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ON THE W_5-CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

D. G. PRAKASHA, VASANT CHAVAN AND KAKASAB MIRJI

Abstract. The object of the paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on W_5-curvature tensor. We characterize W_5-flat, ϕ-W_5-flat and ϕ-W_5-semisymmetric generalized Sasakian-space-forms.

1. Introduction

Generalized Sasakian-space-forms have become today a rather special topic in contact Riemannian geometry, but many contemporary works are concerned with the study of its properties and their related curvature tensor. The study of generalized Sasakian-space-forms was initiated by Algre et al., in [1] and then it was continued by many other authors. A generalized Sasakian-space-form is an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. In such case we will write the manifold as $M(f_1, f_2, f_3)$. This kind of manifolds appears as a natural generalization of the Sasakian-space-forms by taking: $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The ϕ-sectional curvature of generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$. Moreover, cosymplectic space-form and Kenmotsu space-form are also considered as particular types of generalized Sasakian-space-form. Generalized Sasakian-space-forms have

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been studied in a number of papers from several points of view (for instance, [2]-[4], [6]-[8], [9]-[11], [13]-[17], etc).

In the context of generalized Sasakian-space-forms, Kim [11] studied conformally flat and locally symmetric generalized Sasakian-space-forms. Some symmetric properties of generalized Sasakian-space-forms with projective curvature tensor were studied by De and Sarkar [6] and Sarkar and Akbar [16]. In [13], Prakasha shown that every generalized Sasakian space-form is Weyl-pseudosymmetric. Hui [10] studied W_2-curvature tensor in generalized Sasakian-space-forms. Also, Prakasha and Nagaraja [14] studied quasi-conformally flat and quasi-confomally semisymmetric generalized Sasakian-space-forms. In a recent paper [8], De and Majhi studied ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian-space-forms. Conharmonically flat generalized Sasakian-space-forms and conharmonically locally ϕ-symmetric generalized Sasakian-space-forms were studied in [17]. In a recent paper, Hui and Prakasha [9] studied certain properties on the C-Bochner curvature tensor of generalized Sasakian-space-forms. As a continuation of this study, in this paper we plan to characterize flatness and symmetry property of generalized Sasakian-space-forms regarding W_5-curvature tensor.

The paper is organized as follows: after preliminaries in Section 3, we study the W_5-flat generalized Sasakian space-forms. We prove that a generalized Sasakian-space-form is W_5-flat if and only if $f_1 = 3f_2/1-2n = f_3$. In section 4, we study ϕ-W_5-flat generalized Sasakian-space-form and obtain that a generalized Sasakian-space-form of dimension greater than three is ϕ-W_5-flat if and only if it is conformally flat. In the last section, we prove that a generalized Sasakian-space-form is ϕ-W_5-semisymmetric if and only if it is W_5-flat.

2. **Preliminaries**

An odd-dimensional Riemannian manifold (M, g) is said to be an *almost contact metric manifold* [5] if there exist on M a $(1,1)$ tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1,\ \eta(\phi X) = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for arbitrary vector fields X and Y. In view of (2.1) and (2.2), we have

$$g(\phi X, Y) = -g(X, \phi),\quad g(\phi X, X) = 0.$$
$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y).$$

Again, we know that in a generalized Sasakian space-form

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.3}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M and f_1, f_2, f_3 are smooth functions on the manifold. The Ricci operator Q and Ricci tensor S of the manifold of dimension $(2n+1)$ are respectively given by

$$QX = (2nf_1 + 3f_2 - f_3)X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi, \tag{2.4}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y). \tag{2.5}$$

In addition to the relation (2.3)-(2.5), for an $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M(f_1,f_2,f_3)$ the following relations also hold [1]:

$$\eta(R(X,Y)Z) = (f_1-f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \tag{2.6}$$

$$R(X,Y)\xi = (f_1-f_3)\{\eta(Y)X-\eta(X)Y\}, \tag{2.7}$$

$$R(\xi,X)Y = (f_1-f_3)\{g(X,Y)\xi-\eta(Y)X\}. \tag{2.8}$$

The W_5-curvature tensor on a $(2n+1)$-dimensional generalized Sasakian-space-form $M(f_1,f_2,f_3)$ is given by [12]

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{2n}\left\{g(X,Z)S(Y,U) - g(Y.U)S(X,Z)\right\}. \tag{2.9}$$

For $n \geq 1$, $M(f_1,f_2,f_3)$ is locally W_5-flat if and only if the W_5-curvature tensor vanishes, Also, notice that W_5-curvature tensor is symmetric with change of pairs of the vector fields and does not satisfies the cyclic property. A relativistic significance of W_5-curvature tensor has been explored by Pokhariyal [12],

In view of (2.6)-(2.8), it can be easily construct that in a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M(f_1,f_2,f_3)$, the W_5-curvature tensor satisfies the following conditions:

$$\eta(W_5(X,Y)Z) = (f_1-f_3)\{g(Y,Z)\eta(X)\} - \frac{1}{2n}\eta(Y)S(X,Z), \tag{2.10}$$

$$W_5(X,Y)\xi = (f_1-f_3)\{\eta(Y)X-2\eta(X)Y\} + \frac{1}{2n}\eta(X)QY, \tag{2.11}$$

$$\eta(W_5(X,Y)\xi) = 0. \tag{2.12}$$

3. W_5-flat generalized Sasakian-space-forms

Definition 3.1. A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is called W_5-flat if it satisfies the condition

$$W_5(X,Y)Z = 0,$$

for any vector fields X, Y and Z on the manifold.

Let $M(f_1,f_2,f_3)$ be a $(2n+1)$-dimensional $(n>1)$ W_5-flat generalized Sasakian space-form. Then, by Definition 3.1) and (2.9), we get

$$R(X,Y)Z = \frac{1}{2n}\left\{S(X,Z)Y - g(X,Z)QY\right\}. \tag{3.1}$$

In view of (2.6) and (2.7), the above equation takes the form

$$R(X,Y)Z = -\frac{1}{2n}[3f_2+(2n-1)f_3]\left\{\eta(X)\eta(Z)Y - g(X,Z)\eta(Y)\xi\right\}. \tag{3.2}$$

Using (2.3) in (3.2) yields

$$\begin{aligned} & f_1\{g(Y,Z)X-g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + \; & f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ & -\frac{1}{2n}[3f_2+(2n-1)f_3]\left\{\eta(X)\eta(Z)Y - g(X,Z)\eta(Y)\xi\right\}. \end{aligned} \tag{3.3}$$

Taking $Z=\phi Z$ in (3.3), we have

$$\begin{aligned} & f_1\{g(Y,\phi Z)X-g(X,\phi Z)Y\} \\ + & f_2\{g(X,\phi^2 Z)\phi Y-g(Y,\phi^2 Z)\phi X+2g(X,\phi Y)\phi^2 Z\} \\ + & f_3\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi \\ = & \frac{1}{2n}[3f_2+(2n-1)f_3]\{g(X,\phi Z)\eta(Y)\xi\}. \end{aligned}$$

If we take $Y=\xi$, then we obtain from the above equation

$$-2n(f_1-f_3)g(X,\phi Z)\xi=[3f_2+(2n-1)f_3]g(X,\phi Z)\xi. \tag{3.4}$$

Since $g(X,\phi Z)\xi\neq 0$, in general. Thus from (3.4), it follows that

$$2nf_1+3f_2-f_3=0. \tag{3.5}$$

Again, we take $X=\xi$ in (3.3), we obtain

$$\begin{aligned} & (f_1-f_3)\{g(Y,Z)\xi-\eta(Z)Y\} \\ = & \frac{1}{2n}[3f_2+(2n-1)f_3]\eta(Z)\{Y-\eta(Y)\xi\}. \end{aligned} \tag{3.6}$$

Taking inner product with ξ of (3.6), we obtain

$$(f_1-f_3)\{g(Y,Z)-\eta(Z)\eta(Y)\}=0. \tag{3.7}$$

This implies that

$$f_1=f_3. \tag{3.8}$$

Since $g(Y,Z)\neq\eta(Y)\eta(Z)$, in general. From (3.5) and (3.8), it is easy to see that

$$f_3=\frac{3f_2}{1-2n}. \tag{3.9}$$

Thus in view of (3.8) and (3.9), we have

$$f_1=\frac{3f_2}{1-2n}=f_3. \tag{3.10}$$

Conversely, suppose that (3.10) holds. Then from (2.4) and (2.5), we have $QX=0$ and $S(X,Y)=0$, respectively.
Making use of this in (2.9), we get

$$W_5'(X,Y,Z,U)=R'(X,Y,Z,U), \tag{3.11}$$

where $W_5'(X,Y,Z,U)=g(W_5(X,Y)Z,U)$ and $R'(X,Y,Z,U)=g(R(X,Y)Z,U)$.
Putting $Y=Z=e_i$ in (3.11) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$\sum_{i=1}^{2n+1}W_5'(X,e_i,e_i,U)=S(X,U). \tag{3.12}$$

Next, because of (2.3) and (3.11), we have

$$\begin{aligned} & W'(X,Y,Z,U) \\ = & f_1\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\} \\ + & f_2\{g(X,\phi Z)g(\phi Y,U)-g(Y,\phi Z)g(\phi X,U)+2g(X,\phi Y)g(\phi Z,U)\} \\ + & f_3\{\eta(X)\eta(Z)g(Y,U)-\eta(Y)\eta(Z)g(X,U) \\ + & g(X,Z)\eta(Y)\eta(U)-g(Y,Z)\eta(X)\eta(U)\}. \end{aligned} \tag{3.13}$$

Now, putting $Y = Z = e_i$ in (3.13) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}(3.14)\qquad & \sum_{i=1}^{2n+1} W_5'(X, e_i, e_i, U) \\ = \ & 2nf_1 g(X,U) + 3f_2 g(\phi X, \phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X,U)\}.\end{aligned}$$

By virtue of $S(X,U) = 0$, (3.12) and (3.14) we have

$$\begin{aligned}(3.15)\qquad & 2nf_1 g(X,U) + 3f_2 g(\phi X, \phi U) \\ - \ & f_3\{(2n-1)\eta(X)\eta(U) + g(X,U)\} = 0.\end{aligned}$$

Putting $X = U = e_i$ in (3.15) and taking summation over i, $1 \leq i \leq 2n+1$, we get $f_1 = 0$. Then in view of (3.10), $f_2 = f_3 = 0$. Therefore, we obtain from (2.3) that

$$(3.16)\qquad R(X,Y)Z = 0.$$

Using (3.15) and $S(X,Y) = QX = 0$, we have from (2.9) that $W_5(X,Y)Z = 0$, That is, $M(f_1, f_2, f_3)$ is W_5-flat. This leads us to state the following:

Theorem 3.1. *A* $(2n+1)$*-dimensional* $(n > 1)$ *generalized Sasakian-space-form is* W_5*-flat if and only if* $f_1 = \frac{3f_2}{1-2n} = f_3$.

4. ϕ $-W_5$-flat generalized Sasakian-space-forms

Definition 4.1. A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is called ϕ-W_5-flat if it satisfies the condition

$$(4.1)\qquad \phi^2 W_5(\phi X, \phi Y)\phi Z = 0,$$

for any vector fields X, Y and Z on the manifold.

First, taking $X = \phi X$, $Y = \phi Y$ and $Z = \phi Z$ in (2.9), we have

$$\begin{aligned}(4.2)\qquad & W_5(\phi X, \phi Y)\phi Z \\ = \ & R(\phi X, \phi Y)\phi Z + \frac{1}{2n}\{g(\phi X, \phi Z)Q\phi Y - S(\phi X, \phi Z)\phi Y\}.\end{aligned}$$

Using (2.4) and (2.5) in (4.2), we get

$$W_5(\phi X, \phi Y)\phi Z = R(\phi X, \phi Y)\phi Z.$$

In virtue of (2.3), we get from above equation

$$\begin{aligned}(4.3)\qquad & W_5(\phi X, \phi Y)\phi Z \\ = \ & f_1\{g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\} \\ + \ & f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}.\end{aligned}$$

Applying ϕ^2 to both sides of (4.3), we have

$$\begin{aligned}(4.4)\qquad & \phi^2 W_5(\phi X, \phi Y)\phi Z \\ = \ & \phi^2[f_1\{g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y\} \\ + \ & f_2\{g(X, \phi Z)\phi^2 Y - g(Y, \phi Z)\phi^2 X + 2g(X, \phi Y)\phi^2 Z\}].\end{aligned}$$

Let $M(f_1,f_2,f_3)$ be a $(2n+1)$-dimensional $(n>1)$ ϕ-W_5-flat generalized Sasakian-space-form. Then, by Definition 4.1 and (4.4), we get

$$\begin{aligned}\phi^2[f_1\{g(Y,Z)\phi X-\eta(Y)\eta(Z)\phi X-g(X,Z)\phi Y+\eta(X)\eta(Z)\phi Y\}\\ + \quad f_2\{g(X,\phi Z)\phi^2 Y-g(Y,\phi Z)\phi^2 X+2g(X,\phi Y)\phi^2 Z\}]=0.\end{aligned}\tag{4.5}$$

By virtue of (2.1) and (2.2), the above equation yields

$$\begin{aligned}f_1\{g(Y,Z)\phi X-\eta(Y)\eta(Z)\phi X-g(X,Z)\phi Y+\eta(X)\eta(Z)\phi Y\}\\ + \quad f_2\{g(X,\phi Z)\phi^2 Y-g(Y,\phi Z)\phi^2 X+2g(X,\phi Y)\phi^2 Z\}=0.\end{aligned}\tag{4.6}$$

Taking inner product with U in (4.6), we obtain

$$\begin{aligned}f_1\{g(Y,Z)g(\phi X,U)-\eta(Y)\eta(Z)g(\phi X,U)-g(X,Z)g(\phi Y,U)\\ + \quad \eta(X)\eta(Z)g(\phi Y,U)\}+f_2\{g(X,\phi Z)g(\phi^2 Y,U)-g(Y,\phi Z)g(\phi^2 X,U)\\ + \quad 2g(X,\phi Y)g(\phi^2 Z,U)\}=0.\end{aligned}\tag{4.7}$$

Putting $Y=Z=e_i$ in (4.7) and taking summation over i, $1\le i\le 2n+1$, we get $3f_2 g(X,\phi U)=0$. Since $g(X,\phi U)\neq 0$, in general. Hence, it follows that

$$f_2=0.\tag{4.8}$$

In (4.7) again putting $Y=U=e_i$, and taking summation over i, $1\le i\le 2n+1$, we get

$$\{f_1+(2n+1)f_2\}g(\phi X,Z)-f_1\{g(X,Z)-\eta(X)\eta(Z)\}\psi=0,\tag{4.9}$$

where $\psi = Trace\ of\ \phi$. Plugging $X=Z=e_i$ in (4.9), and taking summation over i, $1\le i\le 2n+1$, we obtain $\{(2n-1)f_1+(2n+1)f_2\}=0$. Which in view of (4.8) yields $f_1=0$. Hence, we have $f_1=f_2=0$.

Conversely, if $f_1=f_2=0$ then from (4.4) it follows that

$$\phi^2 W_5(\phi X,\phi Y)\phi Z=0.\tag{4.10}$$

That is, $M(f_1,f_2,f_3)$ is $\phi-W_5$-flat. Therefore, the converse holds when $f_1=f_2=0$. Thus we are able to state the following:

Theorem 4.1. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is $\phi-W_5$-flat if and only if $f_1=f_2=0$ holds.*

In [11], U. K. Kim proved that for a (2n+1)-dimensional generalized Sasakian-space-form the following holds:
(i) If $n>1$, then M is conformally flat if and only if $f_2=0$.
(ii) If M is conformally flat and ξ is a Killing vector field, then M is locally symmetric and has constant ϕ-sectional curvature.

In view of the first part of the above theorem of Kim we immediately obtain the following:

Theorem 4.2. *A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form is $\phi-W_5$-flat if and only if it is conformally flat.*

Also, in view of the second part of the above theorem of Kim we get the following:

Theorem 4.3. *A $(2n+1)$-dimensional $(n>1)$ $\phi-W_5$-flat generalized Sasakian-space-form with ξ as a Killing vector field is locally symmetric and has constant ϕ-sectional curvature.*

5. ϕ-W_5-semisymmetric generalized Sasakian-space-forms

Definition 5.1. A $(2n+1)$-dimensional $(n>1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is called ϕ-W_5-semisymmetric if it satisfies the condition

$$W_5(X,Y).\phi = 0, \tag{5.1}$$

for any vector fields X, Y on the manifold.

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n>1)$ ϕ-W_5-semisymmetric generalized Sasakian-space-form. The condition $W_5(X,Y).\phi = 0$ implies that

$$(W_5(X,Y)\cdot\phi)Z = W_5(X,Y)\phi Z - \phi W_5(X,Y)Z = 0, \tag{5.2}$$

for any vector fields X, Y and Z. Now,

$$W_5(X,Y)\phi Z = R(X,Y)\phi Z + \frac{1}{2n}\{g(X,\phi Z)QY - S(X,\phi Z)Y\}. \tag{5.3}$$

Using (2.3), (2.6) and (2.7) in (5.3), we get

$$\begin{aligned} &W_5(X,Y)\phi Z \\ = \;& f_1\{g(Y,\phi Z)X - g(X,\phi Z)Y\} + f_2\{g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y \\ - \;& \eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\} + f_3\{g(X,\phi Z)\eta(Y)\xi \\ - \;& g(Y,\phi Z)\eta(X)\xi\} - \left[\frac{3f_2+(2n-1)f_3}{2n}\right] g(X,\phi Z)\eta(Y)\xi. \end{aligned} \tag{5.4}$$

Similarly,

$$\phi W_5(X,Y)Z = \phi R(X,Y)Z + \frac{1}{2n}\{g(X,Z)\phi QY - S(X,Z)\phi Y\}. \tag{5.5}$$

By virtue of (2.3), (2.6) and (2.7) we obtain from (5.5) that

$$\begin{aligned} &\phi W_5(X,Y)Z \\ = \;& f_1\{g(Y,Z)\phi X - g(X,Z)\phi Y\} + f_2\{g(Y,\phi Z)X - g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi \\ - \;& g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi\} + f_3\{\eta(X)\eta(Z)\phi Y \\ - \;& \eta(Y)\eta(Z)\phi X\} + \left[\frac{3f_2+(2n-1)f_3}{2n}\right]\eta(X)\eta(Z)\phi Y. \end{aligned} \tag{5.6}$$

Substituting (5.3) and (5.5) in (5.2) yields

$$\begin{aligned} &(f_1-f_2)\{g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y\} \\ + \;& (f_2-f_3)\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X - g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)\eta(X)\xi\} \\ - \;& \left[\frac{3f_2+(2n-1)f_3}{2n}\right]\{g(X,\phi Z)\eta(Y)\xi - \eta(X)\eta(Z)\phi Y\} = 0. \end{aligned} \tag{5.7}$$

Putting $Y=\xi$ in (5.7), we obtain

$$\left[\frac{f_3-3f_2-2nf_1}{2n}\right] g(X,\phi Z)\xi = (f_1-f_3)\eta(Z)\phi X. \tag{5.8}$$

Taking inner product with U, we get from (5.8)

$$\left[\frac{f_3-3f_2-2nf_1}{2n}\right] g(X,\phi Z))\eta(U) = (f_1-f_3)\eta(Z)g(\phi X,U). \tag{5.9}$$

Putting $X=U=e_i$ in (5.9), and then taking summation over i, $1\le i\le 2n+1$, we get

$$(f_1-f_3)\eta(Z)\psi = 0, \tag{5.10}$$

where $\psi = Trace\ \ of\ \ \phi$. From (5.10), we get

$$f_1 = f_3. \tag{5.11}$$

Making use of (5.11) in (5.8), we obtain

$$[(1-2n)f_3 - 3f_2]g(X,\phi Z))\xi = 0, \tag{5.12}$$

which implies that

$$f_3 = \frac{3f_2}{1-2n}. \tag{5.13}$$

Thus in view of (5.11) and (5.13), we have

$$f_1 = \frac{3f_2}{1-2n} = f_3. \tag{5.14}$$

Conversely, suppose (5.13) holds. Then in view of Theorem 3.1, we have $W_5 = 0$ and hence $W_5(X,Y).\phi = 0$. Thus we can state the following:

Theorem 5.1. *A (2n+1)-dimensional* $(n > 1)$ *generalized Sasakian space-form is* ϕ*-*W_5*-semisymmetric if and only if* $f_1 = \frac{3f_2}{1-2n} = f_3$.

In [7], De et al., proved the following result:

Theorem 5.2. *A (2n+1)-dimensional* $(n > 1)$ *generalized Sasakian space-form is conharmonically flat if and only if* $f_1 = \frac{3f_2}{1-2n} = f_3$.

Taking into account of Theorem 3.1, Theorem 5.1 and Theorem 5.2, now we may present the following theorem:

Theorem 5.3. *Let* $M^(f_1, f_2, f_3)$ *be a (2n+1)-dimensional* $(n > 1)$ *generalized Sasakian space-form. Then the following statements are equivalent:*

(1) $M(f_1, f_2, f_3)$ *is* W_5*-flat;*
(2) $M(f_1, f_2, f_3)$ *is* ϕ*-*W_5*-semisymmetric;*
(3) $M(f_1, f_2, f_3)$ *is conharmonically flat;*
(4) $f_1 = \frac{3f_2}{1-2n} = f_3$.

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Department of Mathematics, Karnatak University, Dharwad - 580 003, INDIA.
E-mail address: prakashadg@gmail.com, kcvasant@gmail.com, mirjikk@gmail.com