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OPTIMAL SURPLUS, MINIMUM PENSION BENEFITS AND CONSUMPTION PLANS IN A MEAN-VARIANCE PORTFOLIO APPROACH FOR A DEFINED CONTRIBUTION PENSION SCHEME

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Abstract. In this paper, we study the problem of simultaneous maximization of the value of expected terminal surplus and, minimization of risks associated with the terminal surplus in a defined contribution (DC) pension scheme. The surplus, which is discounted, is solved with dynamic programming techniques. The pension plan member (PPM) makes a flow of contributions from his or her stochastic salary into the scheme. The flow of contributions are invested into a market that is characterized by a cash account, an index bond and a stock. The efficient frontier for the discounted and real surplus are obtained. Optimal consumption of the PPM was found to depend on the terminal wealth, random evolution of minimum pension benefit and "variance minimizing" parameter. It was found that as the variance minimizing parameter, tends to zero, the optimal consumption tends to negative infinity. The optimal expected discounted and real surplus, optimal total expected pension benefits and expected minimum pension benefits were obtained. We found that the optimal portfolio depends linearly on the random evolution of PPM's minimum benefits. Some numerical examples of the results are established.

Keywords. pension scheme, mean-variance, stochastic funding, defined contribution, efficient frontier, surplus, minimum pension benefits, optimal consumption

AMS subject classifications. 91B28, 91B30, 91B70, 93E20.

1. Introduction

In this paper, we consider a mean variance portfolio selection problem for a defined contribution pension scheme. We study the optimal surplus process, minimum pension benefit and optimal total benefit that will accrued to a PPM at terminal time. The salary process of the PPM is assumed to be stochastic. The flow of contribution by by the PPM are invested into a market that is composed of cash

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account, index bond and stock. The real and nominal surplus for the stakeholders (i.e., PPM and PFA) are obtained. The consumption process of the PPM at time, t is examined in this paper. The optimal investment allocation strategy can be found by solving a mean and variance optimization problem, see Nkeki (2012). Optimal surplus, optimal pension benefits, minimum pension benefits and optimal consumption plan in a mean-variance portfolio selection approach for a defined contribution pension scheme are considered in this paper.

Haberman and Sung (1994), considered a defined benefit (DB) plans and modeled it as linear-quadratic optimal control problems. Markowitz (1952) studied a meanvariance optimization model and used it to compare securities and portfolios based in a tradeoff between their expected return and its variance. Colombo and Haberman (2005) and Huang and Cairns (2005) considered a mean-variance portfolio problem in pension plans from a static point. Chiu and Li (2006) studied a dynamic case of the model for asset and liability management under the meanvariance criteria. Josa-Fombellida and Rincon-Zapatero (2008) studied the benefits of the DB plan by assuming that the benefits are stochastic, modeled by a geometric Brownian motion. They assumed that benefit is a non-tradeable asset. They also considered the existence of correlation between the sources of uncertainty in the benefits and in the asset returns.

Our paper follows the work of Josa-Fombellida and Rincon-Zapatero (2008). In our own case, we study optimal surplus, minimum pension benefit, optimal total benefit and optimal consumption plan under the context of a defined contribution pension plan. We assume that the salary process of a PPM is stochastic and modeled by a geometric Brownian motion.

There are extensive literature that exist on the area of accumulation phase of DC pension plan and optimal investment strategies. This can be found in Cains *et.al* (2006), Deelstra *et.al* (2000), Korn and Krekel (2001), Blake *et.al* (2008), Battocchio and Menoncin (2004), Boulier *et.al* (2001), Di Giacinto *et.al* (2010), Haberman and Vigna (2002), Vigna (2010), Gao (2008), Devolder *et.al* (2003), Nkeki and Nwozo (2012), Nkeki (2013). For optimal portfolio and life-cycle of a PPM consumption plan, see Nkeki (2011), Nwozo and Nkeki (2011), Merton (1971).

In the context of DC pension plans, the problem of finding the optimal surplus, minimum pension benefits, total pension benefits, and optimal consumption plan, with stochastic funding in a DC pension scheme under mean-variance efficient approach has not been reported in published articles. Hϕjgaard and Vigna (2007) and Vigna (2010) assumed a constant flow of contributions into the pension scheme which will not be applicable to a time-dependent salary earners in pension scheme. We assume that the contribution of the PPM grows as the salary grows over time. In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by Richardson (1989) via the Martingale approach. Bajeux-Besnainou and Portait (1998) used the same approach in a more general framework. Li and Ng (2000) solved a mean-variance optimization problem in a discrete-time multi-period framework. Zhou and Li (2000) considered a mean-variance in a continuous-time framework. They show the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, Vigna (2010), Bielecky *et.al* (2005), Hϕjgaard and Vigna (2007), Chiu and Li (2006), Josa-Fombellida and Rincon-Zapatero (2008).

In this paper, we study a mean-variance approach to portfolio selection problem for optimal surplus, minimum pension benefits, total pension benefits and optimal consumption plan with stochastic salary of a PPM in accumulation phase of a DC pension scheme. Nkeki (2012) considered a mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. The efficient frontier was obtained for three asset classes which include cash account, stock and index bond. The paper assumed that the flow of contributions of the PPM is stochastic. In this paper, we assumed that the salary of the PPM is stochastic.

The remainder of this paper is organized as follows. In section 2, we present financial market models. In section 3, we presents the pension benefits that will accrued to PPM. In section 4, we present the expected discounted flow of contributions, discounted wealth, discounted minimum pension benefit and discounted surplus. Section 5 presents the problem formulation of the paper. In section 6, we present the optimal portfolio and optimal consumption plan of a PPM. Section 7 presents the efficient frontier of the optimal terminal expected surplus. In section 8, we presents optimal pension benefit for a PPM at retirement. Section 9 presents the numerical examples of our models. Finally, section 10 concludes the paper.

2. Financial Market

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0,T]\}$, where $\mathcal{F}_t = \sigma(W^I(s), W^S(s) : s \leq t)$, where $W^S(t)$ and $W^I(t)$ are Brownian motions with respect to stock and index bond at time t. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure.

In this paper, we assume that the pension fund administrator (PFA) manage the fund contributed by the PPMs through pension fund custodians during the planning interval $[0,T]$ by means of a portfolio characterized by a cash account with price process, $B(t)$, index bond with price process, $Z(t, I(t))$ which is correlated geometric Brownian motion, generated by source of inflation risks, $W^I(t)$, where $I(t)$ is the price index at time t and has the dynamics: $dI(t) = j(t)I(t)dt + \sigma_1(t)I(t)dW^I(t)$, $j(t)$ the expected inflation index, which is the difference between nominal interest rate, $r(t)$, real interest rate $R(t)$ (i.e. $j(t) = r(t) - R(t) + \sigma_1(t)\theta_I(t)$) and $\sigma_I(t) = (\sigma_1(t), 0)$. $Z(t, I(t))$ is a zero-coupon bond which pays the price index at maturity, with a payoff

$$Z(t, I(t)) = E_t\left[I(T)\frac{\Lambda(T)}{\Lambda(t)}\right],$$

where

$$\Lambda(t) = B(t)^{-1}H(t)$$

and $H(t)$ satisfies the process

$$H(t) = \exp(-\theta'(t)W(t) - \frac{1}{2}\|\theta(t)\|^2), \tag{2.1}$$

which we assume to be martingale in $\mathbf{P}$, and a stock with price process, $S(t)$ correlated to geometric Brownian motions, $W^I(t)$ and $W^S(t)$, whose evolutions are respectively given by the equations:

$$dB(t) = r(t)B(t)dt, B(0) = 1, \tag{2.2}$$

$$(2.3)\quad dZ(t, I(t)) = Z(t, I(t))((r(t) + \sigma_1(t)\theta_I(t))dt + \sigma_1(t)dW^I(t)), Z(0) = z \in \mathcal{R}_+,$$

$$(2.4)\quad dS(t) = S(t)(\mu(t)dt + \sigma_S(t)qdW^I(t) + \sigma_S(t)\sqrt{1-q^2}dW^S(t)), S(0) = s \in \mathcal{R}_+.$$

Here $r(t) \in \mathcal{R}_+$ denotes the short risk-free interest rate, $\mu(t) \in \mathcal{R}_+$ the mean rate of return of the stock, $\sigma_S(t) \in \mathcal{R}$ the volatility of stock, $\sigma_1(t) \in \mathcal{R}$ the volatility of index bond, $q \in (-1, 1)$ correlation coefficient of sources of risks from inflation, $W^I(t)$ and stock, $W^S(t)$ and $\theta_I(t) \in \mathcal{R}$ the inflation price of risk. Moreover, $\sigma_S(t)$ and $\sigma_1(t)$ are the volatilities for the stock and index bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

The proportion of fund invested in stock, $S(t)$ at time, t is denoted by $\Delta^S(t)$ and proposition fund invested in index bond is $\Delta^I(t)$. The remainder, $1-\Delta^I(t)-\Delta^S(t)$ is invested in cash account at time, t. We suppose the trading strategy $\{\Delta(t) : t \geq 0\}$, with $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is a control process adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t$-measurable, Markovian, and stationary processes, satisfying

$$(2.5)\qquad E\int_0^T \Delta(t)\Delta'(t)dt < \infty,$$

where E is the expectation operator. Let $C(t)$ be the consumption rate process at time t. Then $C(t)$ is an adapted process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, satisfying

$$(2.6)\qquad E\int_0^T C(t)^2 dt < \infty.$$

Let $Y(t)$ be the salary process of a PPM at time t, then $Y(t)$ satisfies the following stochastic differential equation:

$$(2.7)\qquad dY(t) = Y(t)(\beta(t)dt + \sigma_{Y_1}(t)dW^I(t) + \sigma_{Y_2}(t)dW^S(t)), Y(0) = y_0 \in \mathcal{R}.$$

where $\beta(t) \in \mathcal{R}_+$ is the expected growth rate of the salary, $\sigma_{Y_1}(t)$ is volatility of the salary of a PPM arising from the uncertainty of inflation, $W^I(t)$ and $\sigma_{Y_2}(t)$ is volatility of the salary of a PPM arising from the uncertainty of stock market, $W^S(t)$. We can express (2.3), (2.4) and (2.7) in compact form respectively, as follows:

$$(2.8)\quad dZ(t, I(t)) = Z(t, I(t))((r(t) + \sigma_1(t)\theta_I(t))dt + \sigma_Z(t)dW(t)), Z(0) = z \in \mathcal{R}_+,$$

$$(2.9)\qquad dS(t) = S(t)(\mu(t)dt + \sigma(t)dW(t)), S(0) = s \in \mathcal{R}_+,$$

$$(2.10)\qquad dY(t) = Y(t)(\beta(t)dt + \sigma_Y(t)dW(t)), Y(0) = y_0 \in \mathcal{R},$$

where $\sigma_Z(t) = (\sigma_1(t), 0)$, $\sigma(t) = (\sigma_S(t)q, \sigma_S(t)\sqrt{1-q^2})$, $\sigma_Y(t) = (\sigma_{Y_1}(t), \sigma_{Y_2}(t))$ and $W(t) = (W(^I(t), W(^S(t))'$. Suppose the proportion $c \in \mathcal{R}_+$ of the salary process is a contribution of the PPM into the scheme, then $cY(t)$ is the gross amount of fund contributed into the scheme at time t.

Remark 2.1. If the pension PPM's salary is deterministic, then (2.10) becomes $dY(t) = \beta(t)Y(t)dt$.

Then, the volatility matrix

$$(2.11)\qquad \Sigma(t) := \begin{pmatrix} \sigma_1(t) & 0 \\ q\sigma_S(t) & \sigma_S(t)\sqrt{1-q^2} \end{pmatrix}$$

corresponding to the two risky assets and satisfies $det(\Sigma(t)) = \sigma_S(t)\sigma_1(t)\sqrt{1-q^2} \neq 0$. Therefore, the market is complete and there exists a unique market price of risks vector, $\theta(t)$ satisfying

$$\theta(t) := \begin{pmatrix} \theta_I(t) \\ \theta_S(t) \end{pmatrix} = \begin{pmatrix} \theta_I(t) \\ \frac{\mu(t)-r(t)-\theta_I(t)q\sigma_S(t)}{\sigma_S(t)\sqrt{(1-q^2)}} \end{pmatrix} \tag{2.12}$$

where $\theta_S(t)$ is the market price of stock risks. In this paper, we assume that $r(t)$, $\mu(t)$, $\sigma(t)$, $\sigma_S(t)$, $\sigma_1(t)$, $\sigma_Y(t)$, $\sigma_{Y_1}(t)$, $\sigma_{Y_2}(t)$, $\theta_I(t)$, $\theta_S(t)$, $q(t)$, $\beta(t)$, $\sigma_Z(t)$ are constants in time.

Therefore, the fund, $X(t)$ dynamic evolution under the investment policy Δ is

$$\begin{aligned} &dX(t) = (X(t)(r + \Delta(t)\lambda) + c(1-\eta)Y(t) - C(t))dt + X(t)(\Sigma'\Delta'(t))'dW(t), \\ &X(0) = x_0 \in \mathcal{R}_+, \end{aligned} \tag{2.13}$$

where $\lambda = (\sigma_1\theta_I, \mu - r)'$, η denotes the proportion of PPM's contribution that is set aside for administrative cost (AC). It implies that $\eta c Y(t)$ is the AC at time t and the net contribution is $c(1-\eta)Y(t)$ at time t. We observe that when $\eta = 0$, it implies that the PFA do not charge any management costs. If $\eta = 1$, it implies that the entire contributions by the PPM is taken as management costs, which is unrealistic. Since the PFA may not (or may) charge management costs for the operation, we assume that $0 \leq \eta < 1$.

3. Pension Benefits

In this section, we consider the minimum pension benefits, $P^m(t)$ at time t that will accrued to a PPM up to the final time, T. Let $P(t)$ be the total pension benefits of the contributor at time, $t \in [0, T]$. It is assumed that the value of minimum benefits a PPM can get at retirement should not be less than the value of contributions made into the scheme.

The PPM makes a flow of contribution to the pension fund. This flow consists of a lump sum at time 0, denoted by x_0, and a continuously paid premium, at a rate denoted by $cY(t)(t), t \in [0, T]$. The value at time 0 of the cash given by the contributor (i.e., PPM) to the pension scheme is equal to:

$$\bar{X}_0 = x_0 + c(1-\eta)E\left[\int_0^T \Lambda(s)Y(s)ds\right].$$

At time T, the PFA will provide a benefit which consists of two parts: The first part $P^m(T)$ is the minimum pension benefit, which means that the total benefit will be greater than $P^m(T)$ almost surely. The minimum pension benefit is not a constant (it is a stochastic minimum pension benefit), but a nonnegative random variable that is F_T-measurable, which is L^p integrable with $p > 2$. The second part of the benefit is a fixed fraction of the surplus $\Theta_T(P^m(T))$ (the difference between the terminal wealth $X(T)$ of the managed portfolio and the minimum pension benefit $P^m(T)$. Indeed, we suppose that the PFA receives a fixed fraction of the surplus, as a way to encourage him/her (see Jensen and Sϕrensen, 1999). Let h denotes the fixed fraction of the surplus that will be kept by the PFA. Then, the total benefit of the PPM at time T equals:

$$\begin{aligned} P(T) &= P^m(T) + (1-h)(X(T) - P^m(T)) \\ &= P^m(T) + \Theta_T(P^m(T)), \end{aligned}$$

where $\Theta_T(P^m(T)) = (1-h)V(T)$ is the surplus function at the final time, T and $V(T) = X(T) - P^m(T)$. For $h = 0$, it implies that the PFA does not keep any profit from the surplus, so introduction of the minimum pension benefit is more an obstacle for the PPM, since minimum pension benefits may induce a significant utility loss for quadratic risk tolerant investors (see Jensen and Sϕrensen, 1999 for relative risk averse investor). On the other hand, if $h = 1$, it implies that the PPM will receive only the minimum pension benefit, no matter the final surplus, which is not reasonable. In order to avoid these trivial cases, we therefore assume that $h \in (0,1)$. One of the aims of this paper is to find the optimal discounted benefit that will accrued to the PPM at the final time, T. This is obtained from the discounted surplus and discounted minimum pension benefit at the final time, T.

Definition 3.1. The flow of expected discounted minimum pension benefits for $t \leq T$ is defined by

$$P^m(t) = E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)} cY(u)du\right], t \geq 0. \tag{3.1}$$

where $E_t = E(\cdot|\mathcal{F}_t)$ is the conditional expectation with respect to the Brownian filtration $\{\mathcal{F}\}_{t\geq 0}$.

Definition 3.2. The flow of expected pension benefits, $P(t)$ is defined by

$$P(t) = \begin{cases} P^m(t), \text{ if } 0 \leq t \leq T_0 < T, \\ P^m(t) + \Theta(t, V(t)|_{t-T,t}), \text{ if } t \geq T, \end{cases} \tag{3.2}$$

where T_0 is the time of voluntary retirement and $\Theta(\cdot,\cdot)$ is the surplus function. At time $t \geq T$ the surplus depends on the fund wealth level in time period $[t-T, t]$.

Proposition 3.1. *Let $P^m(t)$ be the value of flow of the minimum pension benefits that will accrued to PPM at time t, then*

$$P^m(t) = \frac{cY(t)}{\delta}(e^{\delta T} - 1), \tag{3.3}$$

where $\delta = \beta - \xi - \sigma_Y\theta$, $\xi \in [0, r]$ is the instantaneous guaranteed rate of return and $cY(t)$ is flow of contributions of PPM at time t.

Proof: By definition 3.1, we have that

$$\begin{aligned} P^m(t) &= E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)} cY(u)du\right] \\ &= cY(t)E_t\left[\int_0^T \frac{\Lambda(u)}{\Lambda(t)}\frac{Y(u)}{Y(t)}du\right]. \end{aligned}$$

Applying change of variable and Markovian rule on the above equation, we have

$$P^m(t) = cY(t)E\left[\int_0^T \frac{\Lambda(\tau)}{\Lambda(0)}\frac{Y(\tau)}{Y(0)}d\tau\right], \tag{3.4}$$

Applying parallelogram law and martingale principles on (3.4), we have

$$P^m(t) = cY(t)E\int_0^T e^{(\beta-\xi-\sigma_Y\theta)\tau}d\tau. \tag{3.5}$$

Integrating, we have

$$P^m(t) = \frac{cY(t)(e^{(\beta-\xi-\sigma_Y\theta)T}-1)}{\beta-\xi-\sigma_Y\theta}. \tag{3.6}$$

Therefore,

$$P^m(t) = \frac{cY(t)(e^{\delta T}-1)}{\delta}. \tag{3.7}$$

where $\delta = \beta - \xi - \sigma_Y\theta$.

This implies that the final minimum pension benefits for a PPM is

$$P^m(T) = \frac{cY(T)(e^{\delta T}-1)}{\delta}, \tag{3.8}$$

and the present value of a PPM's future minimum pension benefit is

$$P^m(0) = P_0^m = \frac{cy_0(e^{\delta T}-1)}{\delta}. \tag{3.9}$$

Taking the differential of both sides of (3.7), we have

$$dP^m(t) = P^m(t)(\beta dt + \sigma_Y dW(t)). \tag{3.10}$$

Corollary 3.1. *Let $P^m(T_0)$ be the minimum pension benefits for a PPM who retired voluntarily from the scheme and T_0 the time of voluntary retirement, then*

$$P^m(T_0) = \frac{cY(T_0)(e^{\delta T_0}-1)}{\delta}, 0 < T_0 < T. \tag{3.11}$$

4. Expected Discounted Flow of Contributions

In this section, we presents the expected discounted flow of PPM's contributions at time t.

Definition 4.1. The expected value of flow of a PPM's net contribution is defined as

$$\Phi(t) = E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} c(1-\eta)Y(u)du\right]. \tag{4.1}$$

Theorem 4.1. *Suppose $\Phi(t)$ is the expected value of a PPM's net contributions, then*

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T-t)}-1)}{\alpha}, \tag{4.2}$$

where $\alpha = \beta - r - \sigma_Y\theta$.

Proof: By definition 4.1, we have

$$\Phi(t) = c(1-\eta)Y(t)E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)}\frac{Y(u)}{Y(t)}du\right]. \tag{4.3}$$

Applying change of variable and Markovian rule on (4.3), we have

$$\Phi(t) = c(1-\eta)Y(t)E\left[\int_0^T \frac{\Lambda(\tau)}{\Lambda(0)}\frac{Y(\tau)}{Y(0)}d\tau\right], \tag{4.4}$$

Applying parallelogram law and martingale principles on (3.4) and then integrate, we have

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T-t)}-1)}{\alpha}, \tag{4.5}$$

where $\alpha = \beta - r - \sigma_Y\theta$. The present value of a PPM's future contribution is obtain as

$$\Phi(0) = \frac{c(1-\eta)y_0(e^{\alpha T}-1)}{\alpha}. \tag{4.6}$$

Taking the differential of both sides of (4.5), we have

$$d\Phi(t) = \Phi(t)((r+\sigma_Y\theta)dt + \sigma_Y dW(t)) - c(1-\eta)Y(t)dt. \tag{4.7}$$

Corollary 4.1. *Let $\Phi(T_0)$ be the value of the contributions of a PPM who will retired voluntarily at time period T_0, then*

$$\Phi(t) = \frac{c(1-\eta)Y(t)(e^{\alpha(T_0-t)}-1)}{\alpha}, 0 \le t \le T_0. \tag{4.8}$$

It implies that the present value of the PPM's contributions that retired voluntarily from the scheme is

$$\Phi(0) = \frac{c(1-\eta)y_0(e^{\alpha T_0}-1)}{\alpha}. \tag{4.9}$$

4.1. **Discounted Wealth, Contribution, Minimum Pension Benefit and Surplus Process.** In this subsection, we consider the discounted wealth, discounted contributions and discounted minimum pension benefit of a PPM at time t. The discounted surplus process of the stakeholder is also established in this subsection. The discounted wealth of a PPM is given by (4.10).

$$d(\Lambda(t)X(t)) = \Lambda(t)X(t)(\Sigma'\Delta'(t) - \theta)'dW(t) + (c(1-\eta)\Lambda(t)Y(t) - \Lambda(t)C(t))dt \tag{4.10}$$

(4.11) gives the discounted contributions of a PPM at time t and is given by

$$d(\Lambda(t)\Phi(t)) = \Lambda(t)\Phi(t)(\sigma_Y' - \theta)'dW(t) - (c(1-\eta)\Lambda(t)Y(t)dt. \tag{4.11}$$

The discounted minimum pension benefits is given by

$$d(\Lambda(t)P^m(t)) = \Lambda(t)P^m(t)(\sigma_Y' - \theta)'dW(t). \tag{4.12}$$

Setting $\tilde{X}(t) = \Lambda(t)X(t)$, $\tilde{Y}(t) = \Lambda(t)Y(t)$, $\tilde{C}(t) = \Lambda(t)C(t)$, $\tilde{\Phi}(t) = \Lambda(t)\Phi(t)$, $\tilde{P}^m(t) = \Lambda(t)P^m(t)$, (4.10)-(4.12) become:

$$d\tilde{X}(t) = \tilde{X}(t)(\Sigma'\Delta'(t) - \theta)'dW(t) + (c(1-\eta)\tilde{Y}(t) - \tilde{C}(t))dt \tag{4.13}$$

$$d\tilde{\Phi}(t) = \tilde{\Phi}(t)(\sigma_Y' - \theta)'dW(t) - (c(1-\eta)\tilde{Y}(t)dt \tag{4.14}$$

$$d\tilde{P}^m(t) = \tilde{P}^m(t)(\sigma_Y' - \theta)'dW(t)dt \tag{4.15}$$

Remark 4.1.

$$d(\Lambda(t)Y(t)) = \Lambda(t)Y(t)(\beta - r - \sigma_Y\theta)dt + \Lambda(t)Y(t)(\sigma_Y' - \theta)'dW(t). \tag{4.16}$$

Solving (4.16), we have

$$E(\Lambda(t)Y(t)) = y_0 e^{(\beta - r - \sigma_Y\theta)t}. \tag{4.17}$$

Solving (4.14), we have

$$E\tilde{\Phi}(t) = \Phi_0 - c(1-\eta)\int_0^t E(\Lambda(s)Y(s))ds. \tag{4.18}$$

Using (4.17) on (4.18), we have

$$E\tilde{\Phi}(t) = \Phi_0 - \frac{cy_0(1-\eta)}{\beta - r - \sigma_Y\theta}\left(e^{(\beta-r-\sigma_Y\theta)t} - 1\right). \tag{4.19}$$

Hence, the value of a PPM's surplus is given as

$$V(t) = X(t) + \Phi(t) - P^m(t). \tag{4.20}$$

Therefore, the value of a PPM's discounted surplus process is given as

$$\tilde{V}(t) = \tilde{X}(t) + \tilde{\Phi}(t) - \tilde{P}^m(t). \tag{4.21}$$

Proposition 4.1. *Suppose $\tilde{X}(t)$ satisfies (4.13), $\tilde{P}^m(t)$ (4.15) and $\tilde{\Phi}(t)$ satisfies (4.14), then the discounted surplus process, $\tilde{V}(t)$ has the following dynamics*

$$\begin{aligned} d\tilde{V}(t) &= [\tilde{X}(t)(\Sigma'\Delta'(t) - \theta)' + (\tilde{\Phi}(t) - \tilde{P}^m(t))(\sigma_Y' - \theta)']dW(t) - \tilde{C}(t)dt, \\ \tilde{V}(0) &= v_0. \end{aligned} \tag{4.22}$$

5. The Mean-Variance Formulation

The objective of the PFA is double. The first objective is to maximize the expected value of fund's (and discounted) assets. The second objective is aim at to minimize the variance of the terminal discounted surplus (and real surplus or simply surplus), $Var(\tilde{V}^*(T))$ (and $Var(V^*(T))$) and the consumption risk, $C^*(t)$ on the interval $[0,T]$. This dual-objective problem reflects the major concern of the stakeholders (in this paper, stakeholders represents the PFA and the PPM only) to increase fund assets in order to pay due pension benefits as at when due, but at the same time not exposed the pension fund to large variations in other to provide stability to the scheme. According to Josa-Fombellida and Rincon-Zapatero (2008), minimization of the contribution risk (though, in this paper, we consider consumption risk) has been considered in other works as Haberman and Sung (1994), Haberman *et al.* (2000) and Josa-Fombellida and Rincon-Zapatero (2001, 2004).

Therefore, this paper is considering a multi-objective optimization problem involving two criteria

$$\min_{(\Delta,C)\in\mathcal{A}}(L_1(\Delta,C), L_2(\Delta,C)) = \min_{(\Delta,C)\in\mathcal{A}}\left(-E(\tilde{V}(T)), E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + Var(\tilde{V}(T))\right) \tag{5.1}$$

subject to (4.22). Here $\mathcal{A}$ is the set of measurable processes (Δ, C), where Δ satisfies (2.5), C satisfies (2.6), and such that (4.22) admit a unique solution that is $\mathcal{F}_t$-measurable adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

An admissible control process (Δ^*, C^*) is Pareto efficient if there exists no admissible (Δ, C) such that

$$L_1(\Delta,C) \leq L_1(\Delta^*,C^*), L_2(\Delta,C) \leq L_2(\Delta^*,C^*)$$

with at least one of the inequalities hold strictly. The pairs $(L_1(\Delta^*,C^*); L_2(\Delta^*,C^*)) \in \mathcal{R}^2$ form the Pareto frontier. We will refer to C^* an efficient consumption rate and

Δ^* an efficient portfolio. The aim of this paper is to find $(E(\tilde{V}^*(T)), Var(\tilde{V}^*(T)))$ (and then deduce $(E(V^*(T)), Var(V^*(T))))$ refer to as an efficient frontier.

According to Da Cunha and Polak (1967) (in Josa-Fombellida and Rincon-Zapatero (2008)), when the objective functionals defining the multi-objective program are convex, the Pareto efficient points can be found by solving a scalar optimal control problem where the dynamics is fixed and the objective functional is a convex combination of the original cost functionals. In our case (4.15) and (4.22) are linear, so both L_1 and L_2 are indeed convex. Hence, the original problem (4.15), (4.22) and (5.1) are equivalent to the scalar problem

$$\min_{(\Delta,C)\in\mathcal{A}} L_1(\Delta,C)+\psi L_2(\Delta,C) = \min_{(\Delta,C)\in\mathcal{A}} -E(\tilde{V}(T))+\psi\left(E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + Var(\tilde{V}(T))\right) \tag{5.2}$$

subject to (4.22) and (4.15), with $\psi > 0$ a weight parameter. As ψ varies within the interval $(0, \infty)$, the solutions of (5.2) describe the Pareto frontier (see Josa-Fombellida and Rincon-Zapatero (2008)). Observe that ψ serves the PFA the opportunity to transfer linearly units of risk to units of expected return, and vice versa. The size of ψ shows which one of the objectives is of major concern for the PFA, to reduce risk or to maximize return.

Problem (4.15), (4.22) and (5.2) are not standard stochastic optimal problem due to the presents of the term $(E(\tilde{V}(T)))^2$ in the variance term, and the dynamic programming approach cannot be applied at this point. Following Zhou and Li (2000), Li and Ng (2000) and Josa-Fombellida and Rincon-Zapatero (2008), we propose an auxiliary problem that transform into a stochastic problem of linear-quadratic case:

$$\min_{(\Delta,C)\in\mathcal{A}} J(\Delta,C) = \min_{(\Delta,C)\in\mathcal{A}} \left(E\int_0^T e^{\rho t}\tilde{C}^2(t)dt + E(\tilde{V}^2(T) - 2\varphi\tilde{V}(T))\right) \tag{5.3}$$

subject to (4.22) and (4.15) and $\varphi \in \mathcal{R}$.

The relationship between problems (4.15), (4.22), (5.2) and (4.15), (4.22), (5.3) is shown in the following result.

Theorem 5.1. *For any $\varphi > 0$, if (Δ^*, C^*) is an optimal control of (4.15), (4.22), (5.2) with associated optimal surplus, V^*, then it is an optimal control of (4.15), (4.22), (5.3) for $\varphi = \frac{1}{2\psi} + E(\tilde{V}^*(T))$.*

Proof: see Josa-Fombellida and Rincon-Zapatero (2008).

6. Optimal Portfolio and Optimal Consumption

In this section, we find the optimal portfolio and optimal consumption rate for a PPM. First, we determine the Hamilton-Jacobi-Bellman equation for our surplus process. We define the follows differential operator:

$$\mathcal{L} = \left(\frac{1}{2}\tilde{\Phi}^2\frac{\partial^2}{\partial\tilde{\Phi}^2} - \tilde{\Phi}\tilde{P}\frac{\partial^2}{\partial\tilde{\Phi}\partial\tilde{P}} + \frac{1}{2}\tilde{P}^2\frac{\partial^2}{\partial\tilde{P}^2}\right)(\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta). \tag{6.1}$$

We define the general value function

$$L(t,\tilde{x},\tilde{\Phi},\tilde{P}) = E[V(T,\tilde{X},\tilde{\Phi},\tilde{P})|\tilde{X}(t) = \tilde{x}, \tilde{\Phi}(t) = \tilde{\Phi}, \tilde{P}^m(t) = \tilde{P}]$$

where $L(t,\tilde{x},\tilde{\Phi},\tilde{P})$ is the path of $V(t)$ given the portfolio strategy $\Delta(t)=(\Delta^I(t),\Delta^S(t))$. Let $L(t,\tilde{x},\tilde{\Phi},\tilde{P})$ be a convex function in $V(t)$ such that

$$\begin{array}{l} U(t,\tilde{x},\tilde{\Phi},\tilde{P}) = \min_{\Delta,C} L(t,\tilde{x},\tilde{\Phi},\tilde{P}), \\ \text{subject to (3.8).} \end{array} \tag{6.2}$$

Then $U(t,\tilde{x},\tilde{\Phi},\tilde{P})$ satisfies the HJB equation

$$\begin{array}{l} U_t - \tilde{C}(t)U_{\tilde{x}} + \tilde{C}^2(t)e^{-\rho t} + \frac{1}{2}\tilde{x}^2(\Sigma\Delta(t)\Sigma'\Delta'(t) - 2\Sigma\Delta(t)\theta + \theta'\theta)U_{\tilde{x}\tilde{x}} \\ +2(\tilde{x}\tilde{\Phi}U_{\tilde{x}\tilde{\Phi}} - \tilde{x}\tilde{P}U_{\tilde{x}\tilde{P}})(\Sigma\Delta(t)\sigma_Y' - 2\Sigma\Delta(t)\theta + \theta'\theta) + \mathcal{L}U = 0, \\ \\ \text{subject to: } U(T,\tilde{x},\tilde{\Phi},\tilde{P}) = (\tilde{x}-\tilde{P})^2 - 2\varphi(\tilde{x}-\tilde{P}). \end{array} \tag{6.3}$$

Proposition 6.1. *The optimal rate of consumption and the optimal investment in the risky assets (index bond and stock) are respectively given by*

$$\Delta^{'*}(t) = \frac{(\Sigma\Sigma')^{-1}(\tilde{x}\Sigma\theta U_{\tilde{x}\tilde{x}} - 2(\tilde{\Phi}U_{\tilde{x}\tilde{\Phi}} - \tilde{P}U_{\tilde{x}\tilde{P}})(\Sigma\sigma_Y' - 2\Sigma\theta))}{\tilde{x}U_{\tilde{x}\tilde{x}}}. \tag{6.4}$$

$$\tilde{C}^*(t) = \frac{1}{2}U_{\tilde{x}}e^{\rho t}. \tag{6.5}$$

Substituting (6.4) and (6.5) into (6.3), we have

$$\begin{array}{l} U_t - \frac{1}{2}U_{\tilde{x}}^2 e^{\rho t} - \frac{1}{2}\theta'\theta(\tilde{x}^2-1)U_{\tilde{x}\tilde{x}} - 2[\theta'\Sigma'M(\Sigma\sigma_Y' - 2\Sigma\theta)\tilde{\Phi} \\ -2\tilde{x}\tilde{\Phi}(\sigma_Y\theta - 2\theta'\theta) - \tilde{x}\tilde{\Phi}\theta'\theta]U_{\tilde{x}\tilde{\Phi}} + 2\tilde{P}[\theta'\Sigma'M(\Sigma\sigma_Y' - 2\Sigma\theta) \\ +2\tilde{x}(2\theta'\theta - \sigma_Y\theta) - \tilde{x}\theta'\theta]U_{\tilde{x}\tilde{P}} - 4(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{P}\tilde{\Phi}\frac{U_{\tilde{x}\tilde{\Phi}}U_{\tilde{x}\tilde{P}}}{U_{\tilde{x}\tilde{x}}} \\ +[2(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{P}^2 - 4\tilde{P}^2(3\sigma_Y\theta - 2\theta'\theta - \sigma_Y\sigma_Y')]\frac{U_{\tilde{x}\tilde{P}}^2}{U_{\tilde{x}\tilde{x}}} \\ +[2(\sigma_Y\sigma_Y' - 4\theta'\theta)\tilde{\Phi}^2 + 4\tilde{\Phi}^2(3\sigma_Y\theta - 2\theta'\theta - \sigma_Y\sigma_Y')]\frac{U_{\tilde{x}\tilde{\Phi}}^2}{U_{\tilde{x}\tilde{x}}} + \mathcal{L}U = 0, \\ \\ \text{subject to: } U(T,\tilde{x},\tilde{\Phi},\tilde{P}) = (\tilde{x}-\tilde{P})^2 - 2\varphi(\tilde{x}-\tilde{P}). \end{array} \tag{6.6}$$

We assume a quadratic solution of the form:

$$\begin{array}{l} U(t,\tilde{x},\tilde{\Phi},\tilde{P}) = \phi_0(t) + \tilde{P}\phi_{\tilde{P}}(t) + \tilde{\Phi}\phi_{\tilde{\Phi}}(t) + \tilde{x}\phi_{\tilde{x}}(t) + \tilde{x}\tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t) \\ +\tilde{x}\tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\tilde{P}\phi_{\tilde{\Phi}\tilde{P}}(t) + \tilde{\Phi}^2\phi_{\tilde{\Phi}\tilde{\Phi}}(t) + \tilde{x}^2\phi_{\tilde{x}\tilde{x}}(t) + \tilde{P}^2\phi_{\tilde{P}\tilde{P}}(t). \end{array} \tag{6.7}$$

Finding the partial derivatives of (6.7) with respect to t, $\tilde{x}$, $\tilde{P}$, $\tilde{\Phi}$, $\tilde{x}\tilde{x}$, $\tilde{x}\tilde{\Phi}$, $\tilde{x}\tilde{P}$, $\tilde{\Phi}\tilde{P}$, $\tilde{P}\tilde{P}$, $\tilde{\Phi}\tilde{\Phi}$ as follows:

$$\begin{array}{l} U_t = \dot{\phi}_0(t) + \tilde{P}\dot{\phi}_{\tilde{P}}(t) + \tilde{\Phi}\dot{\phi}_{\tilde{\Phi}}(t) + \tilde{x}\dot{\phi}_{\tilde{x}}(t) + \tilde{x}\tilde{\Phi}\dot{\phi}_{\tilde{x}\tilde{\Phi}}(t) \\ +\tilde{x}\tilde{P}\dot{\phi}_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\tilde{P}\dot{\phi}_{\tilde{\Phi}\tilde{P}}(t) + \tilde{\Phi}^2\dot{\phi}_{\tilde{\Phi}\tilde{\Phi}}(t) + \tilde{x}^2\dot{\phi}_{\tilde{x}\tilde{x}}(t) + \tilde{P}^2\dot{\phi}_{\tilde{P}\tilde{P}}(t), \end{array} \tag{6.8}$$

$$U_{\tilde{x}} = \phi_{\tilde{x}}(t) + \tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{x}\phi_{\tilde{x}\tilde{x}}(t), \tag{6.9}$$

$$U_{\tilde{\Phi}} = \phi_{\tilde{\Phi}}(t) + \tilde{x}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{\Phi}\phi_{\tilde{\Phi}\tilde{\Phi}}(t), \tag{6.10}$$

$$U_{\tilde{P}} = \phi_{\tilde{P}}(t) + \tilde{x}\phi_{\tilde{x}\tilde{P}}(t) + \tilde{\Phi}\phi_{\tilde{\Phi}\tilde{P}}(t) + 2\tilde{P}\phi_{\tilde{P}\tilde{P}}(t), \tag{6.11}$$

$$\begin{array}{l} U_{\tilde{x}\tilde{x}} = 2\phi_{\tilde{x}\tilde{x}}(t), U_{\tilde{\Phi}\tilde{\Phi}} = 2\phi_{\tilde{\Phi}\tilde{\Phi}}(t), U_{\tilde{P}\tilde{P}} = 2\phi_{\tilde{P}\tilde{P}}(t), \\ U_{\tilde{x}\tilde{\Phi}} = 2\phi_{\tilde{x}\tilde{\Phi}}(t), U_{\tilde{x}\tilde{P}} = 2\phi_{\tilde{x}\tilde{P}}(t), U_{\tilde{\Phi}\tilde{P}} = 2\phi_{\tilde{\Phi}\tilde{P}}(t). \end{array} \tag{6.12}$$

The following ordinary differential equations are obtained for the above coefficients of $\tilde{x}$, $\tilde{P}$, $\tilde{\Phi}$, $\tilde{x}\tilde{x}$, $\tilde{x}\tilde{\Phi}$, $\tilde{x}\tilde{P}$, $\tilde{\Phi}\tilde{P}$, $\tilde{P}\tilde{P}$, $\tilde{\Phi}\tilde{\Phi}$ in (6.6):

$$\dot{\phi}_0(t) = \frac{1}{4}e^{\rho t}\phi_{\tilde{x}}^2(t) - \theta'\theta\phi_{\tilde{x}\tilde{x}}(t), \phi_0(T) = 0,$$

$$\dot{\phi}_{\tilde{P}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{P}}(t) - 4\theta'\Sigma' M(\Sigma\sigma_Y' - 2\Sigma\theta)\phi_{\tilde{x}\tilde{P}}(t), \phi_{\tilde{P}}(T) = 2\varphi,$$

$$\dot{\phi}_{\tilde{\Phi}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{\Phi}}(t) + 4\theta'\Sigma' M(\Sigma\sigma_Y' - 2\Sigma\theta)\phi_{\tilde{x}\tilde{\Phi}}(t), \phi_{\tilde{\Phi}}(T) = 0,$$

$$\dot{\phi}_{\tilde{\Phi}\tilde{P}}(t) = \tfrac{1}{2}e^{\rho t}\phi_{\tilde{x}\tilde{\Phi}}(t)\phi_{\tilde{x}\tilde{P}}(t) + 2(\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{P}\tilde{\Phi}}$$
$$+8(\sigma_Y\sigma_Y' - 4\theta'\theta)\tfrac{\phi_{\tilde{x}\tilde{P}}(t)\phi_{\tilde{x}\tilde{\Phi}}(t)}{\phi_{\tilde{x}\tilde{x}}(t)}, \phi_{\tilde{\Phi}\tilde{P}}(T) = 0,$$

$$\dot{\phi}_{\tilde{\Phi}\tilde{\Phi}}(t) = \tfrac{1}{4}e^{\rho t}\phi^2_{\tilde{x}\tilde{\Phi}}(t) - (\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{\Phi}\tilde{\Phi}}$$
$$+4(\sigma_Y\sigma_Y' + 3\sigma_Y\theta + 8\theta'\theta)\tfrac{\phi^2_{\tilde{x}\tilde{\Phi}}}{\phi_{\tilde{x}\tilde{x}}}, \phi_{\tilde{\Phi}\tilde{\Phi}}(T) = 0,$$

$$\dot{\phi}_{\tilde{P}\tilde{P}}(t) = \tfrac{1}{4}e^{\rho t}\phi^2_{\tilde{x}\tilde{P}}(t) - (\sigma_Y\sigma_Y' - 2\sigma_Y\theta + \theta'\theta)\phi_{\tilde{P}\tilde{P}}$$
$$+12(\sigma_Y\sigma_Y' - 2\sigma_Y\theta)\tfrac{\phi^2_{\tilde{x}\tilde{P}}}{\phi_{\tilde{x}\tilde{x}}}, \phi_{\tilde{P}\tilde{P}}(T) = 0,$$

$$\dot{\phi}_{\tilde{x}}(t) = e^{\rho t}\phi_{\tilde{x}}(t)\phi_{\tilde{x}\tilde{x}}(t), \phi_{\tilde{x}}(T) = -2\varphi, \tag{6.13}$$

$$\dot{\phi}_{\tilde{x}\tilde{\Phi}}(t) = e^{\rho t}\phi_{\tilde{x}\tilde{\Phi}}(t)\phi_{\tilde{x}\tilde{x}}(t) - 4(2\sigma_Y\theta - 3\theta'\theta)\phi_{\tilde{x}\tilde{\Phi}}(t), \phi_{\tilde{x}\tilde{\Phi}}(T) = 0, \tag{6.14}$$

$$\dot{\phi}_{\tilde{x}\tilde{P}}(t) = e^{\rho t}\phi_{\tilde{x}\tilde{P}}(t)\phi_{\tilde{x}\tilde{x}}(t) - 4(3\theta'\theta - 2\sigma_Y\theta)\phi_{\tilde{x}\tilde{P}}(t), \phi_{\tilde{x}\tilde{P}}(T) = -2. \tag{6.15}$$

$$\dot{\phi}_{\tilde{x}\tilde{x}}(t) = e^{\rho t}\phi^2_{\tilde{x}\tilde{x}}(t) + \theta'\theta\phi_{\tilde{x}\tilde{x}}(t), \phi_{\tilde{x}\tilde{x}}(T) = 1. \tag{6.16}$$

Therefore, from (6.6) the optimal controls must be

$$\Delta'^*(t) = (\Sigma\Sigma')^{-1}\Sigma\theta - 2\left(\tilde{\Phi}\tfrac{\phi_{\tilde{x}\tilde{\Phi}}(t)}{\tilde{x}\phi_{\tilde{x}\tilde{x}}(t)} - \tilde{P}\tfrac{\phi_{\tilde{x}\tilde{P}}(t)}{\tilde{x}\phi_{\tilde{x}\tilde{x}}(t)}\right)(\Sigma\Sigma')^{-1}(\Sigma\sigma_Y' - 2\Sigma\theta). \tag{6.17}$$

$$\tilde{C}^*(t) = (\phi_{\tilde{x}}(t) + \tilde{\Phi}\phi_{\tilde{x}\tilde{\Phi}}(t) + \tilde{P}\phi_{\tilde{x}\tilde{P}}(t) + 2\tilde{x}\phi_{\tilde{x}\tilde{x}}(t))e^{\rho t}. \tag{6.18}$$

Solving (6.13), we have

$$\phi_{\tilde{x}}(t) = \frac{2\varphi(\theta'\theta + \rho)}{e^{-\theta'\theta(T-t)+\rho t} - e^{\rho T} - (\theta'\theta + \rho)}, \tag{6.19}$$

Solving (6.14), we have

$$\phi_{\tilde{x}\tilde{\Phi}}(t) = 0, \tag{6.20}$$

Solving (6.15), we have

$$\phi_{\tilde{x}\tilde{P}}(t) = \frac{2e^{4(3\theta'\theta - 2\sigma_Y\theta)(T-t)[e^{-\theta'\theta(T-t)+\rho t} - e^{\rho T} - (\theta'\theta+\rho)]}}{\theta'\theta + \rho}. \tag{6.21}$$

Solving (6.16), we have

$$\phi_{\tilde{x}\tilde{x}}(t) = \frac{(\theta'\theta + \rho)e^{-\theta'\theta(T-t)}}{(\theta'\theta + \rho) - e^{-\theta'\theta(T-t)+\rho t} + e^{\rho T}}. \tag{6.22}$$

Substituting (6.19)-(6.22) into (6.17) and (6.18), we have the following optimal portfolio and optimal discounted consumption for the PPM at time t:

$$\Delta'^*(t) = M\Sigma\theta - \tfrac{4P^m(t)M(\Sigma\sigma_Y' - 2\Sigma\theta)}{X^*(t)} \times$$
$$\tfrac{e^{(13\theta'\theta + 8\sigma_Y\theta)(T-t)}(\theta'\theta + \rho - e^{-\theta'\theta(T-t)+\rho t} + e^{\rho T})^2}{(\theta'\theta+\rho)^2}. \tag{6.23}$$

The first part of (6.23) is the Merton portfolio process. The second part is the variational part which is the intertemporal hedging term that offset any shock to the stochastic salary of a quadratic risk PPM in the scheme.

$$\begin{aligned}\tilde{C}^*(t) = {} & \tfrac{2\tilde{P}^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho}\\ & +\tfrac{2(\theta'\theta+\rho)e^{\rho t}(\tilde{X}^*(t)e^{-\theta'\theta(T-t)}-\varphi)}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}.\end{aligned} \tag{6.24}$$

At time $t=0$, we have

$$\Delta'^*(0) = M\Sigma\theta - \tfrac{4P^m(0)M(\Sigma\sigma_Y'-2\Sigma\theta)}{x_0}\tfrac{e^{(13\theta'\theta+8\sigma_Y\theta)T}(\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T})^2}{(\theta'\theta+\rho)^2}. \tag{6.25}$$

$$\tilde{C}^*(0) = \tfrac{2\tilde{P}^m(0)e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho} + \tfrac{2(\theta'\theta+\rho)(\tilde{x}_0e^{-\theta'\theta T}-\varphi)}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}}. \tag{6.26}$$

The terminal discounted consumption can be obtained by setting $t=T$ as follows:

$$\tilde{C}^*(T) = -2\tilde{P}^m(T)e^{\rho T} + \frac{2(\theta'\theta+\rho)e^{\rho T}(\tilde{X}^*(T)-\varphi)}{\theta'\theta+\rho}. \tag{6.27}$$

6.1. **Optimal Consumption of a PPM.** In this subsection, we consider the optimal consumption process of a PPM at time t. It is given by

$$C^*(t) = \Lambda(t)^{-1}\tilde{C}^*(t) = \tilde{C}^*(t)e^{(r+\|\theta\|^2)t+\theta'W(t)}. \tag{6.28}$$

It implies that

$$\begin{aligned}C^*(t) = {} & \tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho}\\ & +\tfrac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)\varphi e^{(\rho+r+\|\theta\|^2)t+\theta'W(t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}.\end{aligned} \tag{6.29}$$

The positive term $\theta'\theta$ captures the uncertainty of the financial markets. (6.29) shows that when the market become bearish, it induces the PPM not make more contributions into the pension fund and consume more, and vice versa. It is also observed that when the preference consumption rate, ρ increases, the consumption process increases over time, for all other parameters remain fixed.

At time $t=0$, (6.29) becomes

$$\begin{aligned}C^*(0) = {} & \tfrac{2P_0^m e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho}\\ & +\tfrac{2(\theta'\theta+\rho)x_0e^{-\theta'\theta T}}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)\varphi}{\theta'\theta+\rho-e^{-\theta'\theta T}+e^{\rho T}}.\end{aligned} \tag{6.30}$$

At time $t=T$, (6.29) becomes

$$\begin{aligned}C^*(T) = {} & -\tfrac{2P^m(T)e^{\rho T}[(\theta'\theta+\rho)]}{\theta'\theta+\rho} + \tfrac{2(\theta'\theta+\rho)X^*(T)e^{\rho T}}{\theta'\theta+\rho}\\ & -\tfrac{2(\theta'\theta+\rho)\varphi e^{(\rho+r+\|\theta\|^2)T+\theta'W(T)}}{\theta'\theta+\rho}.\end{aligned} \tag{6.31}$$

We can express (6.29) in terms of the parameter ψ (which represents the variance minimizer) as follows:

$$\begin{aligned}C^*(t) = {} & \tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho}\\ & +\tfrac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} - \tfrac{2(\theta'\theta+\rho)(1+2\psi E(\tilde{V}^*(T)))e^{(\rho+r+\|\theta\|^2)t+\theta'W(t)}}{2\psi(\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T})}.\end{aligned} \tag{6.32}$$

It is observe that as ψ becomes smaller and smaller for all other parameters remain constant, consumption rate reduces and vice versa. It imply that

$$\begin{aligned}\lim_{\psi\to\infty} C^*(t) = &\tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)+\rho t}[e^{-\theta'\theta(T-t)+\rho t}-e^{\rho T}-(\theta'\theta+\rho)]}{\theta'\theta+\rho}\\ &+\tfrac{2(\theta'\theta+\rho)X^*(t)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}},\end{aligned} \tag{6.33}$$

and

$$\lim_{\psi\to 0} C^*(t) = -\infty. \tag{6.34}$$

This is an intuitive result, since the PPM will consume more over time when the market is volatile and consume less when the market is not volatile. Observe that the taste of consumption will be negative if the market is absolutely riskless.

6.2. **Special Cases: $\theta = (0,0)'$, $\rho \neq 0$; $\rho = 0$, $\theta \in \mathcal{R}^2_+$. Special Case I:** Suppose $\theta = (0,0)'$ and $\rho \neq 0$, then (6.29) becomes

$$C^*(t) = \tfrac{2P^m(t)e^{\rho t}[e^{\rho t}-e^{\rho T}-\rho]}{\rho} + \tfrac{2\rho X^*(t)e^{\rho t}}{\rho-e^{\rho t}+e^{\rho T}} - \tfrac{2\rho\varphi e^{(\rho+r)t}}{\rho-e^{\rho t}+e^{\rho T}}. \tag{6.35}$$

(6.35) shows the consumption level when the investment is not in the risky assets. It implies that consumption level of the investor do not depends on the uncertainty of the market over time, but upon the riskless asset. In this case, the initial consumption level and the terminal consumption level are given respectively in (6.36) and (6.37). At $t = 0$, (6.35) becomes

$$C^*(0) = \tfrac{2P^m_0[1-e^{\rho T}-\rho]}{\rho} + \tfrac{2\rho x_0}{\rho-1+e^{\rho T}} - \tfrac{2\rho\varphi}{\rho-1+e^{\rho T}}. \tag{6.36}$$

At $t = T$, (6.35) becomes

$$C^*(T) = -2P^m(T)e^{\rho T} + 2X^*(T)e^{\rho T} - 2\varphi e^{(\rho+r)T}. \tag{6.37}$$

Special Case II: Suppose $\rho = 0$, and $\theta \in \mathcal{R}^2_+$, then (6.29) becomes

$$\begin{aligned}C^*(t) = &\tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}[e^{-\theta'\theta(T-t)}-1-\theta'\theta]}{\theta'\theta}\\ &+\tfrac{2\theta'\theta X^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1} - \tfrac{2\theta'\theta\varphi e^{(r+\|\theta\|^2)t+\theta' W(t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1}.\end{aligned} \tag{6.38}$$

(6.38) shows the consumption level when the sharpe ratio $\theta'\theta$ is not zero and the discount factor, ρ is zero. It is observe that consumption level strictly depend on the risky assets with respect to the riskless one. We observe that the market is booming, consumption reduces, and vice versa. In this case, the initial and terminal consumption level are given respectively in (6.39) and (6.40).

Similarly, at $t = 0$, (6.38) becomes

$$\begin{aligned}C^*(0) = &\tfrac{2P^m_0 e^{4(3\theta'\theta-2\sigma_Y\theta)T}[e^{-\theta'\theta T}-1-\theta'\theta]}{\theta'\theta}\\ &+\tfrac{2\theta'\theta x_0 e^{-\theta'\theta T}}{\theta'\theta-e^{-\theta'\theta T}+1} - \tfrac{2\theta'\theta\varphi}{\theta'\theta-e^{-\theta'\theta T}+1}.\end{aligned} \tag{6.39}$$

At $t = T$, (6.38) becomes

$$C^*(T) = -2P^m(T) + 2X^*(T) - 2\varphi e^{(r+\|\theta\|^2)T+\theta' W(T)}. \tag{6.40}$$

We therefore have the following propositions.

Proposition 6.2. *Let*

$$C^*_\rho(t) = \frac{2P^m(t)e^{\rho t}[e^{\rho t} - e^{\rho T} - \rho]}{\rho} + \frac{2\rho X^*(t)e^{\rho t}}{\rho - e^{\rho t} + e^{\rho T}} - \frac{2\rho\varphi e^{(\rho+r)t}}{\rho - e^{\rho t} + e^{\rho T}}$$

and

$$\begin{aligned}C_\theta^*(t) = &\tfrac{2P^m(t)e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}[e^{-\theta'\theta(T-t)}-1-\theta'\theta]}{\theta'\theta}\\ &+\tfrac{2\theta'\theta X^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1} - \tfrac{2\theta'\theta\varphi e^{(r+\|\theta\|^2)t+\theta' W(t)}}{\theta'\theta-e^{-\theta'\theta(T-t)}+1},\end{aligned}$$

then

$$C^*(t) = \begin{cases} C_\rho^*(t), \text{ if } \theta'\theta = 0,\ \rho \neq 0, \\ C_\theta^*(t), \text{ if } \theta \in \mathcal{R}_+^2,\ \rho = 0. \end{cases} \tag{6.41}$$

7. The Efficient Frontier

In this section, we determine the efficient frontier of the surplus process. Substituting (6.23) and (6.24) into (4.22), we have the dynamics of the surplus as follows:

$$\begin{aligned} d\tilde{V}^*(t) = &[(\tilde{P}^m(t)(1+f(t)) + \tilde{\Phi}(t))(\sigma_Y' - \theta)']dW(t) \\ &-2e^{\rho t}[\tilde{P}^m(t)g(t) + \tfrac{(\theta'\theta+\rho)(\tilde{P}^m(t)-\tilde{\Phi}(t))e^{-\theta'\theta(T-t)} - (\theta'\theta+\rho)\varphi}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}} \\ &+\tfrac{(\theta'\theta+\rho)\tilde{V}^*(t)e^{-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}]dt, \tilde{V}^*(0) = v_0, \end{aligned} \tag{7.1}$$

where

$$f(t) = -4e^{(13\theta'\theta+8\sigma_Y\theta)(T-t)}\frac{(\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T})^2}{(\theta'\theta+\rho)^2},$$

$$g(t) = e^{4(3\theta'\theta-2\sigma_Y\theta)(T-t)}\frac{e^{-\theta'\theta(T-t)+\rho t} - e^{\rho T} - (\theta'\theta+\rho)}{\theta'\theta+\rho}.$$

Re-writing (7.1) in a more compact form, we have

$$d\tilde{V}^*(t) = (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + F(t)'dW(t), \tilde{V}^*(0) = v_0, \tag{7.2}$$

where

$$F(t) = (\tilde{P}^m(t)(1+f(t)) + \tilde{\Phi}(t))(\sigma_Y' - \theta)$$

$$G(t) = -2e^{\rho t}[\tilde{P}^m(t)g(t) + \frac{(\theta'\theta+\rho)(\tilde{P}^m(t)-\tilde{\Phi}(t))e^{-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}]$$

$$\alpha(t) = \frac{2(\theta'\theta+\rho)e^{\rho t}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}$$

$$K(t) = -\frac{2(\theta'\theta+\rho)e^{\rho t-\theta'\theta(T-t)}}{\theta'\theta+\rho-e^{-\theta'\theta(T-t)+\rho t}+e^{\rho T}}. \tag{7.3}$$

Applying Ito Lemma on (7.2), we have

$$\begin{aligned} d\tilde{V}^{*2}(t) = &(2K(t)\tilde{V}^{*2}(t) + 2\tilde{V}^*(t)G(t) + 2\varphi\tilde{V}^*(t)\alpha(t) \\ &+F(t)'F(t))dt + F(t)'dW(t), \tilde{V}^{*2}(0) = v_0^2, \end{aligned} \tag{7.4}$$

Taking the mathematical expectation of (7.2) and (7.4), we have

$$dE(\tilde{V}^*)(t) = (K(t)E(\tilde{V}^*(t)) + E(G(t)) + \varphi\alpha(t))dt, E(\tilde{V}^*)(0) = v_0, \tag{7.5}$$

$$\begin{aligned} dE(\tilde{V}^{*2})(t) = &(2K(t)E(\tilde{V}^{*2})(t) + 2E(\tilde{V}^*)(t)E(G(t)) + 2\varphi E(\tilde{V}^*)(t)\alpha(t) \\ &+E(F(t))'E(F(t)))dt, E(\tilde{V}^{*2})(0) = v_0^2, \end{aligned} \tag{7.6}$$

Solving the ordinary differential equations (ODEs), (7.5) and (7.6), we have followings:

$$E(\tilde{V}^*)(t) = A(t)e^{\int_0^t K(s)ds} + \varphi e^{\int_0^t K(s)ds}\int_0^t e^{-\int_0^t K(s)ds}\alpha(s)ds, \tag{7.7}$$

where

$$A(t) = v_0 + \int_0^t e^{-\int_0^t K(s)ds} G(s)ds.$$

(7.8)

$$E(\tilde{V}^{*2})(t) = v_0^2 e^{2\int_0^t K(\tau)d\tau} + e^{2\int_0^t K(\tau)d\tau} \int_0^t E(F(\tau))' E(F(\tau))d\tau +$$
$$2e^{2\int_0^t K(\tau)d\tau} \int_0^t E(G(\tau)) \left(A(\tau) e^{\int_0^\tau K(u)du} + \varphi e^{\int_0^\tau K(u)du} \int_0^\tau e^{-\int_0^s K(u)du} \alpha(s)ds \right) d\tau +$$
$$2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \alpha(\tau) \left(A(\tau) e^{\int_0^\tau K(u)du} + \varphi e^{\int_0^\tau K(u)du} \int_0^\tau e^{-\int_0^s K(u)du} \alpha(s)ds \right) d\tau$$

Simplifying (7.8), we have

(7.9)

$$E(\tilde{V}^{*2})(t) = v_0^2 e^{2\int_0^t K(\tau)d\tau} + e^{2\int_0^t K(\tau)d\tau} \int_0^t E(F(\tau))' E(F(\tau))d\tau$$
$$+2e^{2\int_0^t K(\tau)d\tau} \int_0^t E(G(\tau)) A(\tau) e^{\int_0^\tau K(u)du} d\tau + 2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \alpha(\tau) A(\tau) e^{\int_0^\tau K(u)du} d\tau$$
$$+2\varphi e^{2\int_0^t K(\tau)d\tau} \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} E(G(\tau)) \alpha(s) ds d\tau$$
$$+2\varphi^2 e^{2\int_0^t K(\tau)d\tau} \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} \alpha(\tau)\alpha(s) ds d\tau$$

Re-writing (7.9) in compact form, we have

$$\begin{aligned} E(\tilde{V}^{*2})(t) = {} & v_0^2 e^{2\int_0^t K(\tau)d\tau} + D_1(t) e^{2\int_0^t K(\tau)d\tau} \\ & +2D_2(t)\varphi e^{2\int_0^t K(\tau)d\tau} + 2D_3(t)\varphi^2 e^{2\int_0^t K(\tau)d\tau} \end{aligned} \tag{7.10}$$

where

$$D_1(t) = \int_0^t E(F(\tau))' E(F(\tau))d\tau + 2\int_0^t E(G(\tau)) A(\tau) d\tau$$
$$D_2(t) = \int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} E(G(\tau)) \alpha(s) ds d\tau + \int_0^t \alpha(\tau) A(\tau) d\tau$$
$$D_3(t) = 2\int_0^t \int_0^\tau e^{\int_0^\tau K(u)du} e^{-\int_0^s K(u)du} \alpha(\tau)\alpha(s) ds d\tau$$

At $t = T$, (7.7) and (7.10) becomes:

$$E(\tilde{V}^{*})(T) = A(T)\gamma + \varphi\gamma\omega, \tag{7.11}$$

where, $\gamma = e^{\int_0^T K(u)du}$, $\omega = \int_0^T e^{-\int_0^T K(s)ds} \alpha(s)ds$.

Lemma 7.1. *Suppose that $K(t)$ satisfies (7.14), then*

$$\gamma = \left(\frac{(\theta'\theta + \rho) e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})} \right)^2.$$

Proof: Using (7.14), we have that

$$\int_0^t K(u)du = 2\log_e \left(\frac{e^{(\theta'\theta+\rho)t} - e^{(\theta'\theta+\rho)T} - (\theta'\theta + \rho) e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})} \right).$$

It implies that

$$e^{\int_0^t K(u)du} = \left(\frac{e^{(\theta'\theta+\rho)t} - e^{(\theta'\theta+\rho)T} - (\theta'\theta + \rho) e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})} \right)^2.$$

Therefore, setting $t = T$, we have

$$\gamma = \left(\frac{(\theta'\theta + \rho) e^{\theta'\theta T}}{1 - e^{\theta'\theta T}(\theta'\theta + \rho + e^{\rho T})} \right)^2.$$

Using Lemma 7.1, the second moments of the surplus process becomes

$$E(\tilde{V}^{*2})(T) = v_0^2\gamma^2 + D_1(T)\gamma^2 + 2D_2(T)\varphi\gamma^2 + D_3(T)\varphi^2\gamma^2 \tag{7.12}$$

Substituting (7.11) into (7.12), we have

$$\begin{aligned} E(\tilde{V}^{*2})(T) = v_0^2\gamma^2 + D_1(T)\gamma^2 + \tfrac{2\gamma D_2(T)}{\omega}(E(\tilde{V}^*(T)) - A(T)\gamma) \\ + \tfrac{D_3(T)}{\omega^2}(E(\tilde{V}^*(T)) - A(T)\gamma)^2 \end{aligned} \tag{7.13}$$

The variance of the discounted surplus process for the stakeholders is

$$\begin{aligned}
Var(\tilde{V}^*(T)) &= E(\tilde{V}^{*2})(T) - (E(\tilde{V}^*)(T))^2 \\
&= v_0^2\gamma^2 + D_1(T)\gamma^2 + \frac{2\gamma D_2(T)}{\omega}(E(\tilde{V}^*(T)) - A(T)\gamma) \\
&+ \frac{D_3(T)}{\omega^2}(E(\tilde{V}^*(T)) - A(T)\gamma)^2 - (E(\tilde{V}^*)(T))^2 \\
&= v_0^2\gamma^2 - A(T)^2\gamma^2 + D_1(T)\gamma^2 \\
&+ 2\gamma\left(\frac{D_2(T)}{\omega} - A(T)\right)(E(\tilde{V}^*(T)) - A(T)\gamma) \\
&+ \left(\frac{D_3(T)}{\omega^2} - 1\right)(E(\tilde{V}^*(T)) - A(T)\gamma)^2 \\
&= v_0^2\gamma^2 - A(T)^2\gamma^2 + D_1(T)\gamma^2 - \frac{\gamma^2\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} \\
&+ \left(\frac{D_3(T)}{\omega^2} - 1\right)[\gamma^2\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)^2} + 2\gamma\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} \\
&\times (E(\tilde{V}^*(T)) - A(T)\gamma) + (E(\tilde{V}^*(T)) - A(T)\gamma)^2] \\
&= \gamma^2 Q + \left(\frac{D_3(T)}{\omega^2} - 1\right)\left[\gamma\frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)}{\left(\frac{D_3(T)}{\omega^2} - 1\right)} + E(\tilde{V}^*(T)) - A(T)\gamma\right]^2
\end{aligned}$$

where

$$Q = v_0^2 - A(T)^2 + D_1(T) - \frac{\left(\frac{D_2(T)}{\omega} - A(T)\right)^2}{\left(\frac{D_3(T)}{\omega^2} - 1\right)}.$$

Therefore, the efficient frontier of discounted surplus is obtain as

$$E(\tilde{V}^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}} + \frac{\sqrt{\sigma^2_{V^*(T)} - \gamma^2 Q}}{\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}. \tag{7.14}$$

From (7.14), shows a kind quadratic relation between optimal discounted surplus and its variance. The minimum possible variance, $Var(\tilde{V}(^*(T)) = \gamma^2 Q \geq 0$, could be attained when the stakeholder borrows money from the total amount of surplus at time $t = 0$ for T years, so that

$$E(\tilde{V}(^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}.$$

We now establish the efficient frontier of the optimal terminal surplus of the stakeholders. The expected surplus and the variance, $\sigma^2_{V^*(T)}$, at time T are related by the following (7.15).

Proposition 7.1. *Suppose (7.14) holds and $E(\Lambda(T)) = e^{-(r+2\|\theta\|^2)T}$, then*

$$E(V^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))e^{(r+2\|\theta\|^2)T}}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}} + \frac{e^{(r+2\|\theta\|^2)T}\sqrt{\sigma^2_{V^*(T)} - \gamma^2 Q}}{\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}. \tag{7.15}$$

From (7.15), shows the quadratic relation between surplus and its variance. The minimum possible variance, $Var(V^*(T)) = \gamma^2 Q \geq 0$, could be attained when the stakeholder borrows money from the total amount of surplus at time $t = 0$ for T years, so that

$$E(V^*(T)) = \frac{\gamma(2\omega A(T) - D_2(T))e^{(r+2\|\theta\|^2)T}}{\omega\sqrt{\left(\frac{D_3(T)}{\omega^2} - 1\right)}}.$$

We observe that if $\frac{D_3(T)}{\omega^2} = 1$, we have infinite slope, if $\frac{D_3(T)}{\omega^2} > 1$, we have real slope and complex slope if $\frac{D_3(T)}{\omega^2} < 1$.

8. Optimal Pension Benefit for a PPM at Retirement

In this section, we consider the optimal benefit that will accrued to the PPM at retirement. By definition, the benefit that will accrued to a PPM at the final time, T is given by

$$P(T) = P^m(T) + \Theta_T(P^m(T)). \tag{8.1}$$

Proposition 8.1. *: Let $\tilde{\Theta}^*_T(\tilde{P}^m(T))$ be the optimal discounted surplus function at the final time, T, then*

$$\tilde{P}^*(T) = \tilde{P}^m(T) + \tilde{\Theta}^*_T(\tilde{P}^m(T)), \tag{8.2}$$

with

$$\tilde{V}^*(T) = v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + \int_0^T F(t)'dW(t). \tag{8.3}$$

Corollary 8.1. *: Let $\Theta^*_T(P^m(T))$ be the optimal surplus function at the final time, T, then*

$$P^*(T) = P^m(T) + \Theta^*_T(P^m(T)), \tag{8.4}$$

with

$$V^*(T) = \frac{1}{\Lambda(T)}\left(v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt + \int_0^T F(t)'dW(t)\right). \tag{8.5}$$

Corollary 8.2. *: Let $E(\tilde{\Theta}^*_T(\tilde{P}^m(T)))$ be the expected optimal discounted surplus function at the final time, T, then*

$$E(\tilde{P}^*(T)) = E(\tilde{P}^m(T)) + E(\tilde{\Theta}^*_T(\tilde{P}^m(T))), \tag{8.6}$$

with

$$E(\tilde{V}^*(T)) = v_0 + \int_0^T (K(t)E(\tilde{V}^*(t)) + E(G(t)) + \varphi\alpha(t))dt. \tag{8.7}$$

Corollary 8.3. *: Let $E(\Theta^*_T(P^m(T)))$ be the optimal expected surplus function at the final time, T, then*

$$E(P^*(T)) = E(P^m(T)) + E(\Theta^*_T(P^m(T))), \tag{8.8}$$

with

$$E(V^*(T)) = \frac{1}{E(\Lambda(T))} E\left[v_0 + \int_0^T (K(t)\tilde{V}^*(t) + G(t) + \varphi\alpha(t))dt\right]. \tag{8.9}$$

9. Numerical Illustration

In this section, we give numerical illustration of our results in the previous sections. The aim of this numerical illustration is to observe the nature of the expected final optimal surplus (both discounted case and the real case) as against the final standard deviation, the initial optimal consumption, optimal final pension benefits, minimum pension benefits, with respect to the terminal time to retirement, the expected final surplus and parameter, ψ given to the minimization of the variance. The values of parameters that we consider are as followings.

$c = 0.15$, $\eta = 0.01$, $r = 0.04$, $\psi = 10$, $\xi = 0.3$, $\rho = 0.01$, $\mu = 0.09$, $\sigma_Y = (0.25, 0.32)$, $\theta_I = 0.02$, $\beta = 0.0292$, $y_0 = 0.8$, $x_0 = 1$, $\sigma_S = 0.35$, $\sigma_1 = 0.23$.

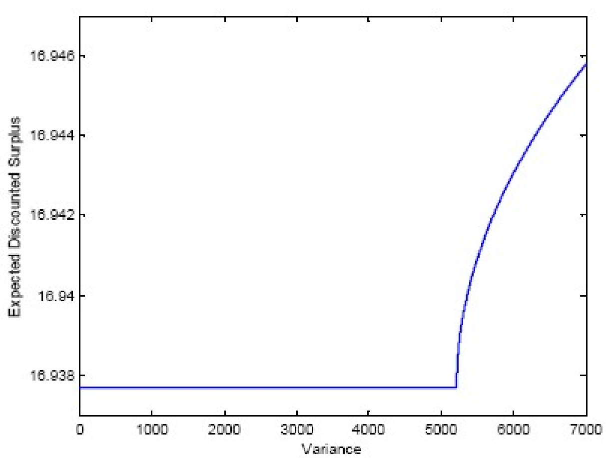

Figure 1. Efficient Frontier

Table 1: Initial Optimal Consumption, C_0, with $\tilde{z} = E(\tilde{V}^*(T))$

$\tilde{z}$	$\psi = 0.1$ $T = 1$	$\psi = 0.1$ $T = 2$	$\psi = 0.1$ $T = 10$	$\psi = 0.1$ $T = 20$	$\psi = 1$ $T = 1$	$\psi = 1$ $T = 2$	$\psi = 1$ $T = 10$	$\psi = 1$ $T = 20$
6	-10.4172	-7.8403	-13.3382	-21.0730	-5.9279	-4.8594	-12.5551	-20.6823
10	-14.4077	-10.4901	-14.0343	-21.4202	-9.9184	-7.5091	-13.2512	-21.0296
20	-24.3838	-17.1144	-15.7745	-22.2883	-19.8945	-14.1334	-14.9914	-21.8877
30	-34.3600	-23.7387	-17.5148	-23.1564	-29.8707	-20.7577	-16.7317	-22.7658
35	-39.3481	-27.0509	-18.3849	-23.5905	-34.8588	-24.0699	-17.6018	-23.1998

Continuation of Table 1

$\tilde{z}$	$\psi = 10$ $T = 1$	$\psi = 10$ $T = 2$	$\psi = 10$ $T = 10$	$\psi = 10$ $T = 20$	$\psi = \infty$ $T = 1$	$\psi = \infty$ $T = 2$	$\psi = \infty$ $T = 10$	$\psi = \infty$ $T = 10$
6	-5.4790	-4.5613	-2.4767	-20.6433	-5.4266	-4.2949	-3.36116	-1.4281
10	-9.4695	-7.2110	-13.1728	-20.9905	-9.4443	-6.9929	-4.1453	-1.8692
20	-19.4456	-13.8353	-14.8131	-21.8586	-19.4886	-13.7380	-6.1057	-2.9719
30	-29.4200	-20.4597	-16.6534	-22.7267	-29.5328	-20.4830	-8.0661	-4.0747
35	-34.4099	-23.7718	-17.5235	-23.1608	-34.5550	-23.8556	-9.0463	-4.6260

Table 2: EODS, EODPB and Minimum Pension Benefit for a PPM

T	EV^*	$E\tilde{V}^*$	$E\tilde{P}^m$	EP^m	$E\tilde{P}^*$ $h = 0.2$	EP^* $h = 0.2$	$E\tilde{P}^*$ $h = 0.3$	EP^* $h = 0.3$	$E\tilde{P}^*$ $h = 0.4$	EP^* $h = 0.4$
1	-2.8513	-2.7400	0.1202	0.1189	-2.1676	-2.2573	-1.8816	-1.9603	-1.5956	-1.6632
2	-0.7245	0.2409	0.2409	0.2357	-0.5314	-0.5807	-0.4349	-0.4786	-0.3383	-0.3766
3	0.2708	0.2402	0.3622	0.3567	0.2646	0.2868	0.2768	0.2948	0.2890	0.3027
4	1.4329	1.2211	0.4844	0.4640	1.0737	1.2391	1.0001	1.1422	0.9264	1.0453
5	2.7673	2.2657	0.6077	0.5758	1.9341	2.3290	1.7683	2.1099	1.6025	1.8907
6	4.2720	3.3605	0.7322	0.6853	2.8348	3.5548	2.5720	3.1963	2.3091	2.8377
10	11.7552	7.8797	1.2463	1.1187	6.5531	9.6279	5.8897	8.5643	5.2264	7.5006
20	40.9465	18.3985	2.7168	2.1890	15.2621	33.1950	13.6940	29.3193	12.1258	25.4435

Continuation of Table 2

T	$E\tilde{P}^*$ $h = 0.5$	EP^* $h = 0.5$	$E\tilde{P}^*$ $h = 0.6$	EP^* $h = 0.6$	$E\tilde{P}^*$ $h = 0.7$	EP^* $h = 0.7$	$E\tilde{P}^*$ $h = 0.8$	EP^* $h = 0.8$
1	-1.3097	-1.3662	-1.0237	-1.0692	-0.7377	-0.7722	-0.4518	-0.4751
2	-0.2418	-0.2745	-0.1453	-0.1725	-0.0487	-0.0704	0.0478	0.03161
3	0.3012	0.3107	0.3134	0.3187	0.3256	0.3267	0.3378	0.3347
4	0.8527	0.9484	0.7791	0.8515	0.7054	0.7546	0.6318	0.6577
5	1.4367	1.6715	1.2709	1.4524	1.1051	1.2332	0.9393	1.0141
6	2.0463	2.4791	1.7835	2.1205	1.5207	1.7620	1.2578	1.4034
10	4.5630	6.4370	3.8997	5.3733	3.2363	4.3097	2.5730	3.2460
20	10.5576	21.5678	8.9895	17.6920	7.4213	13.8163	5.8531	9.9405

EODS denotes Expected Optimal Discounted Surplus, EODPB denotes Expected Optimal Discounted Pension Benefit.

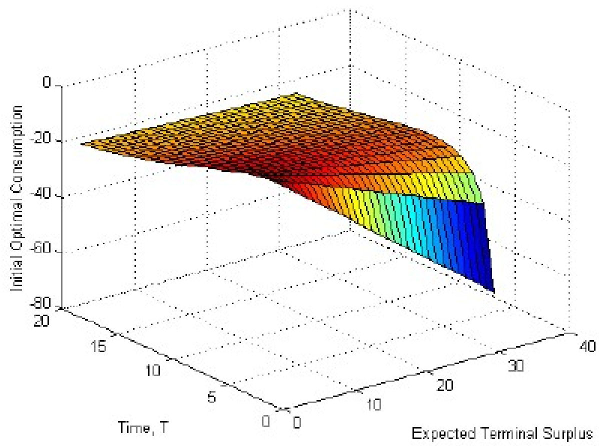

Figure 2. Initial Optimal Consumption

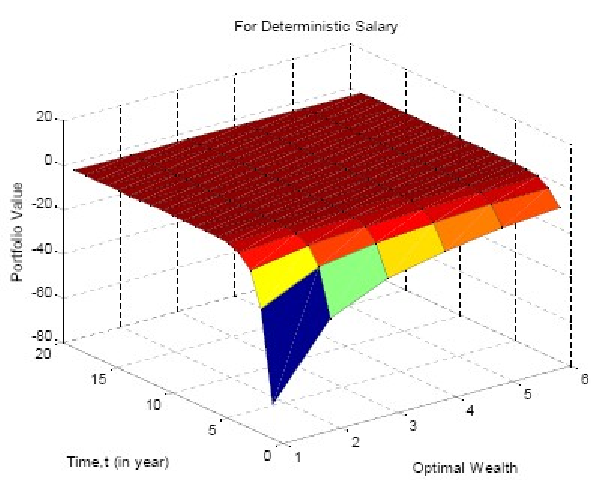

Figure 3. Portfolio Value in Cash Account

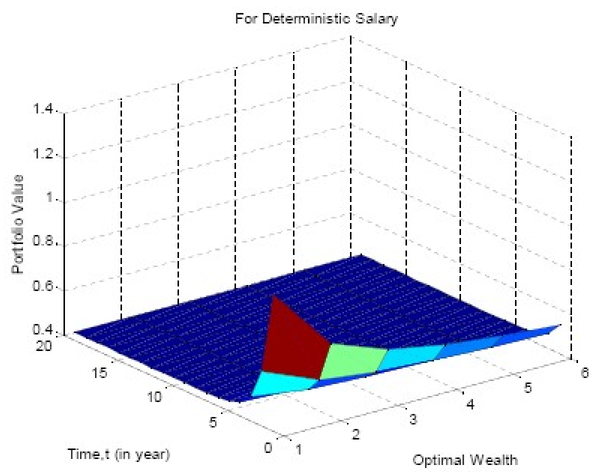

Figure 4. Portfolio Value in Index Bond

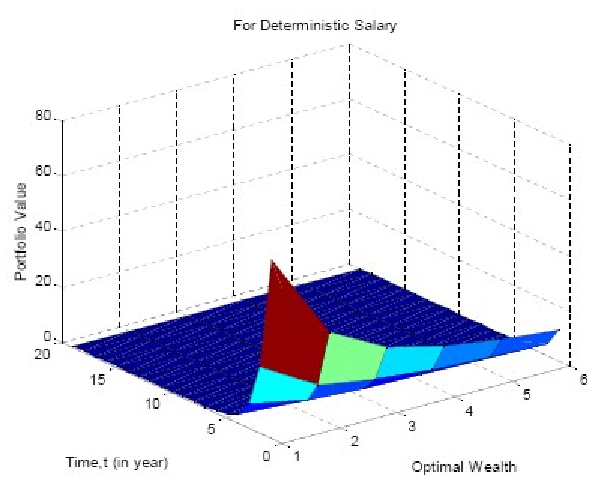

Figure 5. Portfolio Value in Stock

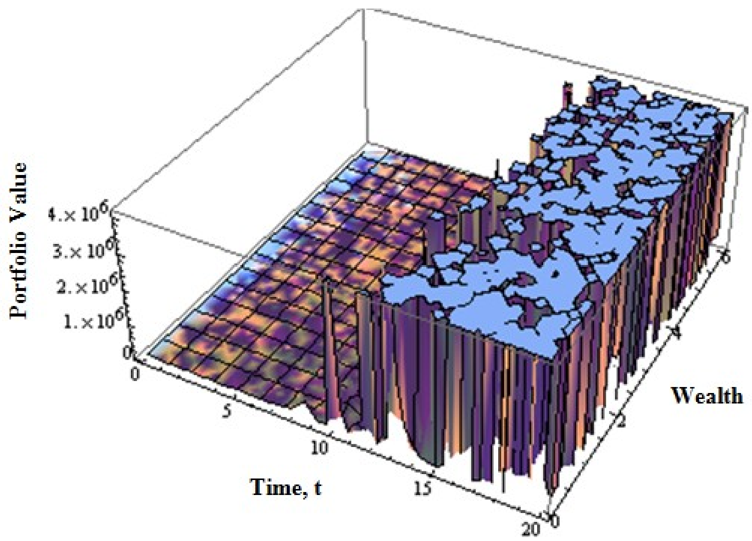

Figure 6. Portfolio Value in Index bond for Stochastic Salary

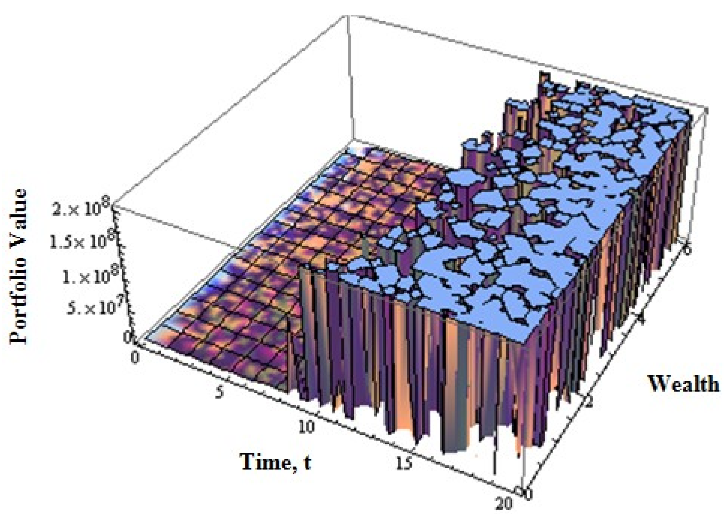

Figure 7. Portfolio Value in Stock for Stochastic Salary

Table 1 shows the initial optimal consumption of a PPM at various values of expected discounted surplus in a varying value of variance minimizer parameter, ψ at time $T = 1, 2, 10$ and 20 years. We observed at different values of ψ and T, that as the expected optimal surplus increases, the initial optimal consumption decreases and vice versa. This shows that the positive growth of the surplus (resulting from the positive growth of the financial market and effective management on the part of the PFA) is capable of discouraging consumption. It was also observed that as ψ increases, the initial optimal consumption increases. This shows that as the market continues to be volatile, consumption rate will continue increase and vice versa.

Table 2 show the expected optimal surplus, expected discounted optimal surplus, expected discounted minimum pension benefit, expected minimum pension benefit, expected discounted total benefit and expected total pension benefit for a PPM, at time $T = 1, 2, 3, 4, 5, 6, 10$ and 20 years, and at different value of h (i.e., the proportion of the surplus that will accrued to the PFA). It was observed that the optimal surplus increases in time, T. It also observed that as the value of h increases, the benefit of the PPM will decrease, and vice versa, which is an obvious result. Hence, what ever the bargain between the PPM and PFA on the sharing of the surplus will be, the values of the PPM minimum and total pension benefit are given in table 2.

Figure **??** shows the efficient frontier of the surplus process. We observed that the shape of the efficient frontier is parabolic in nature. It is also observed that the minimum possible variance is attained when the PFA borrows the amount of about 5200 from the total surplus at time $t = 0$ for time $t = T$. Figure **??** shows the initial optimal consumption of a PPM with $0 - 40$ optimal surplus at time period $0 - 20$ years. It is observed that the initial optimal consumption of a PPM remain negative over the time period, T. This confirm the results obtained in Table 1. Figure **??** shows the portfolio value of the investment in cash account and figure **??** and figure **??** show the portfolio values of a PPM in index bond and stock respectively, under deterministic salary of a PPM. Figure **??** and figure **??** show the portfolio values of a PPM in index bond and stock under stochastic salary. Figure **??**, figure **??** and figure **??** tell us that the fund should be invested in index bond and stock only and that cash account should be shorten and then invest in the risky assets (which include index bond and stock), with high proportion of it being invested in stock over time in order to attain the required target.

10. Conclusion

This paper have studied the management of a stochastic pension funding process of a defined contributory pension scheme. The objectives are to determine the minimum pension benefits, total pension benefits, optimal consumption and optimal investment strategies maximizing the expected terminal surplus and simultaneously minimizing the variance of the terminal surplus. The financial market is made up of cash account, index bond and stock. The salary of the pension plan member is stochastic. The problem was formulated as a modified mean-variance optimization problem and was solved using dynamic programming approach.

The efficient frontier which was found to be nonlinear (i.e., possess a parabolic shape). The optimal investment strategies have two components. The first component depends ultimately on the risky assets and its correlation. The second component is proportional to the ratio of the present expected value of PPM's minimum benefit to the optimal wealth. The second component is the inter-temporal hedging terms that offset any shock to the stochastic finding overtime.

The optimal consumption (both for real and discounted cases) plan have three components. First component depends on the current level of minimum pension benefit, with a coefficient involving the instantaneous variance of salary, preference rate of consumption and risky assets. The second component depends on the optimal wealth, preference rate of consumption and risky assets. The third component is proportional to the present expected value of discounted surplus planned, with coefficient involving preference rate of consumption, variance minimizer, short term interest rate, risky assets and the Brownian motion term, which shows that the consumption of the PPM is stochastic. We found the as the variance minimizing parameter tends to zero, the consumption level tends to negative infinity. This shows that PPM will consume more over time when the market is volatile and consume less when the market is less volatile. Also, the taste of consumption will be negative, if the market is absolutely riskless.

The optimal terminal surplus for the stakeholders was determined in this paper. The pension fund administrator (PFA) was encouraged by sharing the surplus arising from the investment with the PPM. This strategy will go a long way increasing the final benefit that will accrued to the PPM at retirement. The PFA charge propositional administrative costs (AC) for the management of the fund. This costs is on the PPM stochastic contributions into the scheme.

The minimum pension benefit is taken not to be less than the gross contributions of the PPM. It implies that the total benefit must be greater or equal to the minimum pension benefit.

A numerical illustrations show the analytical results and models established in the paper.

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