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A NOTE ON FIBERED QUADRANGLES

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Abstract. In this work, the fibered versions of the diagonal triangle and the quadrangular set of a complete quadrangle in fibered projective planes are introduced. And then some related theorems with them are given.

1. Introduction

Fuzzy set theory was introduced by Zadeh [10] and this theory has been applied in many areas. One of them is projective geometry, see for instance [1,2,5,7,8,9]. A first model of fuzzy projective geometries was introduced by Kuijken, Van Maldeghem and Kerre [7,8]. Also, Kuijken and Van Maldeghem contributed to fuzzy theory by introducing fibered geometries, which is a particular kind of fuzzy geometries [6]. They gave the fibered versions of some classical results in projective planes by using minimum operator. Then the role of the triangular norm in the theory of fibered projective planes and fibered harmonic conjugates and a fibered version of Reidemeister's condition were given in [3]. The fibered version of Menelaus and Ceva's 6-figures was studied in [4].

It is well known that triangles and quadrangles have an important role in projective geometry. A complete quadrangle is a system of geometric objects consisting of any four points in a plane, no three of which are on a common line, and of the six lines connecting each pair of points. The free completion of a configuration containing either a quadrangle or a quadrilateral is a projective plane. In contrast, the free completion of a (non-empty) configuration which does not contain either a quadrangle or a quadrilateral is not a projective plane. Notice that the existence of a quadrangle and the associated diagonal triangle forces any projective plane to have at least seven points. If, in fact, the points of the diagonal triangle are collinear, we obtain a projective plane with seven points. This projective plane is known as the Fano plane.

In the present paper, we consider the fibered versions of classical theorems related to complete quadrangles. We start by defining the fiber version of the diagonal

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triangle of a complete quadrangle in fibered projective planes. And then some related theorems are given between them. It is shown that four f-points intersecting two opposite sides of two f-complete quadrangles are f-collinear when four f-lines spanned by two opposite vertices of two f-complete quadrangles are f-concurrent. Finally, the fiber version of a quadrangular set is defined and related theorems with f-quadrangular sets are given in fibered projective planes.

2. Preliminaries

We first recall some basic notions from fuzzy set theory and fibered geometry. We denote by $\wedge$ a triangular norm on the (real) unit interval $[0,1]$, i.e., a symmetric and associative binary operator satisfying $(a \wedge b) \leq (c \wedge d)$ whenever $a \leq c$ and $b \leq d$, and $a \wedge 1 = a$, for all $a, b, c, d \in [0,1]$.

Definition 2.1. (*see* [6]) Let $\mathcal{P} = (P, B, \circ)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\circ$ (called the *incidence relation*) such that the graph $(P \cup B, \circ)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line, every two distinct lines L, M are incident with exactly one point, and every line is incident with at least three points. A set S of *collinear* points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of *concurrent* lines. We now define fibered points and fibered lines, briefly called f-points and f-lines.

Definition 2.2. *(see* [6]*) Suppose $a \in P$ and $\alpha \in]0,1]$. Then an f-*point *(a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:*

$$(a, \alpha) : P \to [0,1] : \begin{cases} a \to \alpha, \\ x \to 0 \quad if \quad x \in P \backslash \{a\}. \end{cases}$$

Dually, one defines in the same way the f-line (L, β) for $L \in B$ and $\beta \in]0,1]$. The real number α above is called the membership degree *of the f-point (a, α), while the point a is called the* base point *of it. Similarly for f-lines.*

Definition 2.3. *(see* [6]*) Two f-lines (L, α) and (M, β), with $\alpha \wedge \beta > 0$,* intersect *in the unique f-point $(L \cap M, \alpha \wedge \beta)$. Dually, the f-points (a, λ) and (b, μ), with $\lambda \wedge \mu > 0$,* span *the unique f- line $(\langle a, b \rangle, \lambda \wedge \mu)$.*

Definition 2.4. *(see* [6]*) A (nontrivial) fibered projective plane $\mathcal{FP}$ consists of a set FP of f-points of P and a set FB of f-lines of P such that every point and every line of P is the base point and base line of at least one f-point and f-line, respectively (with at least one membership degree different from 1), and such that $\mathcal{FP} = (FP, FB)$ is closed under taking intersections of f-lines and spans of f-points. Finally, a set of f-points are called collinear if each pair of them span the same f-line. Dually, a set of f-lines are called concurrent if each pair of them intersect in the same f-point.*

3. Fibered Version of quadrangles

Definition 3.1. *(see [3]) Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. Choose four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$, and (a_4, α_4) in $\mathcal{FP}$ no three base points of which are collinear. These f-points are called f-vertices. The configuration that consists of these four f-points, the six f-lines $(A_{\{i,j\}}, \beta_{\{i,j\}}) =: (\langle a_i, a_j \rangle, \alpha_i \wedge \alpha_j)$, for $i \neq j$, $i, j \in \{1, 2, 3, 4\}$ (which we call f-sides), and three f-points $(A_{\{i,j\}} \wedge A_{\{k,l\}}, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4)$, with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (the f-diagonal points), is called an* f-complete quadrangle.

Definition 3.2. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. If the vertices of an f-triangle are the f-diagonal points of an f-complete quadrangle, it this f-triangle is called an* f-diagonal triangle.

To simplify notation, we will sometimes omit the binary operator $\wedge$ and write $\alpha\beta$ for $\alpha \wedge \beta$. In this notation, we will also abbreviate $\alpha \wedge \alpha$ to α^2.

Theorem 3.1. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ that is Desarguesian. Let four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and (a_4, α_4) form an f-complete quadrangle and let the three f- points (b_i, β_i), $i = \{1, 2, 3\}$, with $b_1 = a_2a_3 \cap a_1a_4$, $b_2 = a_2a_4 \cap a_1a_3$ and $b_3 = a_1a_2 \cap a_3a_4$, be the associated f-diagonal triangle in $\mathcal{FP}$. The three intersection f-points (c_k, γ_k) of the f-lines $(a_ia_j, \alpha_i \wedge \alpha_j)$ and $(b_ib_j, \beta_i \wedge \beta_j)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear if $\alpha_1^2\alpha_2\alpha_3 = \alpha_1\alpha_2^2\alpha_3 = \alpha_1\alpha_2\alpha_3^2$.*

Proof. Note that by Definition 3.1 $\beta_i = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 =: \beta$, $i = 1, 2, 3$. The lines $a_ib_i, i = 1, 2, 3$ are incident with the point a_4 in $\mathcal{P}$. If Desargues' theorem is applied to the triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, we see that the points $c_k = a_ia_j \wedge b_ib_j$, with $\{i, j, k\} = \{1, 2, 3\}$, are collinear in $\mathcal{P}$. Also, using $\alpha_1^2\alpha_2\alpha_3 = \alpha_1\alpha_2^2\alpha_3 = \alpha_1\alpha_2\alpha_3^2$, the equality

$$\alpha_1^2\alpha_2\alpha_3\beta^4 = \alpha_1\alpha_2^2\alpha_3\beta^4 = \alpha_1\alpha_2\alpha_3^2\beta^4$$

is obtained. So, the three f-points $(c_k, \gamma_k) = (a_ia_j \wedge b_ib_j, \alpha_i\alpha_j\beta^2)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear. □

Corollary 3.1. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base plane $\mathcal{P}$ that is Desarguesian and let $\wedge$ be the minimum triangular norm. Let four f-points $(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)$ and (a_4, α_4) form an f-complete quadrangle and let (b_i, β_i), $i = \{1, 2, 3\}$ be the corresponding f-diagonal triangle, with $b_1 = a_2a_3 \cap a_1a_4$, $b_2 = a_2a_4 \cap a_1a_3$ and $b_3 = a_1a_2 \cap a_3a_4$. Then the three f-points $(c_k, \gamma_k) = (a_ia_j \cap b_ib_j, \alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j)$, with $\{i, j, k\} = \{1, 2, 3\}$, are f-collinear.*

Theorem 3.2. *Suppose we have a fibered projective plane $\mathcal{FP}$ with Desarguesian base plane $\mathcal{P}$. Choose two different f-quadrangles (a_i, α_i) and (b_i, β_i), $i = 1, 2, 3, 4$ in $\mathcal{FP}$. Let the f-lines $(\langle a_i, b_i \rangle, \alpha_i\beta_i)$, for $i \in \{1, 2, 3, 4\}$, be concurrent with intersection points (p, γ) in $\mathcal{FP}$, $a_i \neq b_i \neq p_i \neq a_i$. Let the f-lines $(\langle a_1, a_2 \rangle, \alpha_1\alpha_2)$, $(\langle a_3, a_4 \rangle, \alpha_3\alpha_4)$, $(\langle b_1, b_2 \rangle, \beta_1\beta_2)$ and $(\langle b_3, b_4 \rangle, \beta_3\beta_4)$ meet in the f-point (c_1, γ_1), the f-lines $(\langle a_1, a_4 \rangle, \alpha_1\alpha_4)$, $(\langle a_2, a_3 \rangle, \alpha_2\alpha_3)$, $(\langle b_1, b_4 \rangle, \beta_1\beta_4)$ and $(\langle b_2, b_3 \rangle, \beta_2\beta_3)$ meet in the f-point (c_2, γ_2), the f-lines $(\langle a_2, a_4 \rangle, \alpha_2\alpha_4)$ and $(\langle b_2, b_4 \rangle, \beta_2\beta_4)$ meet in the f-point (c_3, γ_3) and let the f-point (c_4, γ_4) be the intersection point of the f-lines $(\langle a_1, a_3 \rangle, \alpha_1\alpha_3)$ and $(\langle b_1, b_3 \rangle, \beta_1\beta_3)$. Then (c_i, γ_i), $i \in \{1, 2, 3, 4\}$ are collinear in $\mathcal{FP}$ (in particular, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$).*

Proof. One calculates $\gamma = \alpha_i\alpha_j\beta_i\beta_j$, for $\{i,j\} \subseteq \{1,2,3,4\}$,with $i \neq j$. Since the f-lines $(\langle a_1, a_2\rangle, \alpha_1\alpha_2), (\langle a_3, a_4\rangle, \alpha_3\alpha_4), (\langle b_1, b_2\rangle, \beta_1\beta_2)$ and $(\langle b_3, b_4\rangle, \beta_3\beta_4)$ are f-concurrent in (c_1, γ_1), $\gamma_1 = \alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = \alpha_1\alpha_2\beta_1\beta_2 = \alpha_1\alpha_2\beta_3\beta_4 = \alpha_3\alpha_4\beta_1\beta_2 = \alpha_3\alpha_4\beta_3\beta_4 = \gamma$. Similarly, it is seen that $\gamma_2 = \alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = \alpha_1\alpha_4\beta_1\beta_4 = \alpha_1\alpha_4\beta_2\beta_4 = \alpha_2\alpha_3\beta_1\beta_4 = \alpha_2\alpha_3\beta_2\beta_3 = \gamma$, $\gamma_3 = \alpha_2\alpha_4\beta_2\beta_4 = \gamma$, $\gamma_4 = \alpha_1\alpha_3\beta_1\beta_3 = \gamma$. Since $\mathcal{P}$ is a Desarguesian plane, the c_i, $i \in \{1,2,3,4\}$, are collinear and the memberships degrees of them are equal to γ. Hence the f-points (c_i, γ_i), $i \in \{1,2,3,4\}$ are collinear in $\mathcal{FP}$. □

Although the assumptions of the previous Theorem imply a lot of equalities between expressions in the membership degrees of the points a_i and b_j, $i,j \in \{1,2,3,4\}$, they do not imply that all membership degrees should be equal. For instance, if the minimum operator is used, then $\alpha_1 = \alpha_2 = \beta_1 = \beta_3 \leq \alpha_i, \beta_j$, for $i = 1,2$ and $j = 2,4$, satisfies the assumptions.

Definition 3.3. *Suppose we have a fibered projective plane $\mathcal{FP}$ with base projective plane $\mathcal{P}$. Let (a_i, α_i), $i = 1,2,3,4$, be the vertices any f-quadrangle in $\mathcal{FP}$ and let (L, α) be any f-line such that the base line L is not incident with any of the points a_i, $i = 1,2,3,4$. Let (p_1, β_1), (p_2, β_2), (p_3, β_3) be the f-intersection point of the f-line (L, α) with the f-line $(a_1a_2, \alpha_1\alpha_2)$, $(a_1a_3, \alpha_1\alpha_3)$, $(a_1a_4, \alpha_1, \alpha_4)$, respectively, and let (q_1, γ_1), (q_2, γ_2), (q_3, γ_3) be the f-intersection point of the f-line (L, α) with the f-line $(a_3a_4, \alpha_3\alpha_4)$, $(a_2a_4, \alpha_2\alpha_4)$, $(a_2a_3, \alpha_2, \alpha_3)$, respectively. Then these six (not necessarily distinct) points are called an* f-quadrangular set.

The f-quadrangular set may be consist of five or four f-points if the f-line (L, α) happens to pass through one or two f-diagonal points.

Although the six base points p_1, p_2, p_3, q_1, q_2 and q_3 of the f-quadrangular set are collinear in the base plane $\mathcal{P}$, the six f-points (p_1, β_1), (p_2, β_2), (p_3, β_3), (q_1, γ_1), (q_2, γ_2) and (q_3, γ_3) are not necessarily f-collinear in $\mathcal{FP}$. But we do have the following property.

Theorem 3.3. *Suppose we have a fibered projective plane $\mathcal{FP}$ with Desarguesian base plane $\mathcal{P}$. Let, with the notation of Definition 3.3, $\{(p_i, \beta_i), (q_i, \gamma_i)\}$, $i = 1,2,3$, be the f- quadrangular set determined by the f-quadrangle (a_i, α_i), $i = 1,2,3,4$, and the f-line (L, α) in $\mathcal{FP}$. Then the three pairs of f-points $\{(p_i, \beta_i), (q_i, \gamma_i)\}$, $i = 1,2,3$, span the same f-line, namely $(L, \alpha^2\alpha_1\alpha_2\alpha_3\alpha_4)$.*

Proof. Easy calculation. □

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