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WEAK SOLUTIONS VIA LAGRANGE MULTIPLIERS FOR CONTACT MODELS WITH NORMAL COMPLIANCE

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Abstract. We consider a 3D elastostatic frictional contact problem with normal compliance, which consists of a systems of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a frictional contact boundary condition. The frictional contact is modeled by means of a normal compliance condition and a version of Coulomb's law of dry friction. After we state the problem and the hypotheses, we deliver a variational formulation as a mixed variational problem with solution-dependent Lagrange multipliers set. Next, we prove the existence and the boundedness of the weak solutions.

1. Introduction

The present work focuses on a 3D elastostatic frictional contact problem with normal compliance. A normal compliance condition was firstly proposed in [11].

Then, the contact with normal compliance was involved in many models, see e.g. the papers [2, 7, 8, 9, 18].

The model we discuss herein consists of a system of partial differential equations associated with a displacement boundary condition, a traction boundary condition and a frictional contact boundary condition. The frictional contact is modeled by means of a normal compliance condition and a version of Coulomb's law of dry friction. This model was already analyzed in the frame of quasivariational inequalities,

$$a(u, v-u) + j(u, v) - j(u, u) \geq (f, v-u)_X;$$

for details see [19] and the references therein. The novelty in the present paper consists in the variational approach we adopt; herein, we propose a mixed variational formulation in a form of a generalized saddle point problem with solution

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dependent Lagrange multipliers set $\Lambda = \Lambda(u)$,

$$\begin{array}{lcll} a(u,v) + b(v,\lambda) & = & (f,\, v)_X & \text{for all } \ v \in X, \\ b(u, \mu - \lambda) & \leq & 0 & \text{for all } \ \mu \in \Lambda(u); \end{array}$$

Let us refer to [3, 4] for basic elements on the saddle point theory. For recent papers related to mixed variational formulations in contact mechanics see e.g. [6, 13, 14, 15].

The mixed variational formulations are related to modern numerical techniques in order to approximate the weak solutions of contact models and this motivates the research on this direction. Referring to numerical techniques for approximating weak solutions of contact problems via saddle point technique, we send the reader to, e.g., [5, 20, 21].

The main goal of the present paper is to prove the existence and the boundedness of the weak solutions of the considered model, via Lagrange multipliers technique. The results we obtain rely on the abstract results in [12] which, for the convenience of the reader, will be recalled below, in Section 2.

The problem we analyze in the present paper can be viewed as a new application to the abstract results in [12]. A first application was delivered in the antiplane framework, see [12]. A second application was presented in the conference paper [17], for a 3D bilateral contact model with slip-dependent friction (see also [16] for an extended and improved version of the conference paper [17]).

The structure of the paper is as follows. In Section 2 we present abstract auxiliary results. In Section 3 we state the problem and we fix the hypotheses. Then, in Section 4 we prove the existence and the boundedness of the weak solutions of the frictional contact model with normal compliance.

2. Abstract auxiliary results

Let us consider the following abstract mixed variational problem.

Problem 1. *Given $f \in X$, $f \neq 0_X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda(u) \subset Y$ and*

$$a(u,v) + b(v,\lambda) = (f,\, v)_X \qquad \text{for all } \ v \in X, \tag{2.1}$$

$$b(u, \mu - \lambda) \leq 0 \qquad \text{for all } \ \mu \in \Lambda(u). \tag{2.2}$$

We made the following assumptions.

Assumption 1. *$(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two Hilbert spaces.*

Assumption 2. *$a(\cdot,\,\cdot) : X \times X \to \mathbb{R}$ is a symmetric bilinear form such that*
(i_1) there exists $M_a > 0 : \; |a(u,v)| \leq M_a \|u\|_X \|v\|_X$ for all $u, v \in X$,
(i_2) there exists $m_a > 0 : \; a(v,v) \geq m_a \, \|v\|_X^2$ for all $v \in X$.

Assumption 3. *$b(\cdot,\,\cdot) : X \times Y \to \mathbb{R}$ is a bilinear form such that*
(j_1) there exists $M_b > 0 : \; |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,
(j_2) there exists $\alpha > 0 :$ $\displaystyle \inf_{\mu \in Y, \mu \neq 0_Y} \; \sup_{v \in X, v \neq 0_X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 4. *For each $\varphi \in X$, $\Lambda(\varphi)$ is a closed convex subset of Y such that $0_Y \in \Lambda(\varphi)$.*

Assumption 5. *Let $(\eta_n)_n \subset X$ and $(u_n)_n \subset X$ be two weakly convergent sequences, $\eta_n \rightharpoonup \eta$ in X and $u_n \rightharpoonup u$ in X, as $n \to \infty$.*

(k_1) *For each $\mu \in \Lambda(\eta)$, there exists a sequence $(\mu_n)_n \subset Y$ such that $\mu_n \in \Lambda(\eta_n)$ and $\liminf_{n\to\infty} b(u_n, \mu_n - \mu) \geq 0$.*

(k_2) *For each subsequence $(\Lambda(\eta_{n'}))_{n'}$ of the sequence $(\Lambda(\eta_n))_n$, if $(\mu_{n'})_{n'} \subset Y$ such that $\mu_{n'} \in \Lambda(\eta_{n'})$ and $\mu_{n'} \rightharpoonup \mu$ in Y as $n' \to \infty$, then $\mu \in \Lambda(\eta)$.*

Theorem 2.1. *If Assumptions 1-5 hold true, then Problem 1 has a solution. In addition, if $(u, \lambda) \in X \times \Lambda(u)$ is a solution of Problem 1, then*

$$(u, \lambda) \in K_1 \times \big(\Lambda(u) \cap K_2\big),$$

where

$$\begin{aligned} K_1 &= \{v \in X \mid \|v\|_X \leq \frac{1}{m_a}\|f\|_X\}; \\ K_2 &= \{\mu \in Y \mid \|\mu\|_Y \leq \frac{m_a + M_a}{\alpha\, m_a}\|f\|_X\}, \end{aligned}$$

m_a, α and M_a being the constants in Assumptions 2-3.

For the proof of this theorem we refer to [12].

3. The model and hypotheses

3.1. **The statement of the problem.** We consider the following 3D frictional contact model with normal compliance.

Problem 2. *Find* $: \overline{\Omega} \to \mathbb{R}^3$ *and* $: \overline{\Omega} \to \mathbb{S}^3$ *such that*

(3.1) $\quad \mathrm{Div}() +_0 () = \mathbf{0}$ *in* Ω,

(3.2) $\quad () = \mathcal{E}(())$ *in* Ω,

(3.3) $\quad () =$ *on* Γ_1,

(3.4) $\quad () =_2 ()$ *on* Γ_2,

(3.5) $\quad -\sigma_\nu() = p_\nu(u_\nu() - g_a)$ *on* Γ_3,

(3.6) $\quad \|_\tau()\| \leq p_\tau(, u_\nu() - g_a),$

$$_\tau() = -p_\tau(, u_\nu() - g_a)\, \frac{_\tau()}{\|_\tau()\|} \text{ if } _\tau() \neq \text{ on } \Gamma_3.$$

Herein Ω is a bounded domain in $\mathbb{R}^3$, Γ_1, Γ_2, Γ_3 is a partition of the boundary $\partial\Omega := \Gamma$, $\overline{\Omega} = \Omega \cup \Gamma$, $_0 : \Omega \to \mathbb{R}$ denotes the density of the volume forces, $_2 : \Gamma_2 \to \mathbb{R}$ represents the density of the tractions, $= ()$ denotes the infinitesimal strain tensor ($\varepsilon_{i\,j} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ for all $i, j \in \{1, 2, 3\}$) and $\mathcal{E}$ denotes the elastic operator. Here and everywhere below $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$ or $\mathbb{S}^3$. Finally, $u_\nu = \cdot$, $_\tau = -u_\nu$, $\sigma_\nu = ()\cdot$, $_\tau = -\sigma_\nu$, where " $\cdot$ " denotes the inner product of two vectors and is the unit outward normal vector.

Problem 2 has the following structure: (3.1) represents the equilibrium equation, (3.2) represents a constitutive law for linearly elastic materials, (3.3) represents the displacements boundary condition, (3.4) represents the traction boundary condition and (3.5)-(3.6) models the frictional contact with normal compliance, the friction law in (3.6) being a version of the Coulomb law of dry friction, where p_τ is a given nonnegative function. In the normal compliance contact condition (3.5) p_ν is a nonnegative prescribed function which vanishes for negative argument and $g_a > 0$

denotes the gap. When $u_\nu < g_a$ there is no contact and the normal pressure vanishes. When there is contact then $u_\nu - g_a$ is positive and represents a measure of the interpenetration of the asperities. Then, condition (3.5) shows that the foundation exerts a pressure on the body which depends on the penetration.

3.2. **Assumptions.** In order to weakly solve Problem 2 we make the following assumptions.

Assumption 6. $\mathcal{E} = (\mathcal{E}_{ijls}) : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3$,

- $\mathcal{E}_{ijls} = \mathcal{E}_{ijsl} = \mathcal{E}_{lsij} \in L^\infty(\Omega)$,
- *There exists* $m_{\mathcal{E}} > 0$ *such that* $\mathcal{E}_{ijls}\varepsilon_{ij}\varepsilon_{ls} \geq m_{\mathcal{E}} \,\|\|^2, \in \mathbb{S}^3,$ *a.e. in* Ω.

Assumption 7. ${}_0 \in L^2(\Omega)^3, \quad {}_2 \in L^2(\Gamma_2)^3$.

Assumption 8. $p_\nu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$;

- *there exists* $L_\nu > 0 : |p_\nu(, r_1) - p_\nu(, r_2)| \leq L_\nu \,|r_1 - r_2| \quad r_1, r_2 \in \mathbb{R}_+,$ *a.e.* $\in \Gamma_3$;
- *the mapping* $\mapsto p_\nu(, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}_+$;
- $p_\nu(, r) = 0$ *for all* $r \leq 0$ *a.e.* $\in \Gamma_3$.

Assumption 9. $p_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$;

- *there exists* $L_\tau > 0 : |p_\tau(, r_1) - p_\tau(, r_2)| \leq L_\tau \,|r_1 - r_2| \quad r_1, r_2 \in \mathbb{R}_+,$ *a.e.* $\in \Gamma_3$;
- *the mapping* $\mapsto p_\tau(, r)$ *is Lebesgue measurable on* Γ_3, *for all* $r \in \mathbb{R}_+$;
- $p_\tau(, r) = 0$ *for all* $r \leq 0$ *a.e.* $\in \Gamma_3$.

3.3. **Weak formulation.** Let us introduce the following functional space.

$$V = \{\in H^1(\Omega)^3 \mid = 0 \text{ a.e. on } \Gamma_1\}. \tag{3.7}$$

This is a Hilbert space endowed with the following inner product

$$(,)_V = \int_\Omega (()) : (()) \, dx,$$

where " : " denotes the inner product of two tensors.

Everywhere in this paper, for each $\in V$, we denote $w_\nu = \cdot$ and ${}_\tau = -w_\nu$ a.e. on Γ, where denotes the Sobolev trace operator for vectors.

Define $\in V$ using Riesz's representation theorem,

$$(,)_V = \int_\Omega {}_0() \cdot () \, dx + \int_{\Gamma_2} {}_2() \cdot () \, d\Gamma \quad \text{ for all } v \in V. \tag{3.8}$$

Let be a sufficiently regular solution of Problem 2. By a Green formula we get

$$a(,) = (,)_V + \int_{\Gamma_3} () \cdot () \, d\Gamma \quad \text{ for all } \in V \tag{3.9}$$

where

$$a(\cdot,\cdot) : V \times V \to \mathbb{R} \quad a(,) = \int_\Omega \mathcal{E}(()) : (()) \, dx. \tag{3.10}$$

Let us introduce the spaces

$$S = \{|_{\Gamma_3} \quad \in V\}; \tag{3.11}$$

$$D = S'. \tag{3.12}$$

For each $\in S$, we denote $\zeta_\nu = \cdot$ and ${}_\tau = -\zeta_\nu$ a.e. on Γ_3.

Notice that $|_{\Gamma_3}$ denotes the restriction of the trace of the element $\in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3;\mathbb{R}^3)$ where $H^{1/2}(\Gamma_3;\mathbb{R}^3)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^3$. On S we consider the Sobolev-Slobodeckii norm

$$\|\|_S = \Big(\int_{\Gamma_3}\int_{\Gamma_3} \frac{\|() - ()\|^2}{\| - \|^3}\, ds_x\, ds_y\Big)^{1/2};$$

see e.g. [1, 10].

For each $\in V$ we define

$$\begin{aligned}\Lambda() = \{\in D \,|\, \langle, |_{\Gamma_3}\rangle \leq \\ \int_{\Gamma_3}(p_\nu(,\varphi_\nu() - g_a)|v_\nu()| + p_\tau(,\varphi_\nu() - g_a)\|_\tau()\|)d\Gamma \quad \in V\};\end{aligned} \tag{3.13}$$

here and below $\langle\cdot,\cdot\rangle$ denotes the duality pairing between D and S.

Let us define a Lagrange multiplier $\in S$,

$$\langle,\rangle = -\int_{\Gamma_3} () \cdot ()\, d\Gamma. \tag{3.14}$$

Thus, for all $\in V$,

$$\langle, |_{\Gamma_3}\rangle = -\int_{\Gamma_3}(\sigma_\nu() v_\nu() +_\tau () \cdot_\tau ())d\Gamma.$$

By (3.14) and (3.13) we deduce that $\in \Lambda()$.

We also define

$$b : V \times D \to \mathbb{R} \quad b(,) = \langle, |_{\Gamma_3}\rangle. \tag{3.15}$$

Let us rewrite (3.9) as

$$a(,) = (,)_V - \langle, |_{\Gamma_3}\rangle \quad \text{for all } \in V.$$

By the definition of the form $b(\cdot,\cdot)$, we obtain

$$a(,) + b(,) = (,)_V \quad \text{for all } \in V. \tag{3.16}$$

On the other hand, the normal compliance condition (3.5) leads us to the identity

$$\int_{\Gamma_3} \sigma_\nu() u_\nu()\, d\Gamma = -\int_{\Gamma_3} p_\nu(, u_\nu() - g_a)|u_\nu()|\, d\Gamma$$

while the friction law (3.6) leads us to the identity

$$\int_{\Gamma_3} {}_\tau() \cdot_\tau ()\, d\Gamma = -\int_{\Gamma_3} p_\tau(, u_\nu() - g_a)\|_\tau()\|\, d\Gamma.$$

Thus,

$$b(,) = \int_{\Gamma_3}(p_\nu(, u_\nu() - g_a)|u_\nu(x)| + p_\tau(, u_\nu() - g_a)\|_\tau()\|)d\Gamma. \tag{3.17}$$

By (3.13) with = we are led to

$$b(,) \leq \int_{\Gamma_3}(p_\nu(, u_\nu() - g_a)|u_\nu()| + p_\tau(, u_\nu() - g_a)\|_\tau()\|)d\Gamma \quad \text{for all } \in \Lambda(). \tag{3.18}$$

Subtract now (3.17) from (3.18) to obtain the inequality

$$b(, -) \leq 0 \quad \text{for all } \in \Lambda(). \tag{3.19}$$

Therefore, Problem 2 has the following weak formulation.

Problem 3. *Find* $\in V$ *and* $\in \Lambda() \subset S$ *such that* (3.16) *and* (3.19) *hold true.*

Each solution of Problem 3 is called *weak solution* of Problem 2.

3.4. **Existence and boundedness results.**

Theorem 3.1 (An existence result)**.** *If Assumptions 6 -9 hold true, then Problem 2 has a weak solution.*

Proof. As the spaces V and D, see (3.7) and (3.12) are real Hilbert spaces, then Assumption 1 is fulfilled with $X = V$ and $Y = D$.

The form $a(\cdot,\cdot)$ defined in (3.10) verifies Assumption 2 with

$$M_a = \|\mathcal{E}\|_\infty \text{ and } m_a = m_{\mathcal{E}}, \tag{3.20}$$

where

$$\|\mathcal{E}\|_\infty = \max_{0\le i,j,k,l\le d} \|E_{ijkl}\|_{L^\infty(\Omega)}.$$

Let us prove (j_1) in Assumption 3. We have

$$|b(,)| \le \|\|_D\|\|_{H_\Gamma}.$$

We recall that $H_\Gamma = \{\quad \in V\}$ and the Sobolev trace operator $: H^1(\Omega)^3 \to H_\Gamma$ is a linear and continuous operator. Since $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^3}$ are equivalent norms, we deduce that there exists $M_b > 0$ such that (j_1) holds true.

We also recall that there exists a linear and continuous operator $\mathcal{Z}$ such that

$$\mathcal{Z} : H_\Gamma \to H^1(\Omega)^3 \quad (\mathcal{Z}()) = \quad \text{for all } \in H_\Gamma.$$

The operator $\mathcal{Z}$ is called *the right inverse* of the operator . Obviously,

$$(\mathcal{Z}()) = \qquad \text{for all } \in V.$$

For every $\in V$, we denote by * an element of V such that $=^*$ a.e. on Γ_3 and $^* = 0$ a.e. on Γ_2. Therefore, $\||_{\Gamma_3}\|_S = \|^*\|_{H_\Gamma}$.

Since, for each $^* \in V$, $\mathcal{Z}(^*)$ has the same trace as *, we deduce that for each $^* \in V$, $\mathcal{Z}(^*) \in V$.

Let us prove now (j_2) in Assumption 3.

$$\begin{aligned}
\|\|_D &= \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{\langle , |_{\Gamma_3}\rangle}{\||_{\Gamma_3}\|_S} \\
&= \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{\langle ,^* |_{\Gamma_3}\rangle}{\|^*\|_{H_\Gamma}} \\
&\le c \sup_{|_{\Gamma_3}\in S,\, |_{\Gamma_3}\neq 0_S} \frac{b(\mathcal{Z}(^*),)}{\|\mathcal{Z}(^*)\|_V} \\
&\le c \sup_{\in V,\, \neq_V} \frac{b(,)}{\|\|_V},
\end{aligned}$$

where $c > 0$. We can take

$$\alpha = \frac{1}{c}. \tag{3.21}$$

Obviously, $0_D \in \Lambda()$. Also, $\Lambda()$ is a closed convex subset of the space D. Hence, Assumption 4 is fulfilled.

Let us verify Assumption 5. To start, let $(_n)_n \subset V$ and $(_n)_n \subset V$ be two weakly convergent sequences, $_n \rightharpoonup$ in V and $_n \rightharpoonup$ in V, as $n \to \infty$. Let us take $\in \Lambda()$.

In order to check (k_1) in Assumption 5, we define $(_n)_n$ as follows: for each $n \geq 1$,

$$
\begin{aligned}
\langle _n, \rangle \quad = \quad & \int_{\Gamma_3} p_\nu(, \eta_{n\,\nu}() - g_a)\, \mathrm{sgn}\, u_{n\,\nu}()\, \zeta_\nu()\, d\Gamma \\
& + \int_{\Gamma_3} p_\tau(, \eta_{n\,\nu}() - g_a)(_{n\,\tau}()) \cdot_\tau ()\, d\Gamma \\
& - \int_{\Gamma_3} p_\nu(, \eta_\nu() - g_a)|u_{n\,\nu}()|\, d\Gamma \\
& - \int_{\Gamma_3} p_\tau(, \eta_\nu() - g_a)\|_{n\,\tau}()\|\, d\Gamma \\
& + \langle, _n|_{\Gamma_3}\rangle, \qquad \in S,
\end{aligned}
$$

where

$$
() = \begin{cases} \frac{}{\|\|} & \text{if } \neq; \\ & \text{if } =, \end{cases}
$$

and, as usually,

$$
\mathrm{sgn}(r) = \begin{cases} 1 & \text{if } r > 0; \\ 0 & \text{if } r = 0; \\ -1 & \text{if } r < 0. \end{cases}
$$

Taking into account (3.13), we deduce that, for each positive integer n, we have $_n \in \Lambda(_n)$.

Since $_n \rightharpoonup$ in V and $_n \rightharpoonup$ in V as $n \to \infty$, we deduce that

$$
_{n\,\tau}() \to_\tau () \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty,
$$

$$
u_{n\,\nu}() \to u_\nu() \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty,
$$

$$
p_\nu(, \eta_{n\,\nu}() - g_a) \to p_\nu(, \eta_\nu() - g_a) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty
$$

and

$$
p_\tau(, \eta_{n\,\nu}() - g_a) \to p_\tau(, \eta_\nu() - g_a) \text{ a.e. on } \Gamma_3 \text{ as } n \to \infty.
$$

Setting $=_n|_{\Gamma_3}$ we can write

$$
\begin{aligned}
\langle _n -, _n|_{\Gamma_3}\rangle = & \int_{\Gamma_3} (p_\nu(, \eta_{n\,\nu}() - g_a) - p_\nu(, \eta_\nu() - g_a))|u_{n\,\nu}()|\, d\Gamma \\
+ & \int_{\Gamma_3} (p_\tau(, \eta_{n\,\nu}() - g_a) - p_\tau(, \eta_\nu() - g_a))\|_{\tau\,n}()\|\, d\Gamma.
\end{aligned}
$$

Hence, passing to the limit as $n \to \infty$, we get

$$
\begin{aligned}
& \liminf_{n\to\infty} b(_n, _n -) = \lim_{n\to\infty} \int_{\Gamma_3} (p_\nu(, \eta_{n\,\nu}() - g_a) - p_\nu(, \eta_\nu() - g_a))|u_{n\,\nu}()|\, d\Gamma \\
& + \lim_{n\to\infty} \int_{\Gamma_3} (p_\tau(, \eta_{n\,\nu}() - g_a) - p_\tau(, \eta_\nu() - g_a))\|_{n\,\tau}()\|) d\Gamma \\
& = 0.
\end{aligned}
$$

Using again the properties of the trace operator and the assumptions on the friction bound we deduce that (k_2) in Assumption 5 is also verified.

We apply now Theorem 2.1. $\square$

Let us introduce the notation:

$$K_1 = \{ \in V \mid \|\|_V \leq \frac{1}{m_a} \|\|_V \}; \tag{3.22}$$

$$K_2 = \{ \in D \mid \|\|_D \leq \frac{m_a + M_a}{\alpha\, m_a} \|\|_V \}. \tag{3.23}$$

Theorem 3.2 (A boundedness result). *If* $(,)$ *is a weak solution of Problem 2, then*

$$(,) \in K_1 \times \big(\Lambda() \cap K_2\big)$$

where K_1 *and* K_2 *are given by* (3.22)-(3.23)*,* V *given by* (3.7)*,* D *given by* (3.12)*, given by* (3.8)*,* m_a *and* M_a *being the constants in* (3.20) *and* α *being the constant in* (3.21).

The proof is a straightforward consequence of Theorem 2.1.

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