarks introduced in Section 1.

## 4. APPLICATIONS

First, the following bilateral generating functions are obtained as applications of Corollary 2.1:

**I.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.1 and using relation (1.4) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ L_n^{(\alpha+q)}(w)\frac{t^q}{q!}\left(\frac{t}{w}\right)^n = \sum_{n=0}^{\infty}(\alpha+1)_n\ L_n(x,y)\frac{\left(\frac{t}{w}\right)^n}{n!}, \tag{4.1}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{n,q=0}^{\infty} L_{q+n}(x,y)\ L_n^{(\alpha+q)}(w)\frac{t^q}{q!}\left(\frac{t}{w}\right)^n = \left(\frac{w}{w-yt}\right)^{\alpha+1}\ {}_1F_1\Big(\alpha+1;1;\frac{-xt}{w-yt}\Big). \tag{4.2}$$

**II.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.2 and using relation (1.7) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty}(\alpha+\beta+q+n+1)_q\ L_{q+n}(x,y)\ P_n^{(\alpha+q,\beta+q)}(w)\frac{t^q}{q!}\left(\frac{-2t}{1+w}\right)^n$$

$$= \sum_{n=0}^{\infty}(\beta+1)_n\ L_n(x,y)\frac{\left(\frac{2t}{1+w}\right)^n}{n!}, \tag{4.3}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty}(\alpha+\beta+q+n+1)_q\ L_{q+n}(x,y)\ P_n^{(\alpha+q,\beta+q)}(w)\frac{t^q}{q!}\left(\frac{-2t}{1+w}\right)^n$$

$$= \left(\frac{1+w}{1+w-2yt}\right)^{\beta+1}\ {}_1F_1\Big(\beta+1;1;\frac{-2xt}{1+w-2yt}\Big). \tag{4.4}$$

**III.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.3 and using relation (1.9) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\ y_n(w,1-\alpha-q-2n,\beta)\frac{t^q}{q!}\frac{(-t)^n}{n!} = \sum_{n=0}^{\infty}(\alpha+1)_n\ L_n(x,y)\frac{\left(\frac{wt}{\beta}\right)^n}{n!}, \tag{4.5}$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} L_{q+n}(x,y)\, y_n(w,1-\alpha_q-2n,\beta)\frac{t^q}{q!}\frac{(-t)^n}{n!} = \left(\frac{\beta}{\beta-ywt}\right)^{\alpha+1}\ {}_1F_1\Big(\alpha+1;1;\frac{-xwt}{\beta-ywt}\Big). \tag{4.6}$$

**IV.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.4 and using relation (1.11) in

Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (\nu)_q \; L_{q+n}(x,y) \; C_n^{\nu+q}(w) \frac{2t^q}{q!} \left(\frac{t}{1-w}\right)^n = \sum_{n=0}^{\infty} (2\nu)_n \; L_n(x,y) \frac{\left(\frac{t}{1-w}\right)^n}{n!}, \quad (4.7)$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (\nu)_q \; L_{q+n}(x,y) \; C_n^{\nu+q}(w) \frac{2t^q}{q!} \left(\frac{t}{1-w}\right)^n = \left(\frac{1-w}{1-w-yt}\right)^{2\nu} {}_1F_1\Big(2\nu; 1; \frac{-xt}{1-w-yt}\Big). \quad (4.8)$$

**V.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.5 and using relation (1.13) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (p+1)_q \; L_{q+n}(x,y) \; g_n^{(p+q)}(w) \frac{t^q}{q!} \; (-t)^n = \sum_{n=0}^{\infty} L_n(x,y) \; (-wt)^n, \quad (4.9)$$

which on using relation (1.19) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (p+1)_q \; L_{q+n}(x,y) \; g_n^{(p+q)}(w) \frac{t^q}{q!} \; (-t)^n = \frac{1}{1+ywt} \; \exp\Big(\frac{xwt}{1+ywt}\Big). \quad (4.10)$$

**VI.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.6 and using relation (1.15) in Eq. (2.8), we get

$$\sum_{q,n=0}^{\infty} (a+q+2n)_q \; L_{q+n}(x,y) \; R_n(a+q,w) \frac{t^q}{q!} \; (-t)^n = \sum_{n=0}^{\infty} L_n(x,y) \frac{(wt)^n}{n!}, \quad (4.11)$$

which on using relation (1.18) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} (a+q+2n)_q \; L_{q+n}(x,y) \; R_n(a+q,w) \frac{t^q}{q!} \; (-t)^n = \exp(ywt) \; C_0(xwt). \quad (4.12)$$

Next, the following trilateral generating function is obtained as applications of result (3.7):

**VII.** Taking $N = 1$ and $\{A_{q,n}\}_{q,n=0}^{\infty}$ as in Remark 1.1 and using relation (1.4) in Eq. (3.7), we get

$$\sum_{q,n=0}^{\infty} \sum_{k=0}^{[\frac{q}{p}]} \binom{\mu+j+k-2}{k} L_{q+n-pk}(x,y) \; L_n^{(\alpha+q-pk)}(w) \; y_j(v,\mu+k,\beta) \frac{t^{q-pk}\eta^k}{(q-pk)!} \left(\frac{t}{w}\right)^n$$
$$= (1-\eta)^{1-\mu-j} \; y_j\left(\frac{v}{1-\eta}, \mu, \beta\right) \sum_{n=0}^{\infty} (\alpha+1)_n \; L_n(x,y) \frac{\left(\frac{t}{w}\right)^n}{n!}, \quad (4.13)$$

which on using relation (1.21) in the r.h.s. gives

$$\sum_{q,n=0}^{\infty} \sum_{k=0}^{[\frac{q}{p}]} \binom{\mu+j+k-2}{k} L_{q+n-pk}(x,y) \; L_n^{(\alpha+q-pk)}(w) \; y_j(v,\mu+k,\beta) \frac{t^{q-pk}\eta^k}{(q-pk)!} \left(\frac{t}{w}\right)^n$$

$$= (1-\eta)^{1-\mu-j} \left(\frac{w}{w-yt}\right)^{\alpha+1} y_j\left(\frac{v}{1-\eta}, \mu, \beta\right) {}_1F_1\Big(\alpha+1; 1; \frac{-xt}{w-yt}\Big). \quad (4.14)$$

Similarly other trilateral generating functions can be obtained as applications of result (3.7) with the help of Remarks 1.2–1.6 and relation (1.21).

Finally, it is worthy to note that, by taking $y = 1$ and using relation (1.23) the results obtained in this section give many bilateral and trilateral generating functions for the classical Laguerre polynomials $L_n(x)$ associated with other classical polynomials.

REFERENCES

[1] G. Dattoli, A. Torre, Operational methods and two variable Laguerre polynomials, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **132** (1998) 1-7.

[2] G. Dattoli, A. Torre, Exponential operators, quasi-monomials and generalized polynomials, *Radiat. Phys. Chem.* **57** (2000) 21-26.

[3] B. González, J. Matera, H.M. Srivastava, Some q-generating functions and associated generalized hypergeometric polynomials, *Math. Comput. Modelling* **34**(2001) 133–175.

[4] S.-D. Lin, I.-C. Chen, H.M. Srivastava, Certain classes of finite-series relationships and generating functions involving the generalized Bessel polynomials, *Appl. Math. Comput.* **137** (2003) 261-275.

[5] S.-D. Lin, H.M. Srivastava, P.Y. Wang, Some families of hypergeometric transformations and generating relations, *Math. Comput. Model.* **36** (2002) 445-459.

[6] M.A. Özarslan, Some families of generating functions for the extended Srivastava polynomials, *Appl. Math. Comput.* **218** (2011) 959-964.

[7] M.A. Özarslan, A. Altin, Some families of generating functions for the multiple orthogonal polynomials associated with modified Bessel $K$-functions, *J. Math. Anal. Appl.* **297** (2004) 186–193.

[8] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960, reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[9] H.M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972) 1–6.

[10] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.

DEPARTMENT OF MATHEMATICS, ADEN UNIVERSITY,
ADEN, YEMEN

*E-mail address*: gonah1977@yahoo.com