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ON $\mathcal{I}_2$-ASYMPTOTICALLY λ^2-STATISTICAL EQUIVALENT DOUBLE SEQUENCES

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Abstract. In this paper, we introduce the concept of $\mathcal{I}_2-$asymptotically λ^2-statistically equivalence of multiple L for the double sequences (x_{kl}) and (y_{kl}). Also we give some inclusion relations.

1. Introduction

Pobyvanets [14] introduced the concept of asymptotically regular matrices which preserve the asymptotic equivalence of two nonnegative numbers sequences. In 1993, Marouf [9] presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson extended these concepts by prensenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later these definitions extended to λ-sequences by Savas and Başarır in [18]. Esi and Acıkgöz [1] extended the definitions prensented in [18] to double λ^2-sequence.

2. Preliminaries and Background

In this section, we recall some definitions and notations, which form the base for the present study.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of natural numbers $\mathbb{N}$. A subsets of natural numbers $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in E\}\right| \text{ exists.}$$

Definition 2.1. [4] A real or complex number sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0.$$

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In this case, we write $S-\lim x=L$ or $x_k \to L\,(S)$, and S denotes the statistically convergent sequences.

Definition 2.2. [7] A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$,
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

Definition 2.3. [7] A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$,
(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

Lemma 2.1. [7] *If $\mathcal{I}$ is proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets*

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.4. [7] A sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $\mathcal{I}-$convergent to a real number L if

$$\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I},$$

for every $\varepsilon > 0$. In this case we write $\mathcal{I}$-$\lim x = L$.

Definition 2.5. [10] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $S_\lambda-$convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |k \in I_n : |x_k - L| \geq \varepsilon| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 2.6. [15] A double sequence $x = (x_{kl})$ has a Pringsheim limit L (denoted by $P - \lim x = L$) provided that for given $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$, whenever $k, l > n$. We describe such an $x = (x_{kl})$ more briefly as "P-convergent".

The double sequence $(x_{k,l})$ is bounded if there exists a positive integer M such that $|x_{kl}| < M$ for all k and l. We denote all bounded double sequence by l_∞^2.

Definition 2.7. [11] A real double sequence $x = (x_{kl})$ is to be statistically convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{(k,l) : k \leq m \text{ and } l \leq n : |x_{k,l} - L| \geq \varepsilon\}| = 0,$$

denoted by $S^L - \lim x = L$.

Now we give a brief history for asymptotical equivalence for single and double sequences.

Definition 2.8. [15] Two non-negative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be $P-$asymptotically double equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = L,$$

denoted by $(x_{kl}) \sim^{P} (y_{kl})$ and simply asymptotically double equivalent if $L = 1$.

Definition 2.9. [1] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically double statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left| \{ \ k \leq m, l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \ \} \right| = 0,$$

denoted by $(x_{kl}) \sim^{S^L} (y_{kl})$ and simply asymptotically double statistical equivalent if $L = 1$.

Definition 2.10. [6] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically $\mathcal{I}-$equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I},$$

denoted by $(x_{kl}) \sim^{\mathcal{I}^L} (y_{kl})$ and simply asymptotically $\mathcal{I}-$equivalent if $L = 1$.

Definition 2.11. [6] Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be asymptotically $\mathcal{I}$-statistically equivalent of multiple L provided that for every $\varepsilon > 0$, and for every $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \leq m, \ l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I},$$

denoted by $(x_{kl}) \overset{S^L(\mathcal{I})}{\sim} (y_{kl})$ and simply asymptotically $\mathcal{I}-$statistical equivalent if $L = 1$.

Definition 2.12. [5] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. Two non-negative sequences (x_k) and (y_k) are $S_\lambda-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

Definition 2.13. [5] Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. Two non-negative sequences (x_k) and (y_k) are strong $\lambda-$asymptotically equivalent of multiple L provided that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{x_k}{y_k} - L \right| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

The double sequence (λ_{mn}) of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{mn} + 1, \lambda_{m,n+1} \leq \lambda_{mn} + 1,$$

$$\lambda_{mn} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1$$

and

$$I_{mn} = \{(k,l) : m - \lambda_{mn} + 1 \leq k \leq m,\ n - \lambda_{mn} + 1 \leq l \leq n\}.$$

Definition 2.14. [1] For double λ^2-sequence; two non-negative double sequences (x_{kl}) and $(y_{k,l})$ are said to be λ^2-asymptotically double statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left| k \in I_n,\ l \in I_m : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right| = 0,$$

$\left(\text{denoted by } (x_{kl}) \overset{S^L_{\lambda^2}}{\sim} (y_{kl}) \right).$

Definition 2.15. [1] For double λ^2-sequence; two non-negative double sequences (x_{kl}) and $(y_{k,l})$ are said to be strong λ^2-asymptotically double equivalent of multiple L provided that

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| = 0,$$

$\left(\text{denoted by } (x_{kl}) \overset{N^L_{\lambda^2}}{\sim} (y_{kl}) \right).$

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N}\times\mathbb{N}$. A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times i$ belongs to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

3. Main results

In this section we define $\mathcal{I}_2-$asymptotically λ^2-statistically equivalent, strongly $\lambda^2_{\mathcal{I}_2}-$asymptotically equivalent, strongly Cesaro asymptotically $\mathcal{I}_2-$equivalent of double sequences and obtain some analogous results from these new definitons point of views.

Definition 3.1. For double $\lambda^2 = (\lambda_{mn})$-sequence; two nonnegative sequences (x_{kl}) and (y_{kl}) are said to be $\mathcal{I}_2-$asymptotically λ^2-statistically equivalent of multiple L if for every $\varepsilon, \delta > 0$,

$$\left\{ (m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2,$$

denoted by $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{k,l})$.

Definition 3.2. For double $\lambda^2 = (\lambda_{mn})$-sequence; two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be strongly $\lambda^2_{\mathcal{I}_2}-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{ (m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2,$$

$\left(\text{denoted by } x_{kl} \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} y_{kl}\right)$.

Definition 3.3. Two non-negative double sequences (x_{kl}) and (y_{kl}) are said to be strongly Cesaro asymptotically $\mathcal{I}_2-$equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2,$$

$\left(\text{denoted by } x_{kl} \overset{\mathcal{I}_2[C,1]^L}{\sim} y_{kl}\right)$.

Theorem 3.1. *Let $\lambda^2 = (\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If $(x_{kl}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ then $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$.*

Proof. Assume that $(x_{k,l}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ and $\varepsilon > 0$. Then,

$$\begin{aligned}\sum_{(k,l)\in I_{mn}}\left|\tfrac{x_{kl}}{y_{kl}} - L\right| &\geq \sum_{\substack{(k,l)\in I_{mn}\\ \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon}}\left|\tfrac{x_{kl}}{y_{kl}} - L\right| \\ &\geq \varepsilon.\left|\left\{(k,l)\in I_{m,n} : \left|\tfrac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right|,\end{aligned}$$

and so,

$$\frac{1}{\varepsilon.\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn} : \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right|.$$

Then for any $\delta > 0$,

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn} : \left|\tfrac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\qquad\subseteq \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\tfrac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon.\delta\right\}.\end{aligned}$$

Since right hand belongs to $\mathcal{I}_2$, then left hand also belongs to $\mathcal{I}_2$ and this completes the proof. □

Theorem 3.2. *Let $\lambda^2 = (\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If (x_{kl}) and (y_{kl}) are bounded sequences and $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$ then $(x_{kl}) \overset{V^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$.*

Proof. Let (x_{kl}) and (y_{kl}) are bounded sequences and let $(x_{kl}) \overset{S^L_{\lambda^2}(\mathcal{I}_2)}{\sim} (y_{kl})$. Then there is a M such that

$$\left|\frac{x_{kl}}{y_{kl}} - L\right| \leq M$$

for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. For each $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| &= \frac{1}{\lambda_{mn}} \sum_{\substack{(k,l)\in I_{mn} \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \\ &\quad + \frac{1}{\lambda_{mn}} \sum_{\substack{(k,l)\in I_{mn} \\ \left|\frac{x_{kl}}{y_{kl}} - L\right| < \varepsilon}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \\ &\leq M.\frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

And define the sets

$$D_1 = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\}$$

and

$$D_2 = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{\varepsilon}{2} \right\} \right| \geq \frac{\varepsilon}{2M} \right\}.$$

If $(m,n) \notin D_2$, then $\frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{\varepsilon}{2} \right\} \right| < \frac{\varepsilon}{2M}$. Also we can get

$$\begin{aligned} \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| &\leq \frac{M}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $(m,n) \notin D_1$. Consequently, we have

$$\begin{aligned} &\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \sum_{(k,l)\in I_{mn}} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \\ &\qquad \subseteq \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \frac{\varepsilon}{2} \right\} \right| \geq \frac{\varepsilon}{2M} \right\} \in \mathcal{I}_2. \end{aligned}$$

Therefore $(x_{kl}) \overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim} (y_{kl})$. $\square$

The following example shows that if (x_{kl}), (y_{kl}) are not bounded, then theorem 2 can not be true.

Example 3.1. Let (x_{kl}) and (y_{kl}) be two double sequences as follows:

$$(x_{kl}) = \begin{cases} kl, & \text{if } k_{m-1} < k \leq k_{m-1} + \left[\sqrt{\lambda_m}\right],\ l_{n-1} < l \leq l_{n-1} + \left[\sqrt{\lambda_n}\right],\ m,n = 1,2,3,...; \\ 0, & \text{otherwise.} \end{cases}$$

and $(y_{k,l}) = 1$ for all $k, l \in \mathbb{N}$.

It is clear that $(x_{kl}) \notin l_\infty^2$ and for $\varepsilon > 0$,

$$\frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{mn} : \left| \frac{x_{kl}}{y_{kl}} - 1 \right| \geq \varepsilon \right\} \right| \leq \frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}} \to 0 \text{ as } m,n \to \infty, \tag{1.1}$$

This implies that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{mn}:\left|\tfrac{x_{kl}}{y_{kl}}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\tfrac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}}\geq\delta\right\}.$$

By virtue of last part (1.1), the set on the right side is a finite set, and so it belongs to $\mathcal{I}_2$. Consequently, we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{m,n}}\left|\left\{(k,l)\in I_{mn}:\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

Therefore, $(x_{kl})\overset{S_{\lambda^2}^L(\mathcal{I}_2)}{\sim}(y_{kl})$, On the other hand, we shall show that $(x_{kl})\overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim}(y_{kl})$ is not satisfied. Suppose that $(x_{kl})\overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim}(y_{kl})$. Then for every $\delta>0$, we have

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-1\right|\geq\delta\right\}\in\mathcal{I}_2. \tag{1.2}$$

Now,

$$\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\frac{x_{kl}}{y_{kl}}-1\right|=\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}\left(\frac{\left[\sqrt{\lambda_{mn}}\right].\left(\left[\sqrt{\lambda_{mn}}\right]-1\right)}{2}\right)=\frac{1}{2}.$$

It follows for the particular choice $\delta=\frac{1}{4}$ that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{\lambda_{mn}}\sum_{(k,l)\in I_{mn}}\left|\tfrac{x_{kl}}{y_{kl}}-1\right|\geq\tfrac{1}{4}\right\}$$
$$=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\left(\tfrac{\left[\sqrt{\lambda_{mn}}\right].\left(\left[\sqrt{\lambda_{mn}}\right]-1\right)}{\lambda_{m,n}}\right)\geq\tfrac{1}{4}\right\}$$
$$=\{(r,s)\,,\ (r+1,s+1)\,,\ (r+2,s+2)\,,....\}$$

for some $r,s\in\mathbb{N}$ which belongs to $\mathcal{F}(\mathcal{I}_2)$ as $\mathcal{I}_2$ is admissible. This contradicts (1.2) for the choice $\delta=\frac{1}{4}$. Therefore $(x_{kl})\overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\nsim}(y_{kl})$.

Theorem 3.3. *Let $\lambda^2=(\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. If $(x_{kl})\overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim}(y_{kl})$ is then $(x_{kl})\overset{\mathcal{I}_2[C,1]^L}{\sim}(y_{kl})$.*

Proof. Assume that $(x_{kl}) \overset{V_{\lambda^2}^L(\mathcal{I}_2)}{\sim} (y_{kl})$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}}-L\right| &= \frac{1}{mn}\sum_{k,l=1,1}^{m-\lambda_m,\ n-\lambda_n}\left|\frac{x_{kl}}{y_{kl}}-L\right| + \frac{1}{mn}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\
&\leq \frac{1}{\lambda_{mn}}\sum_{k,l=1,1}^{m-\lambda_m,\ n-\lambda_n}\left|\frac{x_{kl}}{y_{kl}}-L\right| + \frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \\
&\leq \frac{2}{\lambda_{mn}}\sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right|
\end{aligned}$$

and so,

$$\begin{aligned}
&\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}}-L\right| \geq \varepsilon\right\} \\
&\subseteq \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \sum_{(k,l)\in I_{m,n}}\left|\frac{x_{kl}}{y_{kl}}-L\right| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2.
\end{aligned}$$

Hence $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$. □

Theorem 3.4. *If* $\liminf \frac{\lambda_{mn}}{mn} > 0$ *then* $(x_{kl}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$ *implies* $(x_{kl}) \overset{S_{\lambda^2}^L(\mathcal{I}_2)}{\sim} (y_{kl})$.

Proof. Assume that $\liminf \frac{\lambda_{mn}}{mn} > 0$. Then, there exists a $\delta > 0$ such that $\frac{\lambda_{mn}}{mn} \geq \delta$ for sufficiently large m, n. For given $\varepsilon > 0$ we have,

$$\begin{aligned}
&\frac{1}{mn}\left\{0\leq k\leq m; 0\leq l\leq n,\ (m,n)\in\mathbb{N}\times\mathbb{N} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\} \\
&\supseteq \frac{1}{mn}\left\{(k,l)\in I_{m,n},\ (m,n)\in\mathbb{N}\times\mathbb{N} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{mn}\left|\left\{0\leq k\leq m; 0\leq l\leq n : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right| \\
&\geq \frac{1}{mn}\left|\left\{(k,l)\in I_{m,n} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right| \\
&\geq \frac{\lambda_{mn}}{mn}.\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right| \\
&\geq \delta.\frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|,
\end{aligned}$$

then for any $\eta > 0$ we get

$$\begin{aligned}
&\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{mn}}\left|\left\{(k,l)\in I_{m,n} : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|\geq\eta\right\} \\
&\subseteq \left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{0\leq k\leq m; 0\leq l\leq n : \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\right|\geq\eta\delta\right\} \in \mathcal{I}_2,
\end{aligned}$$

and this completes the proof. □

Theorem 3.5. *Let $\lambda^2 = (\lambda_{mn})$ be a double sequence and $\mathcal{I}_2$ is a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$, and (x_{kl}) and (y_{kl}) are two non-negative double sequences. Then*

(i) *If $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$ then $(x_{kl}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$,*

(ii) *Let $(x_{kl}), (y_{kl}) \in l^2_\infty$ and $(x_{kl}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$, then $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$.*

Proof. (i) Let $\varepsilon > 0$ and $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$. Then we can write

$$\begin{aligned}\sum_{k,l=1,1}^{m,n}\left|\tfrac{x_{kl}}{y_{kl}} - L\right| &\geq \sum_{\substack{k,l=1,1\\ \left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon}}^{m,n}\left|\tfrac{x_{kl}}{y_{kl}} - L\right| \\ &\geq \varepsilon.\left|\left\{1\leq k\leq m,\ 1\leq l\leq n : \left|\tfrac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon\right\}\right|\end{aligned}$$

$$\Longrightarrow \frac{1}{\varepsilon.mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \frac{1}{mn}\left|\left\{1\leq k\leq m,\ 1\leq l\leq n : \left|\frac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon\right\}\right|.$$

Thus, for any $\delta > 0$,

$$\frac{1}{mn}\left|\left\{1\leq k\leq m,\ 1\leq l\leq n : \left|\frac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon\right\}\right| \geq \delta$$

implies that

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}} - L\right| \geq \varepsilon.\delta.$$

Therefore, we have

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{mn}\left|\left\{1\leq k\leq m,\ 1\leq l\leq n : \left|\tfrac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon\right\}\right|\geq\delta\right\} \\ &\qquad\subset\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\tfrac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon.\delta\right\}.\end{aligned}$$

Since $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$, so that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon.\delta\right\}\in\mathcal{I}_2$$

which implies that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{1\leq k\leq m,\ 1\leq l\leq n : \left|\frac{x_{kl}}{y_{kl}} - L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

This shows that $(x_{kl}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$.

(ii) Suppose that $(x_{kl}), (y_{kl}) \in l^2_\infty$ and $(x_{k,l}) \overset{S^L(\mathcal{I}_2)}{\sim} (y_{kl})$. Then there is an M such that

$$\left|\frac{x_{kl}}{y_{kl}} - L\right| \leq M$$

for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\begin{aligned} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| &= \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon}}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| + \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left| \frac{x_{kl}}{y_{kl}} - L \right| < \varepsilon}}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \\ &\leq \frac{M}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| + \varepsilon. \end{aligned}$$

If we put

$$A(\varepsilon) = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\}$$

and

$$B(\varepsilon_1) = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ 1 \leq k \leq m,\ 1 \leq l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| \geq \frac{\varepsilon_1}{M} \right\},$$

where $\varepsilon_1 = \delta - \varepsilon > 0$, (and δ and ε are independent), then we have $A(\varepsilon) \subset B(\varepsilon_1)$, and so $A(\varepsilon) \in \mathcal{I}_2$. This shows that $(x_{kl}) \overset{\mathcal{I}_2[C,1]^L}{\sim} (y_{kl})$. $\square$

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