KONURALP JOURNAL OF MATHEMATICS
VOLUME 4 NO. 2 PP. 255–270 (2016) ©KJM

ON SHERMAN'S TYPE INEQUALITIES FOR n-CONVEX FUNCTION WITH APPLICATIONS

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ABSTRACT. New generalizations of Sherman's inequality for convex functions of higher order are obtained by using Hermite's interpolating polynomials and Green's function. The Ostrowski and Grüss type bounds for the identity related to generalized Sherman's inequality are established. Some applications are discussed.

1. INTRODUCTION

Let $I \subset \mathbb{R}$ be an interval and $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$, where $m \geq 2$. Let $x_{[i]}$ and $y_{[i]}$ denote the elements of $\mathbf{x}$ and $\mathbf{y}$ sorted in decreasing order. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ or $\mathbf{y}$ is *majorized* by $\mathbf{x}$ and write $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]}, \quad k = 1, ..., m-1, \tag{1.1}$$

and the equation holds for $k = m$.

In majorization theory, the next result, well known as *Majorization theorem,* plays a very important role (see [15]).

Theorem 1.1. *Let $\phi : I \to \mathbb{R}$ be a convex continuous function on an interval I and $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$. If $\mathbf{y} \prec \mathbf{x}$, then*

$$\sum_{i=1}^{m} \phi(y_i) \leq \sum_{i=1}^{m} \phi(x_i).$$

Recently some generalizations of majorization theorem with applications are obtained (see [1]-[5], [12]).

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. majorization, n-convexity, Schur-convexity, Sherman's theorem, Hermite's interpolating polynomial, Čebyšev functional, Grüss type inequalities, Ostrowsky-type inequalities, exponentially convex functions, log-convex functions, means.

The research of the third author has been fully supported by Croatian Science Foundation under the project 5435.

S. Sherman [16], considering a weighted relation of majorization

$$\sum_{i=1}^{k} v_i y_i \leq \sum_{j=1}^{l} u_j x_j,$$

for nonnegative weights u_j and v_i, proved the general result which include the row stochastic $k \times l$ matrix, i.e. matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{kl}(\mathbb{R})$ such that

$$\begin{aligned} a_{ij} &\geq 0 \quad \text{for all } i = 1, ..., k,\ j = 1, ..., l, \\ \sum_{j=1}^{l} a_{ij} &= 1 \quad \text{for all } i = 1, ..., k, \end{aligned}$$

and holds under relations

$$\begin{aligned} y_i &= \sum_{j=1}^{l} x_j a_{ij}, \quad \text{for} \quad i = 1, ..., k, \\ u_j &= \sum_{i=1}^{k} v_i a_{ij}, \quad \text{for} \quad j = 1, ..., l. \end{aligned} \tag{1.2}$$

His result can be formulated as the following theorem.

Theorem 1.2. *Let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^k$, $\mathbf{u} \in [0, \infty)^l$ *and* $\mathbf{v} \in [0, \infty)^k$ *be such that (1.2) holds for some row stochastic matrix* $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$. *Then for every convex function* $\phi : [\alpha, \beta] \to \mathbb{R}$ *we have*

$$\sum_{q=1}^{k} v_q \phi(y_q) \leq \sum_{p=1}^{l} u_p \phi(x_p). \tag{1.3}$$

From Sherman's theorem we can easily get Majorization theorem by setting $k = l$ and $\mathbf{v} = (1, ..., 1)$. Specially, when $k = l$ and all weights $v_i = u_j$ are equal, the condition (1.2), i.e. $\mathbf{u} = \mathbf{vA}$, assures the stochasticity on columns, so in that case we deal with *doubly stochastic matrices.* It is well known that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^l$ is valid

$$\mathbf{y} \prec \mathbf{x} \quad \text{if and only if} \quad \mathbf{y} = \mathbf{xA}$$

for some doubly stochastic matrix $\mathbf{A} \in \mathcal{M}_{ll}(\mathbb{R})$.

The aim of this paper is to establish generalizations of Sherman's result which hold for real, not necessary nonnegative vectors $\mathbf{u}$, $\mathbf{v}$ and matrix $\mathbf{A}$ and for convex functions of higher order. Recently some related results are obtained (see [6], [10]).

The class of convex functions of higher order, i.e. the notion of n-convexity was defined in terms of divided differences by T. Popoviciu. A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, $n \geq 0$, if its nth order divided differences $[x_0, ..., x_n; \phi]$ are nonnegative for all choices of $(n + 1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$. Thus, a 0-convex function is nonnegative, 1-convex function is nondecreasing and 2-convex function is convex in the usual sense. If $\phi^{(n)}$ exists, then ϕ is n-convex iff $\phi^{(n)} \geq 0$ (see [15]).

At the end we point definition and some basic facts about exponential convexity. For more details see [6], [11]. Here I denotes an open interval in $\mathbb{R}$.

Definition 1.1. [14] For a fixed $n \in \mathbb{N}$, a function $\phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} p_i p_j \phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, ..., n$. A function $\phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.1. Let $\phi : I \to \mathbb{R}$ be a given function.

- ϕ is exponentially convex in the Jensen sense on I, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.
- A positive function ϕ is log-convex, i.e. $\log \phi$ is convex, in the Jensen sense on I iff it is 2-exponentially convex in the Jensen sense on I.
- A positive function ϕ is log-convex on I if it is continuous and log-convex in the Jensen sense on I
- A positive exponentially convex function ϕ on I is also log-convex on I.

2. Preliminaries

We use notations and terminology from [7].

Let $-\infty < \alpha < \beta < \infty$ and let $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$ be r $(r \geq 2)$ distinct points. For $\phi \in C^n([\alpha, \beta])$ $(n \geq r)$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists, such that **Hermite conditions** hold

$$\rho_H^{(i)}(a_j) = \phi^{(i)}(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r, \tag{H}$$

where $\sum\limits_{j=1}^{r} k_j + r = n$.

Specially, for $r = 2, 1 \leq m \leq n-1,\ k_1 = m-1$ and $k_2 = n-m-1$ we have **type** $(m, n-m)$ **conditions:**

$$\rho_{(m,n)}^{(i)}(\alpha) = \phi^{(i)}(\alpha),\ 0 \leq i \leq m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \phi^{(i)}(\beta),\ 0 \leq i \leq n-m-1.$$

For $n = 2m$, $r = 2$ and $k_1 = k_2 = m-1$ we have **two-point Taylor conditions:**

$$\rho_{2T}^{(i)}(\alpha) = \phi^{(i)}(\alpha),\ \rho_{2T}^{(i)}(\beta) = \phi^{(i)}(\beta),\ 0 \leq i \leq m-1.$$

Theorem 2.1. *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leq a_1 < a_2 \cdots < a_r \leq \beta$ be r $(r \geq 2)$ distinct points and $\phi \in C^n([\alpha, \beta])$. Then*

$$\phi(t) = \rho_H(t) + R_{H,n}(\phi, t), \tag{2.1}$$

where $\rho_H(t)$ is the Hermite inrepolating polynomial, i.e.

$$\rho_H(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i)}(a_j),$$

H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \Bigg|_{t=a_j} (t-a_j)^k, \tag{2.2}$$

with

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{2.3}$$

and the remainder is given by

$$R_{H,n}(\phi,t)=\int_\alpha^\beta G_{H,n}(t,s)\phi^{(n)}(s)ds,$$

where $G_{H,n}(t,s)$ *is defined by*

$$G_{H,n}(t,s)=\begin{cases}\sum\limits_{j=1}^{l}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\le t,\\ -\sum\limits_{j=l+1}^{r}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\ge t,\end{cases}\tag{2.4}$$

for all $a_l\le s\le a_{l+1}$*;* $l=0,\dots,r$ *with* $a_0=\alpha$ *and* $a_{r+1}=\beta$*.*

Remark 2.1. For type $(m,n-m)$ conditions, from Theorem 2.1 we have

$$\phi(t)=\rho_{(m,n)}(t)+R_{(m,n)}(\phi,t)$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e.

$$\rho_{(m,n)}(t)=\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i)}(\beta),$$

with

$$\tau_i(t)=\frac{1}{i!}(t-\alpha)^i\Big(\frac{t-\beta}{\alpha-\beta}\Big)^{n-m}\sum_{p=0}^{m-1-i}\binom{n-m+p-1}{p}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p,\tag{2.5}$$

$$\eta_i(t)=\frac{1}{i!}(t-\beta)^i\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}\sum_{p=0}^{n-m-1-i}\binom{m+p-1}{p}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^p,\tag{2.6}$$

and the remainder is given by

$$R_{(m,n)}(\phi,t)=\int_\alpha^\beta G_{(m,n)}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s)=\begin{cases}\sum\limits_{j=0}^{m-1}\Big[\sum\limits_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p\Big]\times\\ \frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^{n-m}, & \alpha\le s\le t\le\beta\\ -\sum\limits_{i=0}^{n-m-1}\Big[\sum\limits_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\Big(\frac{\beta-t}{\beta-\alpha}\Big)^q\Big]\times\\ \frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^{m}, & \alpha\le t\le s\le\beta.\end{cases}\tag{2.7}$$

For Type Two-point Taylor conditions, from Theorem 2.1 we have

$$\phi(t)=\rho_{2T}(t)+R_{2T}(\phi,t)$$

where $\rho_{2T}(t)$ is the two-point Taylor interpolating polynomial i.e,

$$\rho_{2T}(t)=\sum_{i=0}^{m-1}\sum_{p=0}^{m-1-i}\binom{m+p-1}{p}\Big[\frac{(t-\alpha)^i}{i!}\Big(\frac{t-\beta}{\alpha-\beta}\Big)^m\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^p\phi^{(i)}(\alpha)$$
$$+\frac{(t-\beta)^i}{i!}\Big(\frac{t-\alpha}{\beta-\alpha}\Big)^m\Big(\frac{t-\beta}{\alpha-\beta}\Big)^p\phi^{(i)}(\beta)\Big]$$

and the remainder is given by

$$R_{2T}(\phi, t) = \int_\alpha^\beta G_{2T}(t, s)\phi^{(n)}(s)ds$$

with

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum\limits_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \le t; \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum\limits_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \ge t; \end{cases} \tag{2.8}$$

where $p(t,s) = \frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s) = p(s,t)$, $\forall\, t, s \in [\alpha, \beta]$.

The following lemma describes the positivity of $G_{H,n}(t,s)$ (see [8], [13]).

Lemma 2.1. *The function $G_{H,n}(t,s)$, defined by (2.4), has the following properties:*

(i) $\frac{G_{H,n}(t,s)}{\omega(t)} > 0$, $a_1 \le t \le a_r$, $a_1 < s < a_r$;

(ii) $G_{H,n}(t,s) \le \frac{1}{(n-1)!(\beta-\alpha)} |\omega(t)|$;

(iii) $\int_\alpha^\beta G_{H,n}(t,s)ds = \frac{\omega(t)}{n!}$.

Green's function of Lagrange type is defined on $[\alpha,\beta] \times [\alpha,\beta]$ by

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \le s \le t \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \le s \le \beta \end{cases}. \tag{2.9}$$

It is convex and continuous in both variables (see [17]).

3. Main results

The next identity related to generalized Sherman's inequality holds.

Theorem 3.1. *Let $n \ge 4$ and $\phi \in C^n([\alpha,\beta])$, $\alpha \le a_1 < a_2 \cdots < a_r \le \beta$ $(r \ge 2)$ be the given points and $k_1, ..., k_r \in \mathbb{N}$ with $\sum\limits_{j=1}^{r} k_j + r = n$. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^k$, $\mathbf{u} \in \mathbb{R}^l$ and $\mathbf{v} \in \mathbb{R}^k$ be such that (1.2) holds for some matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$ whose entries satisfy the condition $\sum\limits_{j=1}^{l} a_{ij} = 1$, $i = 1, ..., k$. Then*

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &= \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j) dt \\ &+ \int_\alpha^\beta \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] G_{H,n-2}(t,s) \phi^{(n)}(s) ds dt, \end{aligned} \tag{3.1}$$

where G, H_{ij} and $G_{H,n-2}$ are defined as in (2.9), (2.2) and (2.4), respectively.

Proof. For any function $\phi \in C^2([\alpha,\beta])$, we can show integration by parts that the following identity holds

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_\alpha^\beta G(x,t)\phi''(t)dt, \tag{3.2}$$

where G is defined by (2.9).

By an easy calculation, applying (3.2) in $\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q)$ and using (1.2), we get

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &= \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] \phi''(t) dt. \end{aligned} \tag{3.3}$$

By Theorem 2.1, the function $\phi''(t)$ can be expressed as

$$\phi''(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j) + \int_{\alpha}^{\beta} G_{H,n-2}(t,s) \phi^{(n)}(s) ds. \tag{3.4}$$

Now, combining (3.3) and (3.4), we get (3.1). □

Using the previous identity we get the following generalization of Sherman's theorem which hold for real, not necessary nonnegative vectors **u**, **v** and matrix **A**.

Theorem 3.2. *Let $n \geq 4$ and $\phi \in C^n([\alpha, \beta])$ be n-convex on $[\alpha, \beta]$, $\alpha = a_1 < a_2 \cdots < a_r = \beta$ $(r \geq 2)$ be the given points and $k_1, ..., k_r \in \mathbb{N}$ with $\sum_{j=1}^{r} k_j + r = n$. Let $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^k$, $\mathbf{u} \in \mathbb{R}^l$ and $\mathbf{v} \in \mathbb{R}^k$ be such that (1.2) holds for some matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$, $i = 1, ..., k$ and*

$$\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \geq 0, \quad t \in [\alpha, \beta]. \tag{3.5}$$

(i) *If k_j is odd for each $j = 2, .., r$, then*

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &\geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j) dt. \end{aligned} \tag{3.6}$$

(ii) *If k_j is odd for each $j = 2, .., r-1$ and k_r is even, then the reverse inequality in (3.6) holds.*

Proof. (i) Since $\phi \in C^n([\alpha, \beta])$ is n-convex, then $\phi^{(n)} \geq 0$.
Clearly, $(t - a_1)^{k_1+1} \geq 0$ for any $t \in [\alpha, \beta]$ and if k_j is odd for each $j = 2, .., r$, then the function ω, defined by (2.3), satisfied $\omega(t) \geq 0$ for any $t \in [\alpha, \beta]$. Therefore, by Lemma 2.1 (i) it follows that $G_{H,n-2}(t, s) \geq 0$. Hence, we can apply Theorem 3.1 to obtain (3.6).
(ii) This part we can prove similarly. □

Under Sherman's assumptions of non-negativity of vectors **u**, **v** and matrix **A** the following generalizations hold.

Theorem 3.3. *Let $n \geq 4$ and $\phi \in C^n([\alpha,\beta])$ be n-convex on $[\alpha,\beta]$, $\alpha = a_1 < a_2 \cdots < a_r = \beta$ $(r \geq 2)$ be the given points and $k_1, ..., k_r \in \mathbb{N}$ with $\sum_{j=1}^{r} k_j + r = n$. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^k$, $\mathbf{u} \in [0,\infty)^l$ and $\mathbf{v} \in [0,\infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If k_j is odd for each $j = 2, .., r$, then (3.6) holds.*

(ii) *If k_j is odd for each $j = 2, .., r-1$ and k_r is even, then the reverse inequality in (3.6) holds.*

(iii) *If (3.6) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \int_{\alpha}^{\beta} G(\cdot,t) H_{ij}(t) \phi^{(i+2)}(a_j) dt \tag{3.7}$$

is convex on $[\alpha,\beta]$, then (1.3) holds.

Proof. (i) Since the function $G(.,t)$, $t \in [\alpha,\beta]$, is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \geq 0, \quad t \in [\alpha,\beta].$$

Applying Theorem 3.2 and Lemma 2.1 (i) we get (3.6).
(ii) Similarly we can prove this part.
(iii) If (3.6) holds, the right hand side of (3.6) can be rewriting in the form

$$\sum_{p=1}^{l} u_p \bar{F}(x_p) - \sum_{q=1}^{k} v_q \bar{F}(y_q),$$

where $\bar{F}$ is defined by (3.7). If $\bar{F}$ is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p \bar{F}(x_p) - \sum_{q=1}^{k} v_q \bar{F}(y_q) \geq 0,$$

i.e. the right hand side of (3.6) is nonnegative, so (1.3) immediately follows. □

As a direct consequence of the previous result, considering particular case of Hermite interpolating polynomial with type $(m, n-m)$ conditions, we get the following corollary.

Corollary 3.1. *Let $n \geq 4$, $1 \leq m \leq n-1$ and $\phi \in C^n([\alpha,\beta])$ be n-convex. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^k$, $\mathbf{u} \in [0,\infty)^l$ and $\mathbf{v} \in [0,\infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If $n-m$ is even, then*

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &\geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \right] \left(\sum_{i=0}^{m-1} \tau_i(t) \phi^{(i+2)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t) \phi^{(i+2)}(\beta) \right) dt, \end{aligned} \tag{3.8}$$

where G, τ_i and η_i are defined as in (2.9), (2.5) and (2.6), respectively.

(ii) *If $n-m$ is odd, then the reverse inequality in (3.8) holds.*

(iii) *If (3.8) holds and the function*

$$\tilde{F}(\cdot)=\int_\alpha^\beta G(\cdot,t)\left(\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i)}(\beta)\right)dt \tag{3.9}$$

is convex on $[\alpha,\beta]$, then (1.3) holds.

Considering particular case of Hermite interpolating polynomial with two-point Taylor conditions we get the next generalizations.

Corollary 3.2. *Let $m\geq 2$ and $\phi\in C^{2m}([\alpha,\beta])$ be $2m$-convex. Let $\mathbf{x}\in[\alpha,\beta]^l$, $\mathbf{y}\in[\alpha,\beta]^k$, $\mathbf{u}\in[0,\infty)^l$ and $\mathbf{v}\in[0,\infty)^k$ be such that (1.2) holds for some row stochastic matrix $\mathbf{A}\in\mathcal{M}_{kl}(\mathbb{R})$.*

(i) *If m is even, then*

$$\sum_{p=1}^{l}u_p\phi(x_p)-\sum_{q=1}^{k}v_q\phi(y_q)\geq\int_\alpha^\beta\left[\sum_{p=1}^{l}u_pG(x_p,t)-\sum_{q=1}^{k}v_qG(y_q,t)\right]F(t)dt, \tag{3.10}$$

where

$$F(t)=\sum_{i=0}^{m-1}\sum_{p=0}^{m-1-i}\binom{m+p-1}{p}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^p\phi^{(i+2)}(\alpha)\right.$$
$$\left.+\frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^p\phi^{(i+2)}(\beta)\right].$$

(ii) *If m is odd, then the reverse inequality in (3.10) holds.*

(iii) *If (3.10) holds and the function*

$$\hat{F}(\cdot)=\int_\alpha^\beta G(\cdot,t)F(t)dt$$

is convex on $[\alpha,\beta]$, then (1.3) holds.

4. Grüss and Ostrowski type inequalities related to generalized Sherman's inequality

P. Cerone and S. S. Dragomir [9], considering the Čebyšev functional

$$T(f,g):=\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt$$

for Lebesgue integrable functions $f,g:[\alpha,\beta]\to\mathbb{R}$, proved the following two results which contain the Grüss and Ostrowski type inequalities.

Theorem 4.1. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be Lebesgue integrable and $g:[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2\in L[\alpha,\beta]$. Then*

$$|T(f,g)|\leq\frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta(x-\alpha)(\beta-x)[g'(x)]^2dx\right)^{\frac{1}{2}}. \tag{4.1}$$

The constant $\frac{1}{\sqrt{2}}$ in (4.1) is the best possible.

Theorem 4.2. *Let $g : [\alpha, \beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{4.2}$$

The constant $\frac{1}{2}$ in (4.2) is the best possible.

To avoid many notations, under assumptions of Theorem 3.1, we define the function $\mathcal{B} : [\alpha, \beta] \to \mathbb{R}$ by

$$\mathcal{B}(s) = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \right] G_{H,n-2}(t,s)dt. \tag{4.3}$$

Then $T(\mathcal{B}, \mathcal{B})$ denotes the Čebyšev functional

$$T(\mathcal{B}, \mathcal{B}) = \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}^2(s)ds - \left(\frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds \right)^2.$$

Theorem 4.3. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $\phi^{(n)}$ be absolutely continuous on $[\alpha, \beta]$ with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L[\alpha, \beta]$ and $\mathcal{B}$ be defined as in (4.3). Then the following representation holds*

$$\begin{aligned} &\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) \\ &= \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j)dt \\ &+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta - \alpha} \int_\alpha^\beta \mathcal{B}(s)ds + R(\phi; \alpha, \beta) \end{aligned} \tag{4.4}$$

and the remainder $R(\phi; \alpha, \beta)$ satisfies the estimation

$$|R(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta-\alpha}}{\sqrt{2}} [T(\mathcal{B}, \mathcal{B})]^{\frac{1}{2}} \left| \int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds \right|^{\frac{1}{2}} \tag{4.5}$$

Proof. Applying Theorem 4.1 for $f \to \mathcal{B}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned} &\left| \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds \cdot \frac{1}{\beta-\alpha} \int_\alpha^\beta \phi^{(n)}(s)ds \right| \\ &\leq \frac{1}{\sqrt{2}} [T(\mathcal{B}, \mathcal{B})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have

$$\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds = \frac{\left(\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)\right)}{\beta-\alpha} \int_\alpha^\beta \mathcal{B}(s)ds + R(\phi; \alpha, \beta),$$

where the remainder $R(\phi; \alpha, \beta)$ satisfies the estimation (4.5).
Now from the identity (3.1) we obtain (4.4). □

Theorem 4.4. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $\phi^{(n+1)} \geq 0$ on $[\alpha, \beta]$ and $\mathcal{B}$ be defined as in (4.3). Then the representation (4.4) holds and $R(\phi; \alpha, \beta)$ satisfies the estimation*

$$|R(\phi;\alpha,\beta)| \leq \|\mathcal{B}'\|_\infty \left\{ \frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha} \right\}. \tag{4.6}$$

Proof. Applying Theorem 4.2 for $f \to \mathcal{B}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned} &\left| \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{B}(s)ds \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds \right| \\ &\leq \frac{1}{2(\beta-\alpha)} \|\mathcal{B}'\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds. \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned} &\int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds = \int_\alpha^\beta \left[2s-(\alpha+\beta)\right]\phi^{(n)}(s)ds \\ &= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right], \end{aligned}$$

using identity (3.1) and the inequality (4.7) we deduce (4.6). □

Theorem 3.2 gives the lower bound for the expression

$$\begin{aligned} &\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q) \\ &- \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt. \end{aligned}$$

The upper bound is presented in the next theorem.

Theorem 4.5. *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, let $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$, $\left|\phi^{(n)}\right|^p \in L_p[\alpha,\beta]$ and $\mathcal{B}$ be defined as in (4.3). Then*

$$\begin{aligned} &\left| \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q) \right. \\ &\left. - \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right] \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt \right| \\ &\leq \left\|\phi^{(n)}\right\|_p \|\mathcal{B}\|_q. \end{aligned} \tag{4.8}$$

The constant $\|\mathcal{B}\|_q$ is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Applying Hölder's inequality to the identity (3.1) we obtain

$$\begin{aligned}(4.9)\qquad &\Bigg|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{k} v_q\phi(y_q)\\ &- \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t)\right]\sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i+2)}(a_j)dt\Bigg|\\ &= \left|\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds\right| \le \left\|\phi^{(n)}\right\|_p \|\mathcal{B}\|_q .\end{aligned}$$

For the proof of the sharpness of the constant $\|\mathcal{B}\|_q$ let us find a function ϕ for which the equality in (4.9) holds.
For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(n)}(s) = \operatorname{sgn}\mathcal{B}(s)\,|\mathcal{B}(s)|\,.$$

For $p = \infty$ take $\phi^{(n)}(s) = \operatorname{sgn}\mathcal{B}(s)$.
For $p = 1$ we prove that

$$(4.10)\qquad \left|\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds\right| \le \max_{s\in[\alpha,\beta]} |\mathcal{B}(s)| \left(\int_\alpha^\beta \left|\phi^{(n)}(s)\right| ds\right)$$

is the best possible inequality.
Assume that $|\mathcal{B}(s)|$ attains its maximum at $s_0 \in [\alpha,\beta]$. First we assume that $\mathcal{B}(s_0) > 0$. For ε small enough we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} 0, & \alpha \le s \le s_0,\\ \frac{1}{\varepsilon\, n!}(s-s_0)^n, & s_0 \le s \le s_0+\varepsilon,\\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\varepsilon \le s \le \beta.\end{cases}$$

Then for ε small enough we have

$$\left|\int_\alpha^\beta \mathcal{B}(s)\phi^{(n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} \mathcal{B}(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{B}(s)ds.$$

Now from (4.10) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{B}(s)ds \le \mathcal{B}(s_0)\int_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = \mathcal{B}(s_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{B}(s)ds = \mathcal{B}(s_0)$$

then the statement follows.
In case $\mathcal{B}(s_0) < 0$, we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1}, & \alpha \le s \le s_0,\\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon,\\ 0, & s_0+\varepsilon \le s \le \beta,\end{cases}$$

and the rest of the proof is the same as above. □

5. Some applications

Motivated by the inequality (3.6), under the assumptions of Theorems 3.2, we define the linear functional $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ by

$$\Lambda(\phi) = \sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q)$$
$$- \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{k} v_q G(y_q,t) \right] \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \phi^{(i+2)}(a_j) dt. \tag{5.1}$$

Remark 5.1. Note that if $\phi \in C^n([\alpha,\beta])$ is n-convex, then by Theorem 3.2 we have

$$\Lambda(\phi) \geq 0.$$

Using the linearity and positivity of defined functional we derive mean-value theorems of the Lagrange and Cauchy type.

Theorem 5.1. *Let $\phi \in C^n([\alpha,\beta])$ and $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ be the linear functional defined by (5.1). Then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Lambda(\phi) = \phi^{(n)}(\xi) \Lambda(\varphi),$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Similar to the proof of Theorem 4.1 in [11]. □

Theorem 5.2. *Let $\phi, \psi \in C^n([\alpha,\beta])$ and $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ be the linear functional defined by (5.1). Then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\Lambda(\phi)}{\Lambda(\psi)} = \frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \tag{5.2}$$

provided that the denominators are non-zero

Proof. Similar to the proof of Corollary 4.2 in [11]. □

Remark 5.2. If $\frac{\phi^{(n)}}{\psi^{(n)}}$ is an invertible function, then we get

$$\xi = \left(\frac{\phi^{(n)}}{\psi^{(n)}} \right)^{-1} \left(\frac{\Lambda(\phi)}{\Lambda(\psi)} \right)$$

which is exactly mean of Chauchy type of the segment $[\alpha,\beta]$.

Applying Exponential convexity method [11], we may interpret our results in the form of exponentially convex functions or in the special case log convex functions. In order to obtain such results, we define the families of functions as follows.

For every choice of $l+1$ mutually different points $x_0, x_1, ..., x_l \in [\alpha,\beta]$ we define

- $\mathcal{F}_1 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_2 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_3 = \{\phi_t : [\alpha,\beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is 2-exponentially convex in the Jensen sense on $I\}$

Theorem 5.3. *Let Λ be the linear functional defined as in (5.1) associated with family $\mathcal{F}_1$. Then the following statements hold:*

(i) *The function* $t \mapsto \Lambda(\phi_t)$ *is* n*-exponentially convex in the Jensen sense on* I.
(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is* n*-exponentially convex on* I.

Proof. (i) We define the function $h : [\alpha, \beta] \to \mathbb{R}$ by

$$h(x) = \sum_{j,k=1}^{n} p_j p_k \phi_{s_{jk}}(x),$$

where $p_j, s_j \in \mathbb{R}$, $j = 1, ..., n$, $s_{jk} = \frac{s_j + s_k}{2}$, $1 \le j, k \le n$, and $\phi_{s_{jk}} \in \mathcal{F}_1$.
Since $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on I, then

$$[x_0, x_1, ..., x_l; h] = \sum_{j,k=1}^{n} p_j p_k \left[x_0, x_1, ..., x_l; \phi_{s_{jk}}\right] \ge 0,$$

i.e. h is l-convex. Therefore, we have

$$\Lambda(h) = \sum_{j,k=1}^{n} p_j p_k \Lambda\left(\phi_{s_{jk}}\right) \ge 0.$$

Hence, the function $t \mapsto \Lambda(\phi_t)$ is n-exponentially convex in the Jensen sense on I.
(ii) Follows from (i) and Definition 1.1. □

The following corollary is an easy consequence of the previous theorem.

Corollary 5.1. *Let* Λ *be the linear functional defined as in (5.1) associated with family* $\mathcal{F}_2$. *Then the following statements hold:*

(i) *The function* $t \mapsto \Lambda(\phi_t)$ *is exponentially convex in the Jensen sense on* I.
(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is exponentially convex on* I.

Corollary 5.2. *Let* Λ *be the linear functional defined as in (5.1) associated with family* $\mathcal{F}_3$. *Then the following statements hold:*

(i) *If the function* $t \mapsto \Lambda(\phi_t)$ *is continuous on* I, *then it is* 2*-exponentially convex on* I. *If* $t \mapsto \Lambda(\phi_t)$ *is additionally positive, then it is also log-convex on* I. *Furthermore, for every choice* $r, s, t \in I$, *such that* $r < s < t$, *it holds*

$$[\Lambda(\phi_s)]^{t-r} \le [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_r)]^{s-r}.$$

(ii) *If the function* $t \mapsto \Lambda(\phi_t)$ *is positive and differentiable on* I, *then for all* $r, s, u, v \in I$ *such that* $r \le u$, $s \le v$, *we have*

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) \le \mu_{u,v}(\Lambda, \mathcal{F}_3), \tag{5.3}$$

where

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) = \begin{cases} \left(\frac{\Lambda(\phi_r)}{\Lambda(\phi_s)}\right)^{\frac{1}{r-s}}, & r \ne s, \\ \exp\left(\frac{\frac{d}{dr}(\Lambda(\phi_r))}{\Lambda(\phi_r)}\right), & r = s. \end{cases} \tag{5.4}$$

Proof. (i) The first part of statement is an easy consequence of Theorem 5.3 and the second one of Remark 1.1.
Since $t \mapsto \Lambda(\phi_t)$ is log-convex on I, i.e. $t \mapsto \log \Lambda(\phi_t)$ is convex on I, then by definition we have

$$(r - t) \log \Lambda(\phi_t) + (t - s) \log \Lambda(f_r) + (s - r) \log \Lambda(f_r) \ge 0$$

for every choice $r, s, t \in I$, such that $r < s < t$. Therefore, we have

$$[\Lambda(\phi_s)]^{t-r} \leq [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_r)]^{s-r}.$$

(ii) Since $t \mapsto \log \Lambda(\phi_t)$ is convex on I, by definition we have

$$\frac{\log \Lambda(\phi_r) - \log \Lambda(\phi_s)}{r-s} \leq \frac{\log \Lambda(\phi_u) - \log \Lambda(\phi_v)}{u-v} \tag{5.5}$$

for $r \leq u$, $s \leq v$, $r \neq u$, $s \neq v$. Therefore, we have

$$\mu_{r,s}(\Lambda, \mathcal{F}_3) \leq \mu_{u,v}(\Lambda, \mathcal{F}_3).$$

Case $r = s$, $u = v$ follows from (5.5) as limiting case. □

Using obtained mean-valued theorems and results regarding the exponential convexity, we may deduce some new classes of two-parameter Cauchy-type means.

For example, consider the family of functions

$$\Omega = \{\varphi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

defined by

$$\varphi_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n\varphi_t}{dx^n}(x) = e^{-x\sqrt{t}} > 0$, the function φ_t is n-convex function for every $t > 0$. Moreover, the function $t \mapsto \frac{d^n\varphi_t}{dx^n}(x)$ is exponentially convex. Therefore, using the same arguments as in proof of Theorem 5.3, we conclude that the function $t \mapsto [x_0, x_1, ..., x_l; \varphi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Then from Corollary 5.1 it follows that $t \mapsto \Lambda(\varphi_t)$ is exponentially convex in the Jensen sense. It is easy to verify that the function $t \mapsto \Lambda(\varphi_t)$ is continuous, so it is exponentially convex.

For this family of functions, with assumption that $[\alpha, \beta] \subset (0,\infty)$ and $t \mapsto \Lambda(\varphi_t)$ is positive, (5.4) becomes

$$\mu_{\eta,\zeta} = \left(\frac{\zeta^n}{\eta^n} . \frac{\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum\limits_{q=1}^{k} v_q e^{-y_q\sqrt{\eta}} - A_1}{\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\zeta}} - \sum\limits_{q=1}^{k} v_q e^{-y_q\sqrt{\zeta}} - B_1} \right)^{\frac{1}{\eta-\zeta}}, \quad \eta \neq \zeta,$$

$$\mu_{\eta,\eta} = \exp\left(\frac{\sum\limits_{q=1}^{k} v_q y_q e^{-y_q\sqrt{\eta}} - \sum\limits_{p=1}^{l} u_p x_p e^{-x_p\sqrt{\eta}} + A_2}{2\sqrt{\eta}\left(\sum\limits_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum\limits_{q=1}^{k} v_q e^{-y_q\sqrt{\eta}} - A_1 \right)} - \frac{n}{\eta} \right),$$

where

$$A_1 = \int_{\alpha}^{\beta} \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)(-1)^{i+2} \eta^{1+\frac{i}{2}} e^{-a_j\sqrt{\eta}} dt,$$

$$A_2 = \int_{\alpha}^{\beta} \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t) \frac{d^{i+2}}{dx^{i+2}} (xe^{-x\sqrt{\eta}})|_{x=a_j} dt,$$

$$B_1 = \int_{\alpha}^{\beta} \left(\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{k} v_q G(y_q, t) \right) \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)(-1)^{i+2} \zeta^{1+\frac{i}{2}} e^{-a_j\sqrt{\zeta}} dt.$$

Using Theorem 5.2 it follows that

$$\mu_{\eta,\zeta}(\Lambda, \Omega) = -\left(\sqrt{\eta} + \sqrt{\zeta}\right) \log \mu_{\eta,\zeta}(\Lambda, \Omega)$$

satisfies

$$\alpha \le \mu_{\eta,\zeta}(\Lambda, \Omega) \le \beta,$$

i.e. $\mu_{\eta,\zeta}(\Lambda, \Omega)$ is mean. By Corollary 5.2, using (5.3), it follows that this mean is monotonic.

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