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COMPUTATION OF MONODROMY MATRIX ON FLOATING POINT ARITHMETIC WITH GODUNOV MODEL

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Abstract. The results computed monodromy matrix on floating point arithmetics according to Wilkinson Model have been given in [1]. In this study, new results have been obtained by examining floating point arithmetics with respect to Godunov Model the results in [1]. These results have been applied to Schur stability of system of linear difference equations with periodic coefficients. Also the effect of floating point arithmetics has been investigated on numerical examples.

Keywords: Floating point, Godunov model, fundamental matrix, monodromy matrix, Schur stability, linear difference equations, periodic coefficients.

1. Introduction

Consider the following linear difference equation system with period T

$$x_{n+1} = A_n x_n, A_n = A_{n+T}, n \in \mathbb{Z}, \tag{1.1}$$

where A_n is N × N dimensional periodic matrix.

It is important to investigate Schur stability in order to know the behaviours of solution without compute the solutions of the system (1.1) [2, 3, 4, 5, 6]. In literature, the parameter is used as Schur stability parameter. It is well-known that

$$\omega_1(A,T) = \left\| \sum_{k=0}^{\infty} (X_T^*)^k (X_T)^k \right\| < \infty \tag{1.2}$$

implies Schur stability of the the system (1.1), where the matrix X_T is monodromy matrix of the the system (1.1) [7], and system (1.1) is Schur stable if and only if the monodromy matrix X_T is Schur stable [2, 3]. According to spectral criterion, the monodromy matrix X_T is Schur stable if and only if each eigenvalue of the monodromy matrix X_T belongs to unit disc ($|\lambda(X_T)| < 1$) [4]. It is clear that

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Schur stability of the the system (1.1) depends on the monodromy matrix X_T in both cases. The computation processes on computer are related to floating point. The errors are produced when computer is used to perform calculations, by nature. Therefore, Schur stability of the the system (1.1) and quality of Schur stability are affected by occured errors on computation of the monodromy matrix X_T. In [1], the results on computation of the monodromy matrix X_T on floating point arithmetics with Wilkinson Model have been given. As parallel the results with Wilkinson Model in [1], the new results have been obtained according to floating point arithmetics with Godunov Model in this study. In Section 2 of this study, floating point numbers and arithmetics with respect to Godunov Model and linear difference equations with periodic coefficients are investigated. Some results on the computation of fundamental matrix of linear difference equations with periodic coefficients in floating point arithmetics are obtained in Section 3. The obtained results are applied to Schur stability of the system (1.1) in Section 4. Finally, these results are supported with numerical examples.

2. Preliminaries

2.1. **Floating Point Numbers and Arithmetic, Godunov Model.** The set

$$\mathbb{F} = \mathbb{F}(\gamma, p_-, p_+, k) = \{0\} \cup \left\{ z | z = \pm\gamma^{p(z)} m_\gamma(z) \right\} \tag{2.1}$$

is called as the set of computer numbers or Format set [2]. The set $\mathbb{F}$ is also characterized by the parameters ε_0, ε_1 and ε_∞, where

$$\varepsilon_0 = \gamma^{p_- - 1},\ \varepsilon_1 = \gamma^{1-k},\ \varepsilon_0 = \gamma^{p_- - 1},\ \varepsilon_\infty = \gamma^{p_+}\left(1 - \frac{1}{\gamma^k}\right) \tag{2.2}$$

are defined (see, for example, [2, 9]). In the represent (2.1), $p_- \in \mathbb{Z}^-, k, p_+ \in \mathbb{Z}^+$ for $p_- \le p \le p_+,\ p \in \mathbb{Z}$ and

$$m_\gamma(z) = \frac{m_1}{\gamma} + \frac{m_2}{\gamma^2} + ... + \frac{m_k}{\gamma^k};\ m_j \in \mathbb{Z}, 0 \le m_j \le \gamma - 1,\ j = 1, 2, \cdots, k\ (m_1 \neq 0) \tag{2.3}$$

is defined [2, 8, 9, 10, 11, 12, 13]. In [2, 9, 14], the operator

$$fl : \mathbb{D} \to \mathbb{F},\ fl(z) = z(1 + \alpha) + \beta;\ \|\alpha\| \le u,\ \|\beta\| \le v,\ \alpha\beta = 0 \tag{2.4}$$

converts the elements of $\mathbb{D} = [-\varepsilon_\infty, \varepsilon_\infty] \cap \mathbb{R}$ to floating point numbers, where

$$u = \begin{cases} \frac{\varepsilon_1}{2}, & \text{rounding} \\ \varepsilon_1, & \text{chopping} \end{cases},\ v = \begin{cases} \frac{\varepsilon_0}{2}, & \text{rounding} \\ \varepsilon_0, & \text{chopping} \end{cases}. \tag{2.5}$$

We have called as Godunov Model, the model which is defined by the equation (2.4). A vector $x = (x_i) \in \mathbb{D}^N$ and a matrix $A = (a_{ij}) \in M_N(\mathbb{D})$ can be stored to memory by floating point as $fl(x) = (fl(x_i))$; $fl(A) = (fl(a_{ij}))$. The upper bound of error that storing vector x by floating point is

$$\|x - fl(x)\| \le u\|x\| + v\sqrt{N}, \tag{2.6}$$

[2, 10], the upper bound of the error storing matrix $fl(A)$ is

$$\|A - fl(A)\| \le u\sqrt{N}\|A\| + vN, \tag{2.7}$$

[2]. The upper bound errors of $fl(AB)$ and $fl(A + B)$ are

$$\|AB - fl(AB)\| \le uN^2\|A\|\|B\| + vN, \tag{2.8}$$

$$\|(A+B) - fl(A+B)\| \leq uN \|A+B\| + vN, \tag{2.9}$$

where u, v are defined by (2.5) and $A, B \in M_N(\mathbb{D})$ [9].

2.2. **Linear Difference Equations with Periodic Coefficients.** The system (1.1) and for given $x_0 \in \mathbb{R}^N$ initial value

$$x_{n+1} = A_n x_n, x_0 - \text{initial vector}, n \geq 0 \tag{2.10}$$

is called *linear difference-Cauchy problem with periodic coefficients.* If I is identity matrix and

$$X_{n+1} = A_n X_n, X_0 = I, n \geq 0 \tag{2.11}$$

is solution of Cauchy problem, then

$$X_n = \prod_{j=0}^{n-1} A_j = A_{n-1}A_{n-2}\cdots A_0, \tag{2.12}$$

is called fundamental matrix of the system (2.10).

$$X_T = \prod_{j=0}^{T-1} A_j = A_{T-1}A_{T-2}\cdots A_0, \tag{2.13}$$

is called *monodromy matrix* of the system (2.10) [2, 3, 7, 15, 16, 17]. The solution of the system (2.10) is

$$X_{kT+m} = X_m X_T^k x_0 \tag{2.14}$$

where $x_0 \in \mathbb{R}^N$ inital value $x_n = X_n x_0$, $n = kT + m$, $0 \leq m \leq T-1$ [3, 7].

3. Computation of Fundamental Matrix

In this chapter, the computation of fundamental matrix X_n that given by (2.10) will be investigated on floating point arithmetics with Godunov model. Let us introduce some definitions and symbols before calculation.

Let

$$Q_{n,s} = \prod_{j=s}^{n-1} A_j;\ Q_{n,s} \times Q_{s,r} = Q_{n,r};\ Q_{n,0} = X_n,\ Q_{n,n} = I\ (I - \text{identity matrix}).$$

$$q_{n,s} = \prod_{j=s}^{n-1} \|A_j\|;\ q_{n,s} \times q_{s,r} = q_{n,r};\ q_{n,0} = q_n,\ q_{n,n} = 1,$$

$$\sum_{j=s}^{r-1} k_j = \begin{cases} \mathbf{0} - matrix, & k_j\text{-matrix function} \\ 0, & k_j\text{-real function} \end{cases},$$

where $r \leq s \leq n$ and r, s, n are natural numbers. Linear Cauchy problem can be written

$$fl(A_{n-1}Y_{n-1}) = Y_n = A_{n-1}Y_{n-1} + \varphi_n;\ Y_0 = I,\ n = 1, 2, 3, \cdots , \tag{3.1}$$

where $A_{n-1} \in M_N(\mathbb{F})$, $Y_n = fl(A_{n-1}Y_{n-1})$ is computation of the matrix X_n by floating point numbers. Matrix φ_n is the computation error of $A_{n-1}Y_{n-1}$ and it is clear that $\varphi_1 = 0$.

It is clear that the solution of difference-Cauchy problem (2.14) is

$$Y_n = X_n + \sum_{k=2}^{n} Q_{n,k}\varphi_k. \tag{3.2}$$

Let us investigate the upper boundary of φ_n in equation (2.14) according to

$$\|\varphi_n\| \leq u\sqrt{N} q_{n,n-1}\|Y_{n-1}\| + vN,\ Y_0 = I,\ n = 2,3,\cdots. \tag{3.3}$$

Theorem 3.1. *The inequality*

$$\|\varphi_n\| \leq u\sqrt{N}\left(1+u\sqrt{N}\right)^{n-2} q_n + uvN^{\frac{3}{2}}\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1} q_{n,j} + vN,\ n = 2,3,\cdots$$

holds, where φ_n is error from (3.3) *and u, v are defined by* (2.5).

Proof. 1. Let consider

$$\|\varphi_k\| \leq u\sqrt{N} q_{k,k-1}\|Y_{k-1}\| + vN,\ \|Y_k\| \leq q_{k,k-1}\|Y_{k-1}\| + \|\varphi_k\|,\ k = 2,3,\cdots$$

from (3.3). Let us write Y_{n-1} and φ_{n-1}

$$\|\varphi_n\| \leq u\sqrt{N} q_{n,n-1}\|Y_{n-1}\| + vN$$

in this inequality.

$$\begin{aligned}
\|\varphi_n\| &\leq u\sqrt{N} q_{n,n-1}\left(q_{n-1,n-2}\|Y_{n-2}\| + \|\varphi_{n-1}\|\right) + vN \\
&\leq u\sqrt{N} q_{n,n-2}\|Y_{n-2}\| + u\sqrt{N} q_{n,n-1}\left(u\sqrt{N} q_{n-1,n-2}\|Y_{n-2}\| + vN\right) + vN \\
&= u\sqrt{N}\left(1+u\sqrt{N}\right) q_{n,n-2}\|Y_{n-2}\| + uvN\sqrt{N} q_{n,n-1} + vN.
\end{aligned}$$

If we write Y_{n-2} and φ_{n-2} in last inequality, we can obtain

$$\begin{aligned}
\|\varphi_n\| &\leq u\sqrt{N}\left(1+u\sqrt{N}\right) q_{n,n-2}\left(q_{n-2,n-3}\|Y_{n-3}\| + \|\varphi_{n-2}\|\right) + uvN\sqrt{N} q_{n,n-1} + vN \\
&\leq u\sqrt{N}\left(1+u\sqrt{N}\right) q_{n,n-3}\|Y_{n-3}\| + u\sqrt{N}\left(1+u\sqrt{N}\right) q_{n,n-2}\left(u\sqrt{N} q_{n-2,n-3}\|Y_{n-3}\| + vN\right) \\
&\quad + uvN\sqrt{N} q_{n,n-1} + vN \\
&\leq u\sqrt{N}\left(1+u\sqrt{N}\right)^2 q_{n,n-3}\|Y_{n-3}\| + uvN\sqrt{N}\left(1+u\sqrt{N}\right) q_{n,n-2} + uvN^{\frac{3}{2}} q_{n,n-1} + vN.
\end{aligned}$$

We can iterate to n same way, and

$$\|\varphi_n\| \leq u\sqrt{N}\left(1+u\sqrt{N}\right)^{n-2} q_n + uvN^{\frac{3}{2}}\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1} q_{n,j} + vN,\ n = 2,3,\cdots$$

is obtained. □

Proof. 2.

$$\|\varphi_n\| \leq u\sqrt{N} q_{n,n-1}\|Y_{n-1}\| + vN,\ n = 2,3,\cdots$$

can be written by (3.3).

$$\|Y_{n-1}\| \leq \left(1+u\sqrt{N}\right)^{n-2} q_{n-1} + vN\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1} q_{n-1,j}$$

can be written from Theorem 3.2. We ordered in this inequality,

$$\begin{aligned}\|\varphi_n\| &\le u\sqrt{N}q_{n,n-1}\left[\left(1+u\sqrt{N}\right)^{n-2}q_{n-1}+vN\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1}q_{n-1,j}\right]+vN\\ &= u\sqrt{N}\left(1+u\sqrt{N}\right)^{n-2}q_n+uvN^{\frac{3}{2}}q_{n,n-1}\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1}q_{n-1,j}+vN\\ &= u\sqrt{N}\left(1+u\sqrt{N}\right)^{n-2}q_n+uvN^{\frac{3}{2}}\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1}q_{n,j}+vN.\end{aligned}$$

So the inequality

$$\|\varphi_n\| \le u\sqrt{N}\left(1+u\sqrt{N}\right)^{n-2}q_n+uvN^{\frac{3}{2}}\sum_{j=2}^{n-1}\left(1+u\sqrt{N}\right)^{n-j-1}q_{n,j}+vN$$

is obtained. □

Theorem 3.2. *The inequality*

$$\|Y_n\| \le \left(1+u\sqrt{N}\right)^{n-1}q_n+vN\sum_{j=2}^{n}\left(1+u\sqrt{N}\right)^{n-j}q_{n,j},\ \ n=1,2,3,\cdots$$

holds, where Y_n *is defined by* (3.1) *and* u, v *are defined by* (2.5).

Proof. Consider

$$\|\varphi_k\| \le u\sqrt{N}q_{k,k-1}\|Y_{k-1}\|+vN;\ \ \|Y_k\| \le q_{k,k-1}\|Y_{k-1}\|+\|\varphi_k\|,\ \ k=2,3,\cdots$$

by (3.3).

$$\|Y_n\| \le q_{n,n-1}\|Y_{n-1}\|+\|\varphi_n\| \le q_{n,n-1}\|Y_{n-1}\|+u\sqrt{N}q_{n,n-1}\|Y_{n-1}\|+vN \tag{3.4}$$

$$= \left(1+u\sqrt{N}\right)q_{n,n-1}\|Y_{n-1}\|+vN \tag{3.5}$$

is obtained by (3.1). It can be obtained Cauchy problem of first-order variable coefficient difference-inequality

$$\|Y_n\| \le \left(1+u\sqrt{N}\right)q_{n,n-1}\|Y_{n-1}\|+vN,\ \ \|Y_1\|=\|A_0\|,\ \ n=2,3,\cdots.$$

By iteration,

$$\begin{aligned}\|Y_n\| &\le \left(1+u\sqrt{N}\right)q_{n,n-1}\left[\left(1+u\sqrt{N}\right)q_{n-1,n-2}\|Y_{n-2}\|+vN\right]+vN\\ &= \left(1+u\sqrt{N}\right)^2q_{n,n-2}\|Y_{n-2}\|+\left(1+u\sqrt{N}\right)vNq_{n,n-1}+vN\\ &\le \left(1+u\sqrt{N}\right)^2q_{n,n-2}\|Y_{n-2}\|+vN\left[1+\left(1+u\sqrt{N}\right)q_{n,n-1}\right]\end{aligned}$$

is written. Y_{n-2} is written in the inequality,

$$\begin{aligned}\|Y_n\| &\le \left(1+u\sqrt{N}\right)^2 q_{n,n-2}\left[\left(1+u\sqrt{N}\right)q_{n-2,n-3}\|Y_{n-3}\|+vN\right]+vN\left[1+\left(1+u\sqrt{N}\right)q_{n,n-1}\right]\\ &= \left(1+u\sqrt{N}\right)^3 q_{n,n-3}\|Y_{n-3}\|+\left(1+u\sqrt{N}\right)^2 vNq_{n,n-2}+vN\left[1+\left(1+u\sqrt{N}\right)q_{n,n-1}\right]\\ &= \left(1+u\sqrt{N}\right)^3 q_{n,n-3}\|Y_{n-3}\|+vN\left[1+\left(1+u\sqrt{N}\right)q_{n,n-1}+\left(1+u\sqrt{N}\right)^2 vNq_{n,n-2}\right].\end{aligned}$$

By iteration in the same way, the inequality

$$\|Y_n\| \le \left(1+u\sqrt{N}\right)^{n-1} q_n + vN\sum_{j=2}^{n}\left(1+u\sqrt{N}\right)^{n-j} q_{n,j}$$

is obtained. □

Theorem 3.3. *The inequality*

$$\|X_n-Y_n\| \le u\sqrt{N}\sum_{k=2}^{n}\left(1+u\sqrt{N}\right)^{k-2} q_n + uvN^{\frac{3}{2}}\sum_{k=2}^{n}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{n,j} + vN\sum_{k=2}^{n} q_{n,k}$$

holds, where the matrix X_n is fundamental matrix of the system (1.1)*, the matrix Y_n is computed fundamental matrix by* (3.1)*, and u, v are defined by* (2.5)*.*

Proof. From (3.2),

$$\|X_n-Y_n\| \le \|A_{n-1}A_{n-2}\cdots A_2\varphi_2 + A_{n-1}A_{n-2}\cdots A_3\varphi_3+\cdots+A_{n-1}\varphi_{n-1}+\varphi_n\|$$

is written, where fundamental matrix X_n of the system (1.1) and computed fundamental matrix Y_n.

$$\|\varphi_k\| \le u\sqrt{N}\left(1+u\sqrt{N}\right)^{k-2} q_k + uvN^{\frac{3}{2}}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{k,j} + vN$$

is known from

$$\|X_n-Y_n\| \le \sum_{k=2}^{n} q_{n,k}\|\varphi_k\|$$

and Theorem 3.1. So

$$\|X_n-Y_n\| \le \sum_{k=2}^{n} q_{n,k}\left[u\sqrt{N}\left(1+u\sqrt{N}\right)^{k-2} q_k + uvN^{\frac{3}{2}}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{k,j} + vN\right]$$

is obtained. We arranged last inequality,

$$\|X_n-Y_n\| \le u\sqrt{N}\sum_{k=2}^{n}\left(1+u\sqrt{N}\right)^{k-2} q_n + uvN^{\frac{3}{2}}\sum_{k=2}^{n}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{n,j} + vN\sum_{k=2}^{n} q_{n,k}$$

is obtained. □

We can write easily Corollary 3.1 from

$$\sum_{k=2}^{n}\left(1+u\sqrt{N}\right)^{k-2} = \frac{\left(1+u\sqrt{N}\right)^{n-1}-1}{u\sqrt{N}}$$

and Theorem 3.3.

Corollary 3.1. *The inequality*

$$\|X_n - Y_n\| \le \left[\left(1+u\sqrt{N}\right)^{n-1} - 1\right] q_n + uvN^{\frac{3}{2}} \sum_{k=2}^{n}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{n,j} + vN\sum_{k=2}^{n} q_{n,k}$$

holds, where the matrix X_n *is fundamental matrix of the system* (1.1) *and the matrix* Y_n *is the computed matrix of the fundamental matrix* X_n *of the system* (1.1)*, and* u, v *are defined by* (2.5)*.*

4. Applying the results to Schur stability of periodic systems

Applying the results to Schur stability of periodic system in section 4 of [1] that obtained with Wilkinson Model is available for results with Godunov Model. The changes due to differences in models can be occured in the computations.

Let

$$y_{n+1} = (A_n + B_n)\, y_n, n \in \mathbb{Z}, \tag{4.1}$$

where $A_n = A_{n+T}$ and $B_n = B_{n+T}$, N-dimensional periodic (T-period). It is called perturbed system of the system (1.1).

Continuity theorem on the monodromy matrix in [16] guarantees Schur stability of the system (4.1) when the system (1.1) or matrix X_T is Schur stable. The following theorem which is application of continuity theorem can easily be obtained as same to Theorem 4.1 in [1].

For $T = 1$, the system (1.1) transforms the system

$$x_{n+1} = Ax_n, n \in \mathbb{Z},$$

and it is called lineer difference equation system with constant coefficients. Therefore, $\omega_1(A, T)$ can be written

$$\omega_1(A, T) = \omega(A), \qquad \omega(A) = \left\|\sum_{k=0}^{\infty}(A^*)^k A^k\right\|.$$

Furthermore, in this case $\omega_1(A, 1)$ is equal to $\omega(X_1) = \omega(A)$ [7, 17].

Theorem 4.1. *If the matrix* Y_T *is Schur stable and the inequality*

$$\|Y_T - X_T\| \le \sqrt{\|Y_T\|^2 + \frac{1}{\omega\left(Y_T\right)}} - \|Y_T\| \tag{4.2}$$

holds, then the matrix X_T *is Schur stable, where the matrix* Y_T *is computed monodromy matrix of* X_T *and the matrix* X_T *is perturbed matrix of* Y_T*.*

We can obtain following corollary by $n = T$ in Corollary 3.1.

Corollary 4.1. *The inequality*

$$\|X_T - Y_T\| \le \left[\left(1+u\sqrt{N}\right)^{T-1} - 1\right] q_T + uvN^{\frac{3}{2}} \sum_{k=2}^{T}\sum_{j=2}^{k-1}\left(1+u\sqrt{N}\right)^{k-j-1} q_{T,j} + vN\sum_{k=2}^{T} q_{T,k}$$

holds, where the matrix X_T *is monodromy matrix of the system* (1.1) *and the matrix* Y_T *is the computed matrix of the monodromy matrix* X_n*.*

The Corollary 4.2 guarantees Schur stability of the system (1.1) (or monodromy matrix X_T) when the *computed matrix* Y_T is Schur Stable.

Corollary 4.2. *Let monodromy matrix X_T of the system* (1.1) *and the computed matrix Y_T of the matrix X_T on floating point arithmetic. If the computed matrix Y_T is Schur stable and the inequality*

$$\Delta < \Delta_s$$

holds then monodromy X_T is Schur stable, where

$$\Delta = \left[\left(1+u\sqrt{N}\right)^{T-1} - 1\right] q_T + uvN^{\frac{3}{2}} \sum_{k=2}^{T} \sum_{j=2}^{k-1} \left(1+u\sqrt{N}\right)^{k-j-1} q_{T,j} + vN \sum_{k=2}^{T} q_{T,k},$$

$$\Delta_s = \sqrt{\|Y_T\|^2 + \frac{1}{\omega(Y_T)}} - \|Y_T\|.$$

Proof. It is clear from Theorem 4.1. □

5. Numerical Examples

The MVC (Matrix Vector Calculator) software has been used in numerical computation to calculate the value $\omega(A)$ of matrix A by function QdaStab [18].

In the examples, let us denote rounding by r, chopping by c, spectral norm of a matrix by $\|A\|$ and let $\Delta^r = \Delta(Y_T^r)$, $\Delta^c = \Delta(Y_T^c)$, $\Delta_s^r = \Delta_s(Y_T^r)$, $\Delta_s^c = \Delta_s(Y_T^c)$.

Example 5.1. Let $\mathbb{F} = \mathbb{F}(10, -3, 3, 3)$ and matrices

$$A_0 = \begin{bmatrix} 0.855 & 0.0005 \\ 0.956 & 0.156 \end{bmatrix}, A_1 = \begin{bmatrix} 0.953 & 0.155 \\ 1.55 & 0.165 \end{bmatrix}$$

where $A_0, A_1 \in M_2(\mathbb{F})$. Let us investigate Schur stability, where $T = 2$. Monodromy matrix X_2 of the system (1.1) has been computed with

$$X_2 = \begin{bmatrix} 0.962995 & 0.0246565 \\ 1.48299 & 0.026515 \end{bmatrix}.$$

And the monodromy matrix X_2 is not Schur stable, since $\omega(X_2) = \infty$.

If the matrix Y_2 is computed matrix in $\mathbb{F}$, the matrices

$$Y_2^r = \begin{bmatrix} 0.963 & 0.0247 \\ 1.48 & 0.0265 \end{bmatrix}, Y_2^c = \begin{bmatrix} 0.962 & 0.0246 \\ 1.48 & 0.0265 \end{bmatrix},$$

are obtained. $\omega(X_2^r) = \infty, \omega(X_2^c) = 2655.69$ and so, computed matrix Y_2 is Schur stable by chopping, but it is not Schur stable by rounding.

Example 5.2. Let $\mathbb{F} = \mathbb{F}(10, -5, 5, 5)$ and matrices

$$A_0 = \begin{bmatrix} 2.002 & 0.1 & 1.675 \\ 1.5 & 0.017 & 0.008955 \\ 0.002 & 3.986 & 0.00245 \end{bmatrix}, A_1 = \begin{bmatrix} 0.005 & 0.6 & 0.04 \\ 0.006 & 0.009842 & 0.0083 \\ 1.2 & 1.986 & 0.00025 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.02 & 0.1982 & 0.03 \\ 0.002 & 0.056 & 0.0475 \\ 0.75622 & 0.03 & 0.0008 \end{bmatrix}$$

where $A_0, A_1, A_2 \in M_3(\mathbb{F})$. Let us investigate Schur stability, where $T = 3$. The matrices

$$Y_3^r = \begin{bmatrix} 0.18495 & 0.014755 & 0.063124 \\ 0.25894 & 0.0095870 & 0.096916 \\ 0.69334 & 0.12980 & 0.012398 \end{bmatrix}, Y_3^c = \begin{bmatrix} 0.18495 & 0.014754 & 0.063123 \\ 0.25893 & 0.0095869 & 0.096916 \\ 0.69333 & 0.12980 & 0.012397 \end{bmatrix},$$

are computed matrices in $\mathbb{F}$. So, the values

$$\omega(Y_3^r) = 1.6621, \Delta_s^r = 0.3213943129, \Delta^r = 0.001247028$$

$$\omega(Y_3^c) = 1.66207, \Delta_s^c = 0.3214029037, \Delta^c = 0.002494142$$

are obtained. It seems that $\Delta^r < \Delta_s^r$ and $\Delta^c < \Delta_s^c$. Therefore, in both cases, Corollary 4.2 guarantees Schur stability of the monodromy matrix X_3 in $\mathbb{F}(10, -5, 5, 5)$.

6. Conclusion

In this study, the effects of floating point arithmetic using Godunov Model on computation of the monodromy matrix X_T were investigated. The bounds were obtained for $\|X_T - Y_T\|$, where the matrix Y_T is the computed value of monodromy matrix. The obtained results were applied to Schur stability of the system (1.1). Further, these results were supported with numerical examples.

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