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THE CHARACTERIZATIONS OF SPACELIKE CURVES IN R_1^4

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Abstract. In this paper, we study the geometry of position vectors of a spacelike curve in the Minkowski 4-space. We give some characterizations for spacelike curves to lie on some subspaces of R_1^4.

1. Introduction

The Frenet frames for spacelike, timelike and null curves have been studied and developed by several authors [5], [3], [11], [1] and [2]. A. Fernandez, A. Gimenez and P. Lucas introduced a Frenet frame with curvature functions for a null curve in a Lorentzian manifold and studied null helices in Lorentzian space forms [2]. C. Coken and U. Ciftci studied null curves in the 4-dimensional Minkowski space R_1^4 , and give some results for psoudospherical null curves and Bertrand null curves.

K. Ilarslan and O. Boyacioglu studied position vectors of a timelike and a null helice in R_1^3 [5]. K. Ilarslan and E. Nesovic gave some characterizations for null curves in R_1^4 and they obtained some relations between null normal curves and null osculating curves as well as between null rectifying curves and null osculating curves [6].

K. Ilarslan studied spacelike normal curves in Minkowski space E_1^3 and gave some characterizations of spacelike normal curves with spacelike, timelike and null principal normal [6]. K. Ilarsalan, E. Nesovic and M. Petrovic-Torgasev characterized non-null and null rectifying curves, lying fully in the Minkowski 3-space [7].

A. T. Ali and M. Onder characterize rectifying spacelike curves in terms of their curvature functions in Minkowski spacetime [3]. M. Onder, H. Kocayigit and M. Kazaz gave some characterizations for spacelike helices in Minkowski spacetime and found the differential equations characterizing the spacelike helices in Minkowski 4-space [11].

M. A. Akgun and A. I. Sivridag studied null Cartan curves in Minkowski 4-space and give some theorems for null Cartan curves to lie on some subspaces of R_1^4 [13].

M. A. Akgun and A. I. Sivridag studied spacelike and timelike curves to lie on some subspaces of R_1^4 and give some theorems in [14] and [15].

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This paper organized following: In section 2 we give some basic knowledge related with curves in Minkowski space-time. Section 3 is the original part of this paper. In this section we investigate the conditions for spacelike curves to lie on some subspaces of R_1^4 and we give some characterizations and theorems for these curves.

2. Preliminaries

Let R_1^4 denote Minkowski space together with a flat Lorentz metric $\langle , \rangle$ of signature $(-,+,+,+)$. A vector X is said to be timelike if $\langle X, X\rangle < 0$, spacelike if $\langle X, X\rangle > 0$ or $X = 0$ and null(lightlike) if $\langle X, X\rangle = 0$ and $X \neq 0$. The norm of a vector $X \in R_1^4$ is denoted by $\|X\|$ and defined by $\|X\| = \sqrt{|\langle X, X\rangle|}$.

A curve α in R_1^4 is called a null curve if $\langle \alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, timelike curve if $\langle \alpha'(s), \alpha'(s)\rangle < 0$ and spacelike curve if $\langle \alpha'(s), \alpha'(s)\rangle > 0$ for all $s \in R$.

Let α be a spacelike curve in R_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$ and let N be null vector and B_1 be null vector. In this case there exists only one Frenet frame $\{T, N, B_1, B_2\}$ for which $\alpha(s)$ is a spacelike curve with Frenet equations

$$\begin{aligned} \nabla_T T &= k_1 N \\ \nabla_T N &= k_2 B_2 \\ \nabla_T B_1 &= -k_1 T + k_3 B_2 \\ \nabla_T B_2 &= -k_3 N - k_2 B_1 \end{aligned}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying the equations

$$\langle B_1, B_1\rangle = \langle N, N\rangle = 0, \quad \langle T, T\rangle = \langle B_2, B_2\rangle = 1, \quad \langle N, B_1\rangle = 1$$

[12]

3. The Characterizations of Spacelike Curves in R_1^4

In this section we will investigate some characterizations of spacelike curves to lie on some subspaces of R_1^4.

Let α be a spacelike curve in R_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$. Then, the subspaces of R_1^4 spanned by $\{T, N\}$, $\{T, B_1\}$, $\{T, B_2\}$, $\{N, B_1\}$, $\{N, B_2\}$, $\{B_1, B_2\}$, $\{T, N, B_1\}$, $\{T, N, B_2\}$, $\{T, B_1, B_2\}$ and $\{N, B_1, B_2\}$.

Case 1) First we will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N \tag{3.1}$$

for some differentiable functions λ and μ of s, which is the arc-length parameter of $\alpha(s)$. Differentiating (3.1) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = \lambda'(s)T + (\lambda(s)k_1(s) + \mu'(s))N + \mu(s)k_2(s)B_2$$

where $\alpha' = T$. Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \\ \mu(s)k_2(s) = 0 \end{cases}$$

If $\mu(s) = 0$ we find $k_1(s) = 0$ and $\lambda(s) = s + c$. So we have

$$\alpha(s) = (s + c)T$$

If $k_2(s)=0$,then we find $\mu(s)=-\int(s+c)k_1(s)ds$. So we have

$$\alpha(s)=(s+c)T-(\int(s+c)k_1(s)ds)N.$$

Thus we have the following theorem.

Theorem 3.1. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T,N\}$ if and only if it is in the form*

$$\alpha(s)=(s+c)T$$

where $k_1(s)=0$ *or*

$$\alpha(s)=(s+c)T-(\int(s+c)k_1(s)ds)N$$

where $k_2(s)=0$

Case 2) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T,B_1\}$. In this case we can write

$$\alpha(s)=\lambda(s)T+\mu(s)B_1 \tag{3.2}$$

for some differentiable functions λ and μ. Differentiating (3.2) with respect to s and by using the Frenet equations we find that

$$\alpha'(s)=(\lambda'(s)-\mu(s)k_1(s))T+\lambda(s)k_1(s)N+\mu'(s)B_1+\mu(s)k_3(s)B_2.$$

Since $\{T,N,B_1,B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases}\lambda'(s)-\mu(s)k_1(s)=1\\ \lambda(s)k_1(s)=0\\ \mu(s)k_3(s)=0\\ \mu'(s)=0\end{cases}$$

From the equation $\lambda(s)k_1(s)=0$, if $\lambda(s)=0$ then we can write $\mu(s)=-\frac{1}{k_1(s)}=$ $cons.$ and $k_3(s)=0$. So we have

$$\alpha(s)=-\frac{1}{k_1(s)}B_1.$$

If $k_1(s)=0$ and $\mu(s)=0$ we find $\lambda(s)=s+c$. So we have

$$\alpha(s)=(s+c)T.$$

If $k_1(s)=k_3(s)=0$ then we find $\mu(s)=c_2$ and $\lambda(s)=s+c_1$. So we have

$$\alpha(s)=(s+c_1)T+c_2B_1.$$

Thus we have the following theorem.

Theorem 3.2. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T,B_1\}$ if and only if it is in the form*

$$\alpha(s)=-\frac{1}{k_1(s)}B_1$$

where $k_3(s)=0$ *or*

$$\alpha(s)=(s+c)T$$

where $k_1(s)=0$ *or*

$$\alpha(s)=(s+c_1)T+c_2B_1$$

where $k_1(s) = k_3(s) = 0$ *and* c, c_1 *and* c_2 *are constants.*

Case 3) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_2, \tag{3.3}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.3) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = \lambda'(s)T + (\lambda(s)k_1(s) - \mu(s)k_3(s))N - \mu(s)k_2(s)B_1 + \mu'(s)B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)k_1(s) - \mu(s)k_3(s) = 0 \\ \mu(s)k_2(s) = 0 \\ \mu'(s) = 0 \end{cases} \tag{3.4}$$

From (3.4) if $\mu(s) = 0$ then we find $\lambda(s) = s + c$ and $k_1(s) = 0$. So we have

$$\alpha(s) = (s + c)T.$$

If $k_2(s) = 0$ then we find $\lambda(s) = s + c_1$ and $\mu(s) = c_2$. So we have

$$\alpha(s) = (s + c_1)T + c_2B_2.$$

Thus we have the following theorem.

Theorem 3.3. *A spacelike curve* α *in* R_1^4 *lies on the subspace spanned by* $\{T, B_2\}$ *if and only if it is in the form*

$$\alpha(s) = (s + c)T.$$

where $k_1(s) = 0$ *or*

$$\alpha(s) = (s + c_1)T + c_2B_2.$$

where $k_2(s) = 0$ *and the curvature functions satisfy the equation* $\frac{k_1(s)}{k_3(s)} = \frac{c_2}{s+c_1}$.

Case 4) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N, B_1\}$. In this case we can write

$$\alpha(s) = \lambda(s)N + \mu(s)B_1 \tag{3.5}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.5) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = -\mu(s)k_1(s)T + \lambda'(s)N + \mu'(s)B_1 + (\lambda(s)k_2(s) + \mu(s)k_3(s))B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} -\mu(s)k_1(s) = 1 \\ \lambda'(s) = 0 \\ \mu'(s) = 0 \\ \lambda(s)k_2(s) + \mu(s)k_3(s) = 0 \end{cases} \tag{3.6}$$

From (3.6) we can write $\lambda(s) = c_1$ and $\mu(s) = -\frac{1}{k_1(s)} = c_2$. . So we have

$$\alpha(s) = c_1N - \frac{1}{k_1(s)}B_1.$$

Thus we have the following theorem.

Theorem 3.4. *A spacelike curve* α *in* R_1^4 *lies on the subspace spanned by* $\{N, B_1\}$ *if and only if it is in the form*

$$\alpha(s) = c_1 N - \frac{1}{k_1(s)} B_1$$

where c_1, c_2 *are constants and the curvature functions satisfy the equation* $c_1 k_2(s) + c_2 k_3(s) = 0$.

Case 5) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s) N + \mu(s) B_2 \tag{3.7}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.7) with respect to s and by using the Frenet equations we find that

$$(3.8)\alpha'(s) = (\lambda'(s) - \mu(s)k_3(s))N - \mu(s)k_2(s)B_1 + (\lambda(s)k_2(s) + \mu'(s))B_2.$$

Since $\alpha(s)$ is a spacelike curve from (3.8) there is a contradiction. Thus we have the following theorem.

Theorem 3.5. *A spacelike curve* α *in* R_1^4 *does not lie on the subspace spanned by* $\{N, B_2\}$.

Case 6) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{B_1, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s) B_1 + \mu(s) B_2 \tag{3.9}$$

for some differentiable functions λ and μ of the parameter s. Differentiating (3.9) with respect to s and by using the Frenet equations we find that

$$\alpha'(s) = -\lambda(s)k_1(s)T - \mu(s)k_3(s)N + (\lambda'(s) - \mu(s)k_2(s))B_1 + (\lambda(s)k_3(s) + \mu'(s))B_2.$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} -\lambda(s)k_1(s) = 1 \\ \mu(s)k_3(s) = 0 \\ \lambda'(s) - \mu(s)k_2(s) = 0 \\ \lambda(s)k_3(s) + \mu'(s) = 0 \end{cases} \tag{3.10}$$

From (3.10) if $\mu(s) = 0$ then we can write $\lambda(s) = -\frac{1}{k_1(s)}$. So we have

$$\alpha(s) = -\frac{1}{k_1(s)} B_1.$$

If $k_3(s) = 0$ we find $\mu(s) = \frac{k_1'(s)}{k_1^2(s)k_2(s)} = cons$. So we have

$$\alpha(s) = (-\frac{1}{k_1(s)})B_1 + (\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2.$$

Theorem 3.6. *A spacelike curve* α *in* R_1^4 *lies on the subspace spanned by* $\{B_1, B_2\}$ *if and only if it is in the form of*

$$\alpha(s) = -\frac{1}{k_1(s)} B_1$$

or

$$\alpha(s) = (-\frac{1}{k_1(s)})B_1 + (\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2$$

where $k_3(s) = 0$.

Case 7) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N, B_1\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_1 \tag{3.11}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.11) with respect to s and by using the Frenet equations we find that

$$\begin{aligned}\alpha'(s) = \; & (\lambda'(s) - \gamma(s)k_1(s))T + (\lambda(s)k_1(s) + \mu'(s))N + \gamma'(s)B_1 \\ & +(\mu(s)k_2(s) + \gamma(s)k_3(s))B_2.\end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) - \gamma(s)k_1(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \\ \gamma'(s) = 0 \\ \mu(s)k_2(s) + \gamma(s)k_3(s) = 0 \end{cases} \tag{3.12}$$

From (3.12) we can write $\gamma(s) = c_1$. If we use the equation $\mu(s)k_2(s)+\gamma(s)k_3(s) = 0$ we find $\mu(s) = -c_1\frac{k_3(s)}{k_2(s)}$. From the equation $\lambda(s)k_1(s) + \mu'(s) = 0$ we obtain $\lambda(s) = c_1\frac{k_3'(s)k_2(s)-k_3(s)k_2'(s)}{k_2^2(s)k_1(s)}$. So we have

$$\alpha(s) = (c_1\frac{k_3'(s)k_2(s) - k_3(s)k_2'(s)}{k_2^2(s)k_1(s)})T - (c_1\frac{k_3(s)}{k_2(s)})N + c_1B_1.$$

Thus we have the following theorem.

Theorem 3.7. *A spacelike curve* α *in* R_1^4 *lies on the subspace spanned by* $\{T, N, B_1\}$ *if and only if it is in the form*

$$\alpha(s) = (c_1\frac{k_3'(s)k_2(s) - k_3(s)k_2'(s)}{k_2^2(s)k_1(s)})T - (c_1\frac{k_3(s)}{k_2(s)})N + c_1B_1.$$

where c_1 *is a constant.*

Case 8) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, N, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)N + \gamma(s)B_2 \tag{3.13}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.13) with respect to s and by using the Frenet equations we find that

$$\begin{aligned}\alpha'(s) = \; & \lambda'(s)T + (\lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_3(s))N + \gamma(s)k_2(s)B_1 \\ & +(\gamma'(s) + \mu(s)k_2(s))B_2.\end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations:

$$\begin{cases} \lambda'(s) = 1 \\ \lambda(s)k_1(s) + \mu'(s) - \gamma(s)k_3(s) = 0 \\ \gamma(s)k_2 = 0 \\ \gamma'(s) + \mu(s)k_2(s) = 0 \end{cases} \tag{3.14}$$

From (3.14) we find $\lambda(s) = s + c$. If $\gamma(s) = 0$ we can write the equations

$$\begin{gathered} \mu(s)k_2(s) = 0 \\ \lambda(s)k_1(s) + \mu'(s) = 0 \end{gathered} \tag{3.15}$$

From (3.15) if $\mu(s) = 0$ then we can write $\lambda(s) = s + c_1$ and $k_1(s) = 0$. So we have

$$\alpha(s) = (s + c_1)T.$$

If $k_2(s) = 0$ then we can write

$$\mu(s) = -\int (s + c_1)k_1(s)ds + c_2.$$

So we have

$$\alpha(s) = (s + c_1)T + (-\int (s + c_1)k_1(s)ds + c_2)N.$$

From (3.14) if $k_2(s) = 0$ then we can write $\gamma(s) = c_2$ and $\mu(s) = c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c$. So we have

$$\alpha(s) = (s + c_1)T + (c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c)N + c_2B_2.$$

Thus we have the following theorem.

Theorem 3.8. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T, N, B_2\}$ if and only if it is in the form*

$$\alpha(s) = (s + c_1)T$$

where $k_1(s) = 0$ or

$$\alpha(s) = (s + c_1)T + (-\int (s + c_1)k_1(s)ds + c_2)N$$

where $k_2(s) = 0$ or

$$\alpha(s) = (s + c_1)T + (c_2 \int k_3(s)ds - \int k_1(s)(s + c_1)ds + c)N + c_2B_2$$

where $k_2(s) = 0$.

Case 9) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{T, B_1, B_2\}$. In this case we can write

$$\alpha(s) = \lambda(s)T + \mu(s)B_1 + \gamma(s)B_2 \tag{3.16}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.16) with respect to s and by using the Frenet equations we find that

$$\begin{aligned} \alpha'(s) &= (\lambda'(s) - \mu(s)k_1(s))T + (\lambda(s)k_1(s) - \gamma(s)k_3(s))N + (\mu'(s) - \gamma(s)k_2(s))B_1 \\ &+ (\mu(s)k_3(s) + \gamma'(s))B_2. \end{aligned}$$

Since $\{T, N, B_1, B_2\}$ is a Frenet frame we have the following equations.

$$\begin{cases} \lambda'(s) - \mu(s)k_1(s) = 1 \\ \lambda(s)k_1(s) - \gamma(s)k_3(s) = 0 \\ \mu'(s) - \gamma(s)k_2(s) = 0 \\ \mu(s)k_3(s) + \gamma'(s) = 0 \end{cases} \tag{3.17}$$

From the equation $\lambda'(s) - \mu(s)k_1(s) = 1$ we can write $\frac{d\lambda(s)}{ds} + \frac{k_1(s)}{k_3(s)}\gamma'(s) = 1$. From the last equation we have

$$\frac{d}{ds}(\frac{k_3(s)}{k_1(s)}\gamma(s)) + \frac{k_1(s)}{k_3(s)}\frac{d\gamma(s)}{ds} = 1 \tag{3.18}$$

By using exchange variable $t=\int_0^s \frac{k_3(s)}{k_1(s)}ds$ in (3.18) we have

$$2\frac{d\gamma(s)}{ds}=1 \tag{3.19}$$

The solution of (3.19) is $\gamma(s)=\frac{t}{2}+c$. Replacing variable $t=\int \frac{k_3(s)}{k_1(s)}ds$ in the last equation we find

$$\gamma(s)=\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c. \tag{3.20}$$

If we use (3.20) in (3.17) we find $\mu(s)=-\frac{1}{2k_1(s)}$ and

$$\lambda(s)=\frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c)$$

So we have

$$\alpha(s)=(\frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c))T-(\frac{1}{2k_1(s)})B_1+(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c)B_2.$$

Thus we have the following theorem.

Theorem 3.9. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{T,B_1,B_2\}$ if and only if it is in the form*

$$\alpha(s)=(\frac{k_3(s)}{k_1(s)}(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c))T-(\frac{1}{2k_1(s)})B_1+(\frac{1}{2}\int_0^s \frac{k_3(s)}{k_1(s)}ds+c)B_2.$$

Case 10) We will investigate the conditions under which the spacelike curve α lies on the subspace spanned by $\{N,B_1,B_2\}$. In this case we can write

$$\alpha(s)=\lambda(s)N+\mu(s)B_1+\gamma(s)B_2 \tag{3.21}$$

for some differentiable functions λ, μ and γ of the parameter s. Differentiating (3.21) with respect to s and by using the Frenet equations we find that

$$\begin{aligned}\alpha'(s) &= -\mu(s)k_1(s)T+(\lambda'(s)-\gamma(s)k_3(s))N+(\mu'(s)-\gamma(s)k_2(s))B_1\\ &+ (\lambda(s)k_2(s)+\mu(s)k_3(s)+\gamma'(s))B_2.\end{aligned}$$

Since $\{T,N,B_1,B_2\}$ is a Frenet frame we have the following equations.

$$\left\{\begin{array}{c}-\mu(s)k_1(s)=1\\ \lambda'(s)-\gamma(s)k_3(s)=0\\ \mu'(s)-\gamma(s)k_2(s)=0\\ \lambda(s)k_2(s)+\mu(s)k_3(s)+\gamma'(s)=0\end{array}\right. \tag{3.22}$$

From (3.22) we can write $\mu(s)=-\frac{1}{k_1(s)}$. From the equation $\gamma(s)=\frac{k_1'(s)}{k_1^2(s)k_2(s)}$ and from the equation $\lambda(s)k_2(s)+\mu(s)k_3(s)+\gamma'(s)=0$ we obtain

$$\lambda(s)=\frac{k_3(s)}{k_1(s)k_2(s)}-\frac{k_1''(s)k_1(s)k_2(s)-k_1'(s)(2k_1'(s)k_2(s)+k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3}$$

So we have

$$\begin{aligned}\alpha(s) &= (\frac{k_3(s)}{k_1(s)k_2(s)}-\frac{k_1''(s)k_1(s)k_2(s)-k_1'(s)(2k_1'(s)k_2(s)+k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3})N\\ &- (\frac{1}{k_1(s)})B_1+(\frac{k_1'(s)}{k_1^2(s)k_2(s)})B_2.\end{aligned}$$

Thus we have the following theorem.

Theorem 3.10. *A spacelike curve α in R_1^4 lies on the subspace spanned by $\{N, B_1, B_2\}$ if and only if it is in the form*

$$\begin{aligned}\alpha(s) &= \left(\frac{k_3(s)}{k_1(s)k_2(s)} - \frac{k_1''(s)k_1(s)k_2(s) - k_1'(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s))}{(k_1(s)k_2(s))^3}\right)N \\ &- \left(\frac{1}{k_1(s)}\right)B_1 + \left(\frac{k_1'(s)}{k_1^2(s)k_2(s)}\right)B_2.\end{aligned}$$

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