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ON RIGHT INVERSE Γ-SEMIGROUP

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Abstract. Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. An element $e \in S$ is said to be α-idempotent for some $\alpha \in \Gamma$ if $e\alpha e = e$. A Γ- semigroup S is called regular Γ-semigroup if each element of S is regular i.e, for each $a \in S$ there exists an element $x \in S$ and there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A regular Γ-semigroup S is called a right inverse Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f\beta e = f\beta e$. In this paper we introduce ip - congruence on regular Γ-semigroup and ip - congruence pair on right inverse Γ-semigroup and investigate some results relating this pair.

1. Introduction

Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if
(i)$a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and
(ii)$(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

A semigroup can be considered to be a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the binary operation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a $\Gamma-$semigroup where $\Gamma = \{1\}$.

In [8] we introduced right inverse Γ-semigroup. In [2] Gomes introduced the notion of congruence pair on inverse semigroup and studied some of its properties. In this paper we introduce the notion of ip - congruence on regular Γ-semigroup, ip - congruence pair on right inverse Γ-semigroup and studied some of its properties. We now recall some definition and results.

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Definition 1.1. *Let S be a Γ-semigroup. An element $a \in S$ is said to be regular if $a \in a\Gamma S\Gamma a$ where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be regular if every element of S is regular.*

Example 1.1. [8] Let M be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then M is a regular Γ semigroup.

Example 1.2. Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p$ is prime $\}$. Let $a, b, c \in S$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup. Let $a = \frac{m}{n} \in S$ where $m > 0$ and $n < 0$. Suppose $m = p_1 p_2 \ldots\ldots\ldots p_k$ where p_i's are prime. Now $\frac{p_1 p_2 \ldots\ldots\ldots p_k}{n}(-\frac{1}{p_1})\frac{n}{p_2 \ldots\ldots\ldots p_{k-1}}(-\frac{1}{p_k})\frac{m}{n} = \frac{p_1 p_2 \ldots\ldots\ldots p_k}{n}$. Thus taking $b = \frac{n}{p_2 \ldots\ldots\ldots p_{k-1}}$, $\alpha = (-\frac{1}{p_1})$ and $\beta = (-\frac{1}{p_k})$ we can say that a is regular. Hence S is a regular Γ-semigroup.

Definition 1.2. *Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-idempotent if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.*

Definition 1.3. *Let S be a Γ-semigroup and $a, b \in S$, $\alpha, \beta \in \Gamma$. b is said to be an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. This is denoted by $b \in V_\alpha^\beta(a)$.*

Theorem 1.1. Let S be a regular Γ-semigroup and $a \in S$. Then $V_\alpha^\beta(a)$ is non-empty for some $\alpha, \beta \in \Gamma$.

Proof: Since S is regular there exist $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$. Now we consider the element $b\beta a\alpha b$. $a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a$ and $(b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b)\beta a)\alpha b\beta a\alpha b = b\beta a\alpha b\beta a\alpha b = b\beta a\alpha b$. Hence $b\beta a\alpha b \in V_\alpha^\beta(a)$.

Definition 1.4. *Let S be a Γ-semigroup. An equivalence relation ρ on S is said to be a right (left) congruence on S if $(a, b) \in \rho$ implies $(a\alpha c, b\alpha c) \in \rho$, $((c\alpha a, c\alpha b) \in \rho)$ for all $a, b, c \in S$ and for all $\alpha \in \Gamma$. An equivalence relation which is both left and right congruence on S is called* congruence *on S.*

Definition 1.5. *A regular Γ-semigroup S is called a right orthodox Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f$ is a β-idempotent.*

Definition 1.6. *A regular Γ-semigroup M is a right orthodox Γ-semigroup if and only if for $a, b \in S$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$, we have $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$.*

Definition 1.7. *A regular Γ-semigroup S is called a right inverse Γ-semigroup if for any α-idempotent e and β-idempotent f of S, $e\alpha f\beta e = f\beta e$.*

Theorem 1.2. Every right inverse Γ-semigroup is a right orthodox Γ-semigroup.

Theorem 1.3. Let S be a regular Γ-semigroup and E_α be the set of all α-idempotents in S. Let $e \in E_\alpha$ and $f \in E_\beta$. Then

$$RS(e, f) = \left\{g \in V_\beta^\alpha(e\alpha f) \cap E_\alpha : g\alpha e = f\beta g = g\right\}$$

is non-empty.

Proof: Since S is regular, there exist $b \in S$ and $\gamma, \delta \in \Gamma$ such that $e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $b\delta e\alpha f\gamma b = b$. Now $(e\alpha f)\beta(f\gamma b\delta e)\alpha(e\alpha f) = e\alpha f\gamma b\delta e\alpha f = e\alpha f$ and $(f\gamma b\delta e)\alpha(e\alpha f)\beta(f\gamma b\delta e) = f\gamma b\delta e\alpha f\gamma b\delta e = f\gamma b\delta e$. Hence $f\gamma b\delta e \in V_\beta^\alpha(e\alpha f)$. Thus $V_\beta^\alpha(e\alpha f) \neq \phi$. Now let $x \in V_\beta^\alpha(e\alpha f)$ and setting $g = f\beta x\alpha e$ we have $g\alpha g = (f\beta x\alpha e)\alpha(f\beta x\alpha e) = f\beta(x\alpha e)\alpha f\beta x)\alpha e = f\beta x\alpha e = g$. Thus $g \in E_\alpha$.

Again $g\alpha e\alpha f\beta g = f\beta x\alpha e\alpha e\alpha f\beta f\beta x\alpha e = f\beta x\alpha e\alpha f\beta x\alpha e = f\beta x\alpha e = g$ and $e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta f\beta x\alpha e\alpha e\alpha f = e\alpha f\beta x\alpha e\alpha f = e\alpha f$ implies that $g \in V_\beta^\alpha(e\alpha f)$. Hence $g\alpha e = f\beta x\alpha e\alpha e = f\beta x\alpha e = g$ and $f\beta g = f\beta f\beta x\alpha e = f\beta x\alpha e = g$. Therefore $RS(e, f) \neq \emptyset$.

Definition 1.8. *Let S be a regular Γ- semigroup and e and f be α and β- idempotents respectively. Then the set $RS(e, f)$ described in the above Theorem is called the right sandwich set of e and f.*

Theorem 1.4. Let S be a regular Γ-semigroup and e and f be α and β-idempotents respectively. Then the set $RS(e, f) = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \ \ and \ \ e\alpha g\alpha f = e\alpha f\}$.

Proof: Let $P = \{g \in V_\beta^\alpha(e\alpha f) : g\alpha e = g = f\beta g \ \ and \ \ e\alpha g\alpha f = e\alpha f\}$ and let $g \in RS(e, f)$. Then $g \in E_\alpha, g\alpha e = g = f\beta g$ and $g \in V_\beta^\alpha(e\alpha f)$. Now $e\alpha g\alpha f = e\alpha g\alpha e\alpha f\beta g\alpha f = e\alpha f\beta g\alpha e\alpha f\beta g\alpha e\alpha f = e\alpha f\beta g\alpha e\alpha f = e\alpha f$. Hence $RS(e, f) \subseteq P$. Next let $g \in P$. Now $g\alpha g = g\alpha e\alpha f\beta g = g$. Hence $g \in E_\alpha$, which shows that $P \subseteq RS(e, f)$ and hence the proof.

Theorem 1.5. Let S be a regular Γ- semigroup and $a, b \in S$.If $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$ and $g \in RS(a'\beta a, b\gamma b')$ then $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Proof: Let $e = a'\beta a$ and $f = b\gamma b'$. Then e is an α-idempotent and f is a δ-idempotent and also g is an α-idempotent. Now $(a\alpha b)\gamma(b'\delta g\alpha a')\beta(a\alpha b) = a\alpha f\delta g\alpha e\alpha b = a\alpha g\alpha b = a\alpha a'\beta a\alpha g\alpha b\gamma b'\delta b = a\alpha e\alpha g\alpha e\alpha b = a\alpha e\alpha f\delta b = a\alpha a'\beta a\alpha b \gamma b'\delta b = a\alpha b$. Again $(b'\delta g\alpha a')\beta(a\alpha b)\gamma(b'\delta g\alpha a') = b'\delta g\alpha e\alpha f\delta g\alpha a' = b'\delta g\alpha g\alpha a' = b'\delta g\alpha a'$. Hence $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$.

Corollary 1.1. For $a, b \in S$, if $V_\alpha^\beta(a)$ and $V_\gamma^\delta(b)$ are nonempty then $V_\gamma^\beta(a\alpha b)$ is nonempty.

Proof: Let $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ then we know that $RS(a'\beta a, b\gamma b') \neq \phi$. For $g \in RS(a'\beta a, b\gamma b')$ and hence we get $b'\delta g\alpha a' \in V_\gamma^\beta(a\alpha b)$. Hence the proof.

2. IP- CONGRUENCE PAIR ON RIGHT INVERSE Γ-SEMIGROUP

In this section we characterize some congruences on a right inverse Γ - semigroup S.

Definition 2.1. *Let S be a Γ-semigroup. A nonempty subset K of S is said to be partial Γ-subsemigroup if for $a, b \in K, a\alpha b \in K$, whenever $V_\alpha^\beta(a) \neq \phi$. for $\alpha, \beta \in \Gamma$.*

Definition 2.2. *A partial Γ-subsemigroup K of S is said to be regular if $V_\alpha^\beta(k) \subseteq K$ for all $k \in K$ and $\alpha, \beta \in \Gamma$.*

Definition 2.3. *A partial Γ-subsemigroup K is said to be full if $E(S) \subseteq K$ where $E(S)$ is the set of all idempotent elements of S.*

Definition 2.4. *A partial Γ-subsemigroup K of S is said to be self conjugate if for all $a \in S, k \in K$ and $a' \in V_\alpha^\beta(a), a'\beta k\gamma a \in K$ whenever $V_\gamma^\delta(k) \neq \phi$ for some $\delta \in \Gamma$.*

Definition 2.5. *A partial Γ-subsemigroup K of S is said to be normal if it is regular, full and self conjugate.*

Definition 2.6. *An equivalence relation ρ on S is said to be left partial congruence if $(a, b) \in \rho$ implies $(c\alpha_3 a, c\alpha_3 b) \in \rho$ whenever $V_{\alpha_3}^{\beta_3}(c)$ is nonempty. Note that every left congruence is a left partial congruence.*

Here we consider these left partial congruence which satisfy the following condition:
$(a, b) \in \rho$ implies $(a\alpha_1 c, b\alpha_2 c) \in \rho$ whenever each of the sets $V_{\alpha_1}^{\beta_1}(a), V_{\alpha_2}^{\beta_2}(b)$ is nonempty for $\alpha_i, \beta_i \in \Gamma, i = 1, 2$. We call this left partial congruence as inverse related partial congruence (ip - congruence).

Example 2.1. *Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. S denotes the set of all mappings from A to B. Here members of S will be described by the images of the elements 1, 2, 3. For example the map $1 \to 4, 2 \to 5, 3 \to 4$ will be written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \to 5, 2 \to 5, 3 \to 4$. A map from B to A will be described in the same fashion. For example $(1, 2)$ denotes $4 \to 1, 5 \to 2$. Now $S = \Big\{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 5, 5), (5, 4, 5), (5, 4, 4), (5, 5, 4)\Big\}$ and let $\Gamma = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f\alpha g$ by $(f\alpha g)(a) = f\alpha\Big(g(a)\Big)$ for all $a \in A$. So $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in S$ and we can show that $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. Hence S is a Γ - semigroup.*

We can also show that it is right inverse. We now give a partition $S = \bigcup_{1\le i\le 5} S_i$ and let ρ be the equivalence relation yielded by the partition where each S_i is given by:
$S_1 = \{(4, 4, 4)\}$,
$S_2 = \{(5, 5, 5)\}$,
$S_3 = \{(4, 5, 4), (5, 4, 5)\}$,
$S_4 = \{(4, 5, 5), (5, 4, 4)\}$,
$S_5 = \{(4, 4, 5), (5, 5, 4)\}$.
Here we see that $(4, 5, 4)\rho(5, 4, 5)$ but $(4, 5, 4)(3, 1)(4, 4, 4) = (4, 4, 4)$ and $(5, 4, 5)$ $(3, 1)(4, 4, 4) = (5, 5, 5)$ i.e ρ is not a congruence.

Now for $f \in S$ we observe the following cases:
(*a*) $(4, 4, 4)\alpha f = (4, 4, 4)$ *for all* $\alpha \in \Gamma$,
(*b*) $(5, 5, 5)\alpha f = (5, 5, 5)$ *for all* $\alpha \in \Gamma$,
(*c*) $(4, 5, 4)(1, 2)f = f$ *and* $(4, 5, 4)(2, 3)f = f'$,
$(5, 4, 5)(2, 3)f = f$ *and* $(5, 4, 5)(1, 2)f = f'$,
(*d*) $(4, 4, 5)(2, 3)f = f$ *and* $(4, 4, 5)(3, 1)f = f'$,
$(5, 5, 4)(3, 1)f = f$ *and* $(5, 5, 4)(2, 3)f = f'$,
(*e*) $(4, 5, 5)(1, 2)f = f$ *and* $(4, 5, 5)(3, 1)f = f'$,
$(5, 4, 4)(3, 1)f = f$ *and* $(5, 4, 4)(1, 2)f = f'$,

From the above cases we can easily verify that ρ is a ip - congruence on S.

Definition 2.7. *An ip - congruence ξ on $E(S)$ of S is said to be normal if for any α-idempotent e and β-idempotent $f, a \in S$ and $a' \in V_{\gamma}^{\delta}(a), (e, f) \in \xi$ implies $(a'\delta e\alpha a, a'\delta f\beta a) \in \xi$ whenever $a'\delta e\alpha a, a'\delta f\beta a \in E(S)$.*

Let ρ be an ip - congruence on a regular Γ - semigroup S then we can define a binary operation on S/ρ as $(a\rho)(b\rho) = (a\alpha b)\rho$ whenever $V_\alpha^\beta(a)$ exists for some $\beta \in \Gamma$. This is well defined because if $a\rho = a'\rho$ and $b\rho = b'\rho$ then

$$\begin{aligned}(a\rho)(b\rho) &= (a\alpha b)\rho \text{ (Since } V_\alpha^\beta(a) \neq \phi \text{ for some } \alpha, \beta \in \Gamma) \\ &= (a\alpha b')\rho \\ &= (a'\alpha_1 b')\rho (\text{Since } V_{\alpha_1}^{\beta_1}(a') \neq \phi \text{ for some } \alpha_1, \beta_1 \in \Gamma) \\ &= (a'\rho)(b'\rho).\end{aligned}$$

The operation is easily seen to be associative, and so S/ρ is a semigroup.

Definition 2.8. *Let ρ be an ip - congruence on a regular Γ-semigroup S. Let $\alpha \in \Gamma$, then the subset $\{a \in S : a\rho \in E(S/\rho)\}$ of S is called kernel of ρ and it is denoted by K.*

Definition 2.9. *Let ρ be an ip - congruence on a regular Γ-semigroup S. Then the restriction of ρ to the subset $E(S)$ is called the trace of ρ and it is denoted by $tr\rho$.*

We now treat S as a right inverse Γ-semigroup throughout the paper.

Definition 2.10. *A pair (ξ, K) consisting of a normal ip - congruence ξ on $E(S)$ and a normal partial Γ- subsemigroup K of S is said to be ip - congruence pair for S if for all $a, b \in S, a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma$*
(i) $e\gamma a \in K, (e, a\alpha a') \in \xi \Rightarrow a \in K$
(ii) $a \in K \Rightarrow (a\alpha e\gamma a', e\gamma a\alpha a') \in \xi$

Given a pair (ξ, K) we define a relation $\rho_{(\xi,K)}$ on S by $(a, b) \in \rho_{(\xi,K)}$ if and only if there exist $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that $a\alpha b' \in K, (a'\beta a, b'\delta b) \in \xi$.

Theorem 2.1. Let S be a right inverse Γ-semigroup. Then for an ip - congruence pair (ξ, K) and a μ-idempotent $e, a\alpha b \in K$ implies $a\alpha e\mu b \in K$ for all $a, b \in S$ and $V_\alpha^\beta(a) \neq \phi$ for some $\beta \in \Gamma$.

Proof: Let $a\alpha b \in K$. Since S is regular there exist $\gamma, \delta \in \Gamma$ such that $V_\gamma^\delta(b) \neq \phi$. Then by Corollary 1.1 , $V_\gamma^\beta(a\alpha b) \neq \phi$. Let $b' \in V_\gamma^\delta(b)$. Then $b\gamma b'$ is a δ-idempotent and since S is a right inverse Γ-semigroup $(b\gamma b')\delta e\mu(b\gamma b') = e\mu(b\gamma b')$. Now $a\alpha e\mu b = a\alpha e\mu b\gamma b'\delta b = a\alpha(b\gamma b')\delta e\mu(b\gamma b')\delta b = (a\alpha b)\gamma(b'\delta e\mu b)$. Since S is right inverse Γ-semigroup $b'\delta e\mu b \in E_\gamma \subseteq K$. Since K is a partial Γ-subsemigroup and $a\alpha b \in K$, $(a\alpha b)\gamma(b'\delta e\mu b) \in K$. So $a\alpha e\mu b \in K$.

Theorem 2.2. Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that $(a, b) \in \rho_{(\xi,K)}$, then there exist $a' \in V_\alpha^\beta(a)$ and $b' \in V_\gamma^\delta(b)$ such that
(i) $a\alpha b' \in K$ and $(a'\beta a, b'\delta b) \in \xi$
(ii) $b\gamma a' \in K$ and so $(b, a) \in \rho_{(\xi,K)}$
(iii) $(b\gamma b', a\alpha a'\beta b\gamma b') \in \xi$ and $(a\alpha a', b\gamma b'\delta a\alpha a') \in \xi$

Proof: (i) Let $a, b \in S$ and $(a, b) \in \rho_{(\xi,K)}$. Then (i) follows from definition of $\rho_{(\xi,K)}$. Now from (i) we have $a\alpha b' \in K$ and $(a'\beta a, b'\delta b) \in \xi$. Let $g \in RS(b'\delta b, a'\beta a)$, then g is a γ-idempotent. So by Theorem 1.5 we have $a\alpha g\gamma b' \in V_\beta^\delta(b\gamma a')$. Also by Theorem 2.1 $a\alpha g\gamma b' \in K$ since $a\alpha b' \in K$ and $g \in E_\gamma$. On the other hand $b\gamma a' \in V_\delta^\beta(a\alpha g\gamma b')$ and so $b\gamma a' \in K$, since K is a normal subsemigroup of S. Therefore $(b, a) \in \rho_{(\xi,K)}$ since ξ is symmetric. Hence (ii) follows.
Again for $g \in RS(b'\delta b, a'\beta a)$, $g = g\gamma b'\delta b = a'\beta a\alpha g$ and $(b'\delta b)\gamma g\gamma(a'\beta a) = (b'\delta b)\gamma$ $(a'\beta a)$ by Theorem 1.4. Hence $b\gamma g\gamma b' \in E_\delta$. Now $b'\delta b = (b'\delta b)\gamma(b'\delta b)\ \xi\ (b'\delta b)\gamma$

$(a'\beta a) = (b'\delta b)\gamma g\gamma(a'\beta a)\ \xi\ (b'\delta b)\gamma g\gamma(b'\delta b)$ and so by normality of ξ we have $b\gamma(b'\delta b)\gamma b'\ \xi\ b\gamma(b'\delta b\gamma g\gamma b'\delta b)\gamma b'$ i.e $b\gamma b'\ \xi\ b\gamma g\gamma b'$. Now $a\alpha g\gamma b' \in V_\beta^\delta(b\gamma a')$ and so we have

$$\begin{aligned} b\gamma b' &\ \xi\ b\gamma g\gamma b' \\ &= b\gamma(a'\beta a\alpha g)\gamma b' \text{ (Since } g \in RS(b'\delta b, a'\beta a)) \\ &= (b\gamma a')\beta(a\alpha a'\beta a)\alpha g\gamma b' \\ &= (b\gamma a')\beta(a\alpha a')\beta(a\alpha g\gamma b') \text{ (Since } a\alpha a' \in E_\beta \text{ and } b\gamma a' \in K) \\ &\ \xi\ (a\alpha a')\beta(b\gamma a')\beta(a\alpha g\gamma b') \text{ (by Definition 2.6 and } a\alpha g\gamma b' \in V_\beta^\delta(b\gamma a')) \\ &= a\alpha a'\beta b\gamma g\gamma b' \\ &\ \xi\ (a\alpha a')\beta(b\gamma b'). \end{aligned}$$

Similarly interchanging the role of a and b we can get the second relation.

Theorem 2.3. Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that $a, b \in \rho_{(\xi,K)}$, then for all $a^* \in V_\alpha^\beta(a)$ and $b^* \in V_\gamma^\delta(b)$, $a\alpha b^* \in K$ and $(a^*\beta a, b^*\delta b) \in \xi$

Proof: Since $(a, b) \in \rho_{(\xi,K)}$, there exist $a' \in V_{\alpha_1}^{\beta_1}(a)$ and $b' \in V_{\gamma_1}^{\delta_1}(b)$ such that all the three conditions of Theorem 2.2 are satisfied. Now

$$\begin{aligned} a'\beta_1 a &= a'\beta_1 a\alpha a^*\beta a \\ &= a'\beta_1 a\alpha a^*\beta a\alpha_1 a'\beta_1 a \\ &\ \xi\ a'\beta_1 a\alpha_1 a^*\beta a\alpha a'\beta_1 a \text{ (Since } \xi \text{ is an ip - congruence and } V_\alpha^\beta(a) \text{ and } V_{\alpha_1}^{\beta_1}(a) \text{ are nonempty.)} \\ &= (a'\beta_1 a)\alpha_1(a^*\beta a)\alpha(a'\beta_1 a) \\ &= (a^*\beta a)\alpha(a'\beta a) \\ &\ \xi\ a^*\beta a\alpha_1 a'\beta a \text{ (Since } \xi \text{ is an ip - congruence and } V_\alpha^\beta(a) \text{ and } V_{\alpha_1}^{\beta_1}(a) \text{ are nonempty.)} \\ &= a^*\beta a. \end{aligned}$$

Similarly we can show that $(b'\delta_1 b, b^*\delta b) \in \xi$. Hence we have $a^*\beta a\ \xi\ a'\beta_1 a\ \xi\ b'\delta_1 b\ \xi\ b^*\delta b$. Hence $(a^*\beta a, b^*\delta b) \in \xi$. We now prove that $a\alpha b^* \in K$. To prove this we proceed by five steps.

Step1: $b\gamma_1 a' \in K$.
Step2: $b'\delta_1 a \in K$.
Step3: $b^*\delta a \in K$.
Step4: $(b\gamma b^*, a\alpha a^*\beta b\gamma b^*) \in \xi$.
Step5: $a\alpha b^* \in K$.

Let $g \in RS(b'\delta_1 b, a'\beta_1 a)$, then g is a γ_1-idempotent and we have $a\alpha_1 g\gamma_1 b' \in V_{\beta_1}^{\delta_1}(b\gamma_1 a')$. Also since $a\alpha_1 b' \in K$ and $g \in E_{\gamma_1}$, by Theorem 2.1 $a\alpha_1 g\gamma_1 b' \in K$. On the other hand $b\gamma_1 a' \in V_{\delta_1}^{\beta_1}(a\alpha_1 g\gamma_1 b')$. Since K is regular we have $b\gamma_1 a' \in K$.

Let $h \in RS(b\gamma_1 b', a\alpha_1 a')$. Then $a'\beta_1 h\delta_1 b \in V_{\alpha_1}^{\gamma_1}(b'\delta_1 a)$ i.e, $b'\delta_1 a \in V_{\gamma_1}^{\alpha_1}(a'\beta_1 h\ \delta_1 b)$. Now since $b\gamma_1 a' \in K$ and K is full self conjugate partial Γ-subsemigroup of S, we have

$(b'\delta_1 b)\gamma_1(a'\beta_1 a)\alpha_1(a'\beta_1 h\delta_1 b) = b'\delta_1((b\gamma_1 a')\beta_1 h)\delta_1 b \in K$.

Now

$$\begin{aligned} h\delta_1(a\alpha_1 a') &= (a\alpha_1 a')\beta_1 h\delta_1(a\alpha_1 a') \\ &\ \xi\ (b\gamma_1 b')\delta_1(a\alpha_1 a')\beta_1 h\delta_1(a\alpha_1 a') \text{(By Theorem 2.2)} \\ &= (b\gamma b')\delta_1 h\delta_1(a\alpha a') \text{ (Since } S \text{ is right inverse)} \\ &= (b\gamma b')\delta_1(a\alpha a') \text{ (Since } h \in RS(b\gamma_1 b', a\alpha_1 a'). \\ &\ \xi\ a\alpha_1 a' \text{ (By Theorem 2.2).} \end{aligned}$$

Again

$$\begin{array}{rcl}(a'\beta_1 h\delta_1 b)\gamma_1(b'\delta_1 a) & = & a'\beta_1 h\delta_1 a\\ & \xi & a\alpha_1 a'\\ & \xi & (b'\delta_1 b)\gamma_1(a'\beta_1 a) \text{ (By Theorem 2.2).}\end{array}$$

Now since S is a right inverse Γ-semigroup, it is right orthodox and hence $(b'\delta_1 b)\gamma_1$ $(a'\beta_1 a)$ is an α_1-idempotent. Thus by Definition 2.10 $a'\beta_1 h\delta_1 b \in K$ and since K is regular, $b'\delta_1 a \in K$.

Now we have $b'\delta_1 a \in K$. Hence we get $b'\delta_1(b\gamma b^*)\delta a \in K$ by Theorem 2.1. Again $b^*\delta a = b^*\delta b\gamma b^*\delta a = b^*\delta(b\gamma_1 b'\delta_1 b)\gamma b^*\delta a = (b^*\delta b)\gamma_1(b'\delta b\gamma b^*\delta a) \in K$ since $b^*\delta b \in E_\gamma \subseteq K$, $V_{\gamma_1}^{\delta_1}(b)$ is nonempty and K is a partial Γ-subsemigroup.

We now prove step 4.

$$\begin{array}{rcl}b\gamma b^* & = & (b\gamma_1 b')\delta_1(b\gamma b^*)\\ & \xi & (a\alpha_1 a')\beta_1(b\gamma_1 b')\delta_1(b\gamma b^*)\\ & = & (a\alpha a^*)\beta(a\alpha_1 a')\beta_1(b\gamma_1 b')\delta_1(b\gamma b^*)\\ & \xi & (a\alpha a^*)\beta(b\gamma_1 b')\delta_1(b\gamma b^*)\\ & = & (a\alpha a^*)\beta(b\gamma b^*).\end{array}$$

Finally we show the last step. Now we have $b^*\delta a \in K$. Since $a^* \in V_\alpha^\beta(a)$ and $b^* \in V_\gamma^\delta(b)$, we have $(a^*\beta b) \in V_\alpha^\gamma(b^*\delta a)$ and hence $a^*\beta b \in K$, since K is regular. Let $x \in RS(a^*\beta a, b^*\delta b)$. Then $b\gamma x\alpha a^* \in V_\delta^\beta(a\alpha b^*)$. Now $((a\alpha a^*)\beta(b\gamma b^*))\delta(b\gamma x\alpha a^*) = a\alpha a^*\beta b\gamma x\alpha a^* = a\alpha((a^*\beta b)\gamma x)\alpha a^* \in K$, since $a^*\beta b \in K, x \in E_\alpha \subseteq K$ and hence $(a^*\beta b)\gamma x \in K$ and also K is self conjugate. Again

$$\begin{array}{rcl}x\alpha(b^*\delta b) & = & (b^*\delta b)\gamma x\alpha(b^*\delta b) \text{ (Since } S \text{ is right inverse)}\\ & \xi & ((b^*\delta b\gamma(a^*\beta a))\alpha x\alpha(b^*\delta b) \text{ (Since } (a^*\beta a, b^*\delta b) \in \xi\\ & = & (b^*\delta b)\gamma(a^*\beta a)\alpha(b^*\delta b) \text{ (Since } x \in RS(a^*\beta a, b^*\delta b).)\\ & \xi & ((b^*\delta b)\gamma(b^*\delta b)\gamma(b^*\delta b)\text{(Since } \xi \text{ is an ip - congruence and}\\ & & \qquad (a^*\beta a, b^*\delta b) \in \xi)\\ & = & b^*\delta b.\end{array}$$

Thus

$$\begin{array}{rcl}b\gamma x\alpha b^* & = & b\gamma(x\alpha(b^*\delta b))\gamma b^*\\ & \xi & b\gamma(b^*b)\gamma b^*\\ & = & b\gamma b^*.\end{array}$$

Now

$$\begin{array}{rcl}(b\gamma x\alpha a^*)\beta(a\alpha b^*) & = & b\gamma(x\alpha(a^*\beta a))\alpha b^*\\ & = & b\gamma x\alpha b^*\\ & \xi & b\gamma b^*\\ & \xi & (a\alpha a^*)\beta(b\gamma b^*).\end{array}$$

Again since S is a right inverse Γ-semigroup, $(a\alpha a^*)\beta(b\gamma b^*)$ is a δ-idempotent and by Definition 2.10(i) $b\gamma x\alpha a^* \in K$ and hence $a\alpha b^* \in K$ since K is regular. Hence the Theorem.

Remark 2.1. *From the previous Theorem, we can say that in the definition 3.11 of $\rho_{(\xi,K)}$ and in the Theorem 2.2 "there exist" can be substituted by "for all".*

Theorem 2.4. Let (ξ, K) be an ip - congruence pair for S and $a, b, c \in S$ and let $a' \in V_{\alpha_1}^{\beta_1}(a)$, $b' \in V_{\alpha_2}^{\beta_2}(b), c' \in V_{\alpha_3}^{\beta_3}(c), g \in RS(c'\beta_3 c, a\alpha_1 a'), h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then $(a'\beta_1 a, b'\beta_2 b) \in \xi$, $a\alpha_1 b' \in K$ implies $(a'\beta_1 g\alpha_3 a, b'\beta_2 h\alpha_3 b) \in \xi$.

Proof: Let (ξ, K) be an ip - congruence pair for S and $a, b \in S$ are such that for some $a' \in V_{\alpha_1}^{\beta_1}(a)$, $b' \in V_{\alpha_2}^{\beta_2}(b), (a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$. Given $c \in S$

and $c' \in V_{\alpha_3}^{\beta_3}(c)$, let $g \in RS(c'\beta_3 c, a\alpha_1 a')$ and $h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then g and h are α_3-idempotents. Choose an arbitrary element $x \in RS(a'\beta_1 a, b'\beta_2 b)$. Then $b\alpha_2 x\alpha_1 a' \in V_{\beta_2}^{\beta_1}(a\alpha_1 b')$. So $a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a' \in E_{\beta_1}$. Also let $t \in RS(g, a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a')$ then $t \in E_{\alpha_3}$ and $t = t\alpha_3 g$ and hence $b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 g \in V_{\beta_2}^{\alpha_3}(g\alpha_3 a\alpha_1 b')$ and $b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b' = (b\alpha_2 x\alpha_1 a')\beta_1(t\alpha_3 g)\alpha_3 a\alpha_1 b' = (b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 g)\alpha_3(g\alpha_3 a\alpha_1 b') \in E_{\beta_2}$. On the other hand $b\alpha_2 x\alpha_1 a' \in K$, since it is an (β_2, β_1)-inverse of $a\alpha_1 b'$ which belongs to K. Now since (ξ, K) is an ip - congruence pair for S, by definition we have $((b\alpha_2 x\alpha_1 a')\beta_1 t\alpha_3(a\alpha_1 b')\,,\, t\alpha_3 b\alpha_2 x\alpha_1 a'\beta_1 a\alpha_1 b') \in \xi$. Again since $x\alpha_1(a'\beta_1 a) = x$ we get

$$(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', t\alpha_3 b\alpha_2 x\alpha_1 b') \in \xi \tag{2.1}$$

for all $x \in RS(a'\beta_1 a, b'\beta_2 b)$

Now since ξ is an ip - congruence and $(a'\beta_1 a, b'\beta_2 b) \in \xi$, we have $b'\beta_2 b\alpha_2 x\alpha_1 b'\beta_2 b$ ξ $a'\beta_1 a\alpha_1 x\alpha_1 b'\beta_2 b = a'\beta_1 a\alpha_1 b'\beta_2 b$ ξ $b'\beta_2 b\alpha_2 b'\beta_2 b = b'\beta_2 b$. Again and hence $(b\alpha_2 x\alpha_1 b')\beta_2(b\alpha_2 x\alpha_1 b') = b\alpha_2 x\alpha_1(b'\beta_2 b\alpha_2 x)\alpha_1 b' = b\alpha_2 x\alpha_1 b'$ and hence $b\alpha_2 x\alpha_1 b' \in E_{\beta_2}$. Hence ξ is normal, we have $(b\alpha_2(b'\beta_2 b\alpha_2 x\alpha_1 b'\beta_2 b)\alpha_2 b', b\alpha_2(b'\beta_2 b)\alpha_2 b') \in \xi$ which implies

$$(b\alpha_2 x\alpha_1 b', b\alpha_2 b') \in \xi \tag{2.2}$$

Similarly we can show that

$$(a\alpha_1 x\alpha_1 a', a\alpha_1 a') \in \xi \tag{2.3}$$

Using (2.1)and(2.2) we get

$$(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', t\alpha_3 b\alpha_1 b') \in \xi \tag{2.4}$$

Since $a\alpha_1 a'\beta_1 t = a\alpha_1 a'\beta_1((a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a')\beta_1 t) = a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t = t$, we have $a'\beta_1 t\alpha_3 a \in E_{\alpha_1}$. Since $(b'\beta_2 b, a'\beta_1 a) \in \xi$, we have

$$\begin{array}{rcl} b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b & \xi & a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 a'\beta_1 a \\ & = & a'\beta_1 a\alpha_1 x\alpha_1 a'\beta_1 t\alpha_3 a \\ & = & a'\beta_1 a\alpha_1(x\alpha_1 a'\beta_1 a)\alpha_1 a'\beta_1 t\alpha_3 a \\ & \xi & a'\beta_1 a\alpha_1 x\alpha_1(b'\beta_2 b)\alpha_2 a'\beta_1 t\alpha_3 a \text{ (Since } \xi \text{ is an} \\ & = & \text{ip - congruence)} \\ & = & a'\beta_1 a\alpha_1 b'\beta_2 b\alpha_2 a'\beta_1 t\alpha_3 a \text{ (Since } x \in \\ & & RS(a'\beta_1 a, b'\beta_2 b)) \\ & \xi & a'\beta_1 a\alpha_1 a'\beta_1 a\alpha_1 a'\beta_1 t\alpha_3 a \\ & = & a'\beta_1 t\alpha_3 a. \end{array}$$

Hence

$$(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, a'\beta_1 t\alpha_3 a) \in \xi \tag{2.5}$$

Next since $g \in RS(c'\beta_3 c, a\alpha_1 a')$, $a\alpha_1 a'\beta_1 g = g$ and hence we have $a'\beta_1 g\alpha_3 a \in E_{\alpha_1}$. Now since $x \in RS(a'\beta_1 a, b'\beta_2 b), a\alpha_1 b'\beta_2 b\alpha_2 x\alpha_1 a' = a\alpha_1 x\alpha_1 a' \in E_{\beta_1}$ and hence $t \in RS(g, a\alpha_1 x\alpha_1 a')$. Thus we have $g\alpha_3 t\alpha_3 a\alpha_1 x\alpha_1 a' = g\alpha_3 a\alpha_1 x\alpha_1 a'$. Now by (2.3) we have $((g\alpha_3 t)\alpha_3 a\alpha_1 x\alpha_1 a', (g\alpha_3 t)\alpha_3 a\alpha_1 a') \in \xi$ i.e, $(g\alpha_3 a\alpha_1 x\alpha_1 a', g\alpha_3 t\alpha_3 a\alpha_1 a') \in \xi$ since $t \in RS(ga\alpha_1 x\alpha_1 a')$ and again using (2.3)we have $g\alpha_3 a\alpha_1 a'$ ξ $g\alpha_3 a\alpha_1 x\alpha_1 a'$ ξ

$g\alpha_3 t\alpha_3 a\alpha_1 a'$ i.e, we get $(g\alpha_3 a\alpha_1 a', g\alpha_3 t\alpha_3 a\alpha_1 a') \in \xi$. Now since S is a right inverse Γ-semigroup $t\alpha_3 g\alpha_3 t = g\alpha_3 t$ and hence we have $g\alpha_3 t\alpha_3 a\alpha_1 a' = t\alpha_3 g\alpha_3 t\alpha_3 a\alpha_1 a' = t\alpha_3 a\alpha_1 a'$ since $t\alpha_3 g = t$. Thus $(g\alpha_3 a\alpha_1 a', t\alpha_3 a\alpha_1 a') \in \xi$ by transitivity of ξ. Now since ξ is normal, we have $(a'\beta_1(g\alpha_3 a\alpha_1 a')\beta_1 a\ ,\ a'\beta_1(t\alpha_3 a\alpha_1 a')\beta_1 a) \in \xi$. i.e,

$$(a'\beta_1 g\alpha_3 a, a'\beta_1 t\alpha_3 a) \in \xi \tag{2.6}$$

Again since S is a right inverse Γ-semigroup and the fact that $t \in RS(g, a\alpha_1 x\alpha_1 a')$ and $g \in RS(c'\beta_3 c, a\alpha_1 a')$ we see that

$$\begin{aligned} t\alpha_3 b\alpha_2 b' &= b\alpha_2 b'\beta_2 t\alpha_3 b\alpha_2 b' \text{ (Since } S \text{ is right inverse Γ-semigroup)} \\ &= b\alpha_2 b'\beta_2 (t\alpha_3 g)\alpha_3 (b\alpha_2 b') \\ &= b\alpha_2 b'\beta_2 (t\alpha_3 g\alpha_3 c'\beta_3 c)\alpha_3 b\alpha_2 b'. \end{aligned}$$

Now since $(a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$, proceeding the same way of Theorem 2.2 we have $(b\alpha_2 b', a\alpha_1 a'\beta_1 b\alpha_2 b') \in \xi$. Now

$$\begin{aligned} t\alpha_3 b\alpha_2 b' &= b\alpha_2 b'\beta_2 t\alpha_3 g\alpha_3 c'\beta_3 c\alpha_3 b\alpha_2 b' \\ &\ \xi\ b\alpha_2 b'\beta_2 t\alpha_3 g\alpha_3 c'\beta_3 c\alpha_3 (a\alpha_1 a'\beta_1 b\alpha_2 b') \text{ (Since } (b\alpha_2 b', a\alpha_1 a'\beta_1 b\alpha_2 b') \in \xi) \\ &= b\alpha_2 b'\beta_2 (g\alpha_3 t\alpha_3 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 g\alpha_3 t\alpha_3 (a\alpha_1 a'\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } g \in RS(c'\beta_3 c, a\alpha_1 a')) \\ &\ \xi\ b\alpha_2 b'\beta_2 g\alpha_3 t\alpha_3 (a\alpha_1 x\alpha_1 a')\beta_1 g\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (by (2.3))} \\ &= b\alpha_2 b'\beta_2 (g\alpha_3 (a\alpha_1 x\alpha_1 a')\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } t \in RS(g, a\alpha_1 x\alpha_1 a')) \\ &\ \xi\ b\alpha_2 b'\beta_2 (g\alpha_3 (a\alpha_1 a')\beta_1 g)\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (By (2.3))} \\ &= b\alpha_2 b'\beta_2 g\alpha_3 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } (a\alpha_1 a')\beta_1 g = g) \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 g\alpha_3 c'\beta_3 c)\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 c'\beta_3 c\alpha_3 g\alpha_3 (a\alpha_1 a'\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 b\alpha_2 b' \text{ (Since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{(since } g \in RS(c'\beta_3 c, a\alpha_1 a')) \\ &= b\alpha_2 b'\beta_2 a\alpha_1 a'\beta_1 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{(since } S \text{ is right inverse)} \\ &= b\alpha_2 b'\beta_2 c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b \\ &= b\alpha_2 b'\beta_2 (c'\beta_3 c\alpha_3 a\alpha_1 a')\beta_1 b\alpha_2 b' \\ &= c'\beta_3 c\alpha_3 a\alpha_1 a'\beta_1 b\alpha_2 b' \text{ (Since } S \text{ is right inverse and hence right orthodox)} \\ &\ \xi\ c'\beta_3 c\alpha_3 b\alpha_2 b' \\ &= c'\beta_3 \alpha_3 h\alpha_3 b\alpha_2 b' \text{(since } h \in RS(c'\beta_3 c, b\alpha_2 b')) \\ &= h\alpha_3 c'\beta_3 c\alpha_3 h\alpha_3 b\alpha_2 b' \text{ (since } S \text{ is right inverse)} \\ &= h\alpha_3 b\alpha_2 b' \text{ (Since } h \in RS(c'\beta_3 c, b\alpha_2 b')) \end{aligned}$$

Hence we have

$$(t\alpha_3 b\alpha_2 b', h\alpha_3 b\alpha_2 b') \in \xi \tag{2.7}$$

Finally from (2.4) and (2.7) we have $(b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b', h\alpha_3 b\alpha_2 b') \in \xi$ and by normality of ξ we have $(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, b'\beta_2 h\alpha_3 b\alpha_2 b'\beta_2 b) \in \xi$ i.e, $(b'\beta_2 b\alpha_2 x\alpha_1 a'\beta_1 t\alpha_3 a\alpha_1 b'\beta_2 b, b'\beta_2 h\alpha_3 b) \in \xi$. It is to be noted that both the elements belong to E_{α_2}. Also by normality of ξ together with (2.5) and (2.6) we have $(a'\beta_1 g\alpha_3 a, b'\beta_2 h\alpha_3 b) \in \xi$. Hence the proof.

Theorem 2.5. If (ξ, K) is an ip - congruence pair for S,then $\rho_{(\xi,K)}$ is an ip - congruence with trace ξ and kernel K. Conversely if ρ is an ip - congruence on S then $(tr\rho, Ker\rho)$is an ip - congruence pair and $\rho = \rho_{(tr\rho, Ker\rho)}$.

Proof. Let (ξ, K) be an ip - congruence pair for S and $\rho_{(\xi,K)}$ and let $\rho = \rho_{(\xi,K)}$. Since $E(S) \subseteq K$ and ξ is reflexive, ρ is also reflexive. Again from Theorem 2.2 and Remark 2.1, we see that ρ is symmetric. We now show that ρ is transitive. For this let us suppose that $(a,b) \in \rho$ and $(b,c) \in \rho$ and let $a' \in V_{\alpha_1}^{\beta_1}(a)$, $b' \in V_{\alpha_2}^{\beta_2}(b), c' \in V_{\alpha_3}^{\beta_3}(c)$. Then we have $(a'\beta_1 a, b'\beta_2 b) \in \xi, (b'\beta_2 b, c'\beta_3 c) \in \xi, a\alpha_1 b' \in K, b\alpha_2 c' \in K$. Since ξ is transitive we have $(a'\beta_1 a, c'\beta_3 c) \in \xi$. We now show that $a\alpha_1 c' \in K$. Now by Theorem 2.2, $b\alpha_2 a' \in K$ and $c\alpha_3 b' \in K$. Hence $c\alpha_3 b'\beta_2 b\alpha_2 a' \in K$, Since K is a Γ-subsemigroup. Let $g \in RS(c'\beta_3 c, b'\beta_2 b)$ and $h \in RS(c'\beta_3 c, a'\beta_1 a)$. By Theorem 2.1 and since $g = g\alpha_3 c'\beta_3 c \in E_{\alpha_3}$, we have,

$$(c\alpha_3 b'\beta_2 b)\alpha_2 (g\alpha_3 c'\beta_3 c)\alpha_3 a' \in K \tag{2.8}$$

Again since $b\alpha_2 g\alpha_3 c' \in V_{\beta_2}^{\beta_3}(c\alpha_3 b'), c\alpha_3 b'\beta_2 b\alpha_2 g\alpha_3 c' \in E_{\beta_3}$. Now $c'\beta_3 c = c'\beta_3 c\alpha_3 c'\beta_3 c \; \xi \; c'\beta_3 c\alpha_3 b'\beta_2 b = c'\beta_3 c\alpha_3 g\alpha_3 b'\beta_2 b \; \xi \; c'\beta_3 c\alpha_3 g\alpha_3 c'\beta_3 c = c'\beta_3 c\alpha_3 g$, since $(b'\beta_2 b, c'\beta_3 c) \in \xi$ and $g \in RS(c'\beta_3 c, b'\beta_2 b)$. Also since $c\alpha_3 g\alpha_3 c' \in E_{\beta_3}$ and ξ is normal, it follows that $(c\alpha_3 (c'\beta_3 c)\alpha_3 c, c\alpha_3 (c'\beta_3 c\alpha_3 g)\alpha_3 c') \in \xi$ i.e,$(c\alpha_3 c', c\alpha_3 g\alpha_3 c') \in \xi$. Similarly since $(c'\beta_3 c, a'\beta_1 a) \in \xi$ and $c\alpha_3 h\alpha_3 c' \in E_{\beta_3}$ we have $(c\alpha_3 c, c\alpha_3 h\alpha_3 c') \in \xi$. By transitivity of ξ , $(c\alpha_3 g\alpha_3 c', c\alpha_3 h\alpha_3 c') \in \xi$. Again $c\alpha_3 (b'\beta_2 b\alpha_2 g)\alpha_3 c' = c\alpha_3 g\alpha_3 c' \; \xi \; c\alpha_3 h\alpha_3 c' = c\alpha_3 (a'\beta_1 a\alpha_1 h)\alpha_3 c'$. i.e,
$(c\alpha_3 b'\beta_2 b\alpha_2 g\alpha_3 c', c\alpha_3 a'\beta_1 a\alpha_1 h\alpha_3 c') \in \xi$. Again since $b\alpha_2 g\alpha_3 c' \in V_{\beta_2}^{\beta_3}(c\alpha_3 b')$, $c\alpha_3 b' \beta_2 b\alpha_2 g\alpha_3 c' \in E_{\beta_3}$ and since $a\alpha_1 h\alpha_3 c' \in V_{\beta_1}^{\beta_3}(c\alpha_3 a')$, from (2.8) and Definition 2.10 we can say that $c\alpha_3 a' \in K$ and by Theorem 2.2 we have $a\alpha_1 c' \in K$. Hence ρ is transitive. Hence ρ is an equivalence relation.
We now prove that ρ is an ip - congruence. Let us suppose that $(a,b) \in \rho$. Then for all $a' \in V_{\alpha_1}^{\beta_1}(a), b' \in V_{\alpha_2}^{\beta_2}(b), (a'\beta_1 a, b'\beta_2 b) \in \xi$ and $a\alpha_1 b' \in K$. Let $c \in S$ and $c' \in V_{\alpha_3}^{\beta_3}(c)$. We now prove that $(c\alpha_3 a, c\alpha_3 b) \in \rho$. Let $g \in RS(c'\beta_3 c, a\alpha_1 a')$ and $h \in RS(c'\beta_3 c, b\alpha_2 b')$. Then $a'\beta_1 g\alpha_3 c' \in V_{\alpha_1}^{\beta_3}(c\alpha_3 a)$ and $b'\beta_2 h\alpha_3 c' \in V_{\alpha_2}^{\beta_3}(c\alpha_3 b)$ and by Theorem 2.4 we have $a'\beta_1 g\alpha_3 c'\beta_3 c\alpha_3 a = a'\beta_1 g\alpha_3 a \; \xi \; b'\beta_2 h\alpha_3 b = b'\beta_2 h\alpha_3 c'\beta_3 c\alpha_3 b$. Also $(c\alpha_3 a)\alpha_1 (b'\beta_2 h\alpha_3 c') = c\alpha_3 (a\alpha_1 b')\beta_2 h\alpha_3 c' \in K$ since $a\alpha_1 b' \in K$ and $h \in E_{\alpha_3}$ and K is self conjugate. Hence by definition of ρ we have $(c\alpha_3 a, c\alpha_3 b) \in \rho$. Next we prove that $(a\alpha_1 c, b\beta_1 c) \in \rho$. For this let $g \in RS(a'\beta_1 a, c\alpha_3 c')$ and $h \in RS(b'\beta_2 b, c\alpha_3 c')$. Then $c'\beta_3 g\alpha_1 a' \in V_{\alpha_3}^{\beta_1}(a\alpha_1 c)$ and $c'\beta_3 h\alpha_2 b' \in V_{\alpha_3}^{\beta_2}(b\alpha_2 c)$. Now

$$
\begin{aligned}
g\alpha_1 c\alpha_3 c' &= g\alpha_1 a'\beta_1 a\alpha_1 c\alpha_3 c' \text{ (Since } g \in RS(a'\beta_1 a, c\alpha_3 c'))\\
&\xi\ g\alpha_1 b'\beta_2 b\alpha_2 c\alpha_3 c'\\
&= g\alpha_1 b'\beta_2 b\alpha_2 h\alpha_2 c\alpha_3 c' \text{ (Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&\xi\ g\alpha_1 (a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{ (Since } \xi \text{ is an ip - congruence and } (a'\beta_1 a, b'\beta_2 b) \in \xi)\\
&= (a'\beta_1 a\alpha_1 g\alpha_1 a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{ (Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 g\alpha_1 a'\beta_1 a\alpha_1 (c\alpha_3 c'\beta_3 h)\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&= a'\beta_1 a\alpha_1 g\alpha_1 (a'\beta_1 a\alpha_1 c\alpha_3 c')\beta_3 h\alpha_2 c\alpha_3 c'\\
&= a'\beta_1 a\alpha_1 g\alpha_1 (c\alpha_3 c'\beta_3 a'\beta_1 a\alpha_1 c\alpha_3 c')\beta_3 h\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 g\alpha_1 c\alpha_3 c'\beta_3 a'\beta_1 a\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&= (a'\beta_1 a\alpha_1 c\alpha_3 c'\beta_3 a'\beta_1 a)\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } g \in RS(a'\beta_1 a, c\alpha_3 c'))\\
&= c\alpha_3 c'\beta_3 (a'\beta_1 a\alpha_1 h)\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= a'\beta_1 a\alpha_1 h\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse and hence right orthodox)}\\
&\xi\ b'\beta_2 b\alpha_2 h\alpha_2 c\alpha_3 c'\\
&= b'\beta_2 b\alpha_2 h\alpha_2 b'\beta_2 b\alpha_2 c\alpha_3 c' \text{(Since } h \in RS(b'\beta_2 b, c\alpha_3 c'))\\
&\xi\ h\alpha_2 b'\beta_2 b\alpha_2 c\alpha_3 c' \text{(Since } S \text{ is right inverse)}\\
&= h\alpha_2 c\alpha_3 c'.
\end{aligned}
$$

Hence

$$(g\alpha_1 c\alpha_3 c', h\alpha_2 c\alpha_3 c') \in \xi \tag{2.9}$$

Now since $g \in RS(a'\beta_1 a, c\alpha_3 c')$ and $h \in RS(b'\beta_2 b, c\alpha_3 c'), c'\beta_3 h\alpha_2 c \in E_{\alpha_3}$ and $c'\beta_3 g\alpha_1 c \in E_{\alpha_3}$. Again by normality of ξ and by (2.9) we have $(c'\beta_3(g\alpha_1 c\alpha_3 c')\beta_3 c,$ $c'\beta_3(h\alpha_2 c\alpha_3 c')\beta_3 c) \in \xi$. i.e, $(c'\beta_3 g\alpha_1 c, c'\beta_3 h\alpha_3 c) \in \xi$. Thus $(c'\beta_3 g\alpha_1 a')\beta_1(a\alpha_1 c)\ \xi$ $(c'\beta_3 h\alpha_2 b')\beta_2(b\alpha_2 c)$. Finally $(a\alpha_1 c)\alpha_3(c'\beta_3 h\alpha_2 b') = a\alpha_1(c\alpha_3 c'\beta_3 h)\alpha_2 b' \in K$ since $a\alpha_1 b' \in K$. Hence $(a\alpha_1 c, b\alpha_2 c) \in \rho$ by definition of ρ.

Let us now show that $tr\rho = \xi$. Let us suppose that e be an α-idempotent and f be a β-idempotent are such that $(e, f) \in \rho$. Then by definition of ρ we have $(e, f) \in \xi$, since $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$. Hence $tr\rho \subseteq \xi$. Conversely let $e \in E_\alpha$ and $f \in E_\beta$ and $(e, f) \in \xi$. We now show that $(e, f) \in \rho$. Since S is right inverse Γ-semigroup, $e\alpha f \in E_\beta \subseteq K$. Again considering $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$ we can say that $(e, f) \in \rho$. Hence $\xi = tr\rho$.

Let us now show that $K = ker\rho$. For that let $a \in Ker\rho$. Then there exists an α-idempotent $e \in S$ such that $(a, e) \in \rho$ and hence $(a'\delta a, e) \in \xi$ for all $a' \in V_\gamma^\delta(a)$ and $a\gamma e \in K$. Then by Theorem 2.2 and Remark 2.1 $e\alpha a' \in K$ and so by definition of (ξ, K) we have $a' \in K$ and hence from regularity of $K, a \in K$.

Conversely suppose that $a \in K$. Let $a' \in V_\alpha^\beta(a)$ then $(a'\beta a, a'\beta a\alpha a'\beta a) \in \xi$ and $a\alpha a'\beta a \in K$ i.e, $(a, a'\beta a) \in \rho$ by definition of ρ. Thus $a \in Ker\rho$. Hence $K = Ker\rho$.

We now prove the converse part of the Theorem. Let us suppose that ρ is a ip - congruence on S. We show that $(tr\rho, Ker\rho)$ is an ip - congruence pair and $\rho = \rho_{(tr\rho, Ker\rho)}$. Let $a, b \in ker\rho$ and let $V_\alpha^\beta(a) \neq \phi$. Hence $a\rho = e\rho$ and $b\rho = f\rho$ for some γ-idempotent e and δ-idempotent f. Now $a\rho e$ implies $a\alpha b\ \rho\ e\gamma b\ \rho\ e\gamma f$. Since S is a right inverse Γ-semigroup $e\gamma f \in E_\delta$ and hence $a\alpha b \in Ker\rho$. Thus $Ker\rho$ is a partial Γ-subsemigroup of S. Clearly $Ker\rho$ contains $E(S)$. Let $a \in Ker\rho$ and $a' \in V_\alpha^\beta(a)$. We show that $a' \in Ker\rho$. Since $a \in Ker\rho$, $a\rho = e\rho$ for some $e \in E_\gamma$.

Now $a' = a'\beta a\alpha a' \ \rho \ a'\beta e\gamma a' = a'\beta e\gamma e\gamma a' \ \rho \ a'\beta a\alpha e\gamma a' \ \rho \ a'\beta a\alpha a\alpha a'$. Since $(a'\beta a)\alpha$ $(a\alpha a') \in E_\beta, a' \in Ker\rho$. Thus $Ker\rho$ is regular. Next let $a \in S$ and $a' \in V^\beta_\alpha(a)$ and $k \in Ker\rho$ where $V^\delta_\gamma(k) \neq \phi$. Since $k \in Ker\rho, k\rho = e\rho$ for some μ-idempotent e. Now since S is a right inverse Γ-semigroup, $(a'\beta e\mu a)\alpha(a'\beta e\mu a) = a'\beta(e\mu a\alpha a'\beta e)\mu a = a'\beta(a\alpha a'\beta e)\mu a = a'\beta e\mu a$ i.e,$a'\beta e\mu a \in E_\alpha$.
Now $a'\beta k\gamma a \ \rho \ a'\beta e\mu a$ and hence $a'\beta k\gamma a \in Ker\rho$ i.e, $Ker\rho$ is self conjugate. Thus $Ker\rho$ is a normal partial Γ-subsemigroup of S. We now prove that $(tr\rho, Ker\rho)$ is an ip - congruence pair for S. Since ρ is a ip - congruence and for $a' \in V^\beta_\alpha(a)$ and $e \in E_\gamma, a'\beta e\gamma a \in E_\alpha, tr\rho$ is a normal ip - congruence. Now let $a \in S$ and $a' \in V^\beta_\alpha(a)$ and $e \in E_\gamma$ be such that $e\gamma a \in ker\rho$ and $(e, a\alpha a') \in tr\rho$. Now $a \ \rho$ $(a\alpha a')\beta a \ \rho \ e\gamma a \ \rho \ f$ for some $f \in E(S)$ since $e\gamma a \in Ker\rho$. Hence condition (i) of Definition 2.10 is satisfied. Next let $a \in Ker\rho$ and $e \in E_\gamma$ and let $a' \in V^\beta_\alpha(a)$. Now since $a \in Ker\rho, a\rho = f\rho$ for some δ-idempotent f and $a'\rho = g\rho$ for some μ-idempotent g.
Now $a\alpha e\gamma a' = a\alpha e\gamma a'\beta a\alpha a' \ \rho \ f\delta e\gamma g\mu f\delta g \ \rho \ f\delta e\gamma f\delta g \ \rho \ e\gamma f\delta g \ \rho \ e\gamma a\alpha a'$. Now since $a\alpha e\gamma a', e\gamma a\alpha a' \in E_\beta$,we have $(a\alpha e\gamma a', e\gamma a\alpha a') \in tr\rho$. Thus condition (ii) of definition 2.10 is also satisfied. Finally we show that $\rho = \rho_{(tr\rho, Ker\rho)}$ i.e, we prove $(a, b) \in \rho$ if and only if for all $a' \in V^{\beta_1}_{\alpha_1}(a)$ and for all $b' \in V^{\beta_2}_{\alpha_2}(b), a\alpha_1 b' \in Ker\rho$ and $(a'\beta_1 a, b'\beta_2 b) \in tr\rho$. Suppose $(a, b) \in \rho$ and $a' \in V^{\beta_1}_{\alpha_1}(a), b' \in V^{\beta_2}_{\alpha_2}(b)$. Now $a\alpha_1 b' \ \rho \ b\alpha_2 b'$ since ρ is an ip - congruence. Again since $b\alpha_2 b'$ is a β_2-idempotent we can say that $a\alpha_1 b' \in Ker\rho$. Now $a'\beta_1 a \ \rho \ a'\beta_1 b = a'\beta_1 b\alpha_2 b'\beta_2 b \ \rho \ a'\beta_1 a\alpha_1 b'\beta_2 b \ \rho$ $(a'\beta_1 a)\alpha_1(b'\beta_2 a) = (a'\beta_1 a)\alpha_1 b'\beta_2 a\alpha_1 a'\beta_1 a \ \rho \ (a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) = (b'\beta_2 b)\alpha_2$ $(a'\beta_1 a) = b'\beta_2 b\alpha_2(a'\beta_1 a) \ \rho \ b'\beta_2(a\alpha_1 a'\beta_1 a) = b'\beta_2 a \ \rho \ b'\beta_2 b$. Now since $a'\beta_1 a$ and $b'\beta_2 b$ are α_1-idempotent and α_2-idempotent respectively, we have $(a'\beta_1 a, b'\beta_2 b) \in tr\rho$. Hence $\rho \subseteq \rho_{(tr\rho, Ker\rho)}$.
Conversely let $(a, b) \in S$ such that for all $a' \in V^{\beta_1}_{\alpha_1}(a), b' \in V^{\beta_2}_{\alpha_2}(b), (a'\beta_1 a, b'\beta_2 b) \in tr\rho$ and $a\alpha_1 b' \in Ker\rho$.
Now

$$\begin{aligned}(a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1(a\alpha_1 b') &= a\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2 b' \\ &= a\alpha_1(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2 b' \\ &= a\alpha_1 b'\end{aligned}$$

and

$$\begin{aligned}(b\alpha_2 a')\beta_1(a\alpha_1 b')\beta_2(b\alpha_2 a') &= b\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1 a' \\ &= b\alpha_2(b'\beta_2 b)\alpha_2(a'\beta_1 a)\alpha_1 a' \\ &= b\alpha_2 a'\end{aligned}$$

Hence $a\alpha_1 b' \in V^{\beta_2}_{\beta_1}(b\alpha_2 a')$. Again since $a\alpha_1 b' \in Ker\rho, b\alpha_2 a' \in Ker\rho$ and let $(a\alpha_1 b') \ \rho \ e$ and $(b\alpha_2 a') \ \rho \ f$ for γ-idempotent e and δ-idempotent f. Now $a = a\alpha_1(a'\beta_1 a)\alpha_1(a'\beta_1 a) \ \rho \ a\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) \ \rho \ (a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1 a \ \rho \ e\gamma f\delta a = f\delta e\gamma f$ $\delta a \ \rho \ (b\alpha_2 a')\beta_1(a\alpha_1 b')\beta_2(b\alpha_2 a')\beta_1 a = b\alpha_2(a'\beta_1 a)\alpha_1(b'\beta_2 b)\alpha_2(a'\beta_1 a) = b\alpha_2(b'\beta_2 b)\alpha_2$ $(a'\beta_1 a) \ \rho \ b\alpha_2(b'\beta_2 b)\alpha_2(b'\beta_2 b = b$. i.e, $(a, b) \in \rho$. Hence the proof.

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