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AN ALTERNATIVE TECHNIQUE FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a new method for solving ordinary differential equations is given by using the generalized Laplace transform $\mathcal{L}_n$. Firstly, the authors introduce a differential operator $\overline{\delta}$ that is called the $\overline{\delta}$-derivative. A relation between the $\mathcal{L}_n$-transform of the $\overline{\delta}$-derivative of a function and the $\mathcal{L}_n$-transform of the function itself are derived. Then, the convolution theorem is proven. Using obtained theorems, a few initial-value problems for ordinary differential equations are solved as illustrations.

1. Introduction, definitions and preliminaries

The Laplace transform is defined by

$$\mathcal{L}\{f(x);y\} = \int_0^\infty \exp(-xy)f(x)dx. \tag{1.1}$$

The following Laplace-type the $\mathcal{L}_2$ transform

$$\mathcal{L}_2\{f(x);y\} = \int_0^\infty x\exp(-x^2y^2)f(x)dx, \tag{1.2}$$

was introduced by Yurekli and Sadek [10]. After then Aghili, Ansari and Sedghi [1] derived the following complex inversion formula

$$\mathcal{L}_2^{-1}\{\mathcal{L}_2\{f(x);y\}\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2\mathcal{L}_2\{f(x);\sqrt{y}\}\exp(yx^2)dy, \tag{1.3}$$

where $\mathcal{L}_2\{f(x);\sqrt{y}\}$ has a finite number of singularities in the left half plane $Re(y) \leq c$. The generalized Laplace transform $\mathcal{L}_n$ and the inverse generalized

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Laplace transform $\mathcal{L}_n^{-1}$ were introduced by Dernek and Aylıkçı in

$$\mathcal{L}_n\{f(x);y\} = \int_0^\infty x^{n-1}\exp(-x^n y^n) f(x)dx \tag{1.4}$$

$$\mathcal{L}_n^{-1}\{F(y);x\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} n\mathcal{L}_n\{f(x);y^{\frac{1}{n}}\}\exp(yx^n)dy, \tag{1.5}$$

respectively. The $\mathcal{L}_n$-transform is related to the Laplace transform with

$$\mathcal{L}_n\{f(x);y\} = \frac{1}{n}\mathcal{L}\{f(x^{\frac{1}{n}});y^n\}. \tag{1.6}$$

Definition 1.1. The $\overline{\delta}$ differential operator $\overline{\delta}$ that we call the $\overline{\delta}$-derivative is defined as

$$\overline{\delta}_x = \frac{1}{x^{n-1}}\frac{d}{dx},\ (n \in \mathbb{N}) \tag{1.7}$$

and

$$\overline{\delta}_x^2 = \overline{\delta}_x\overline{\delta}_x = \frac{1}{x^{2n-2}}\frac{d^2}{dx^2} - \frac{(n-1)}{x^{2n-1}}\frac{d}{dx}. \tag{1.8}$$

The $\overline{\delta}$ derivative operator can be successively applied in a similar fashion for any positive integer power.

Definition 1.2. The convolution of $f(x)$ and $g(x)$ is defined by

$$f(x) * g(x) = \int_0^x \tau^{n-1} g(\tau) f((x^n - \tau^n)^{1/n})d\tau. \tag{1.9}$$

The above integral is often referred to as the convolution integral.

2. The main results

In this section we will give some properties of the $\mathcal{L}_n$-transform that will be used to solve the initial-boundary-value problems for ordinary differential equations.

Here we will derive a relation between the $\mathcal{L}_n$-transform of the $\overline{\delta}$-derivative of a function (1.7) and the $\mathcal{L}_n$-transform of the function itself.

Theorem 2.1. *If $f, f', ..., f^{(k-1)}$ are all continuous functions with a piecewise continuous derivative $f^{(k)}$ on the interval $[0,\infty)$, and if all functions are of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$ for some constant α then*

$$\mathcal{L}_n\{\overline{\delta}_x^k f(x);y\} = (ny^n)^k \mathcal{L}_n\{f(x);y\} - (ny^n)^{k-1} f(0^+)$$

$$-(ny^n)^{k-2}(\overline{\delta}_x f)(0^+) - ... - ny^n(\overline{\delta}_x^{k-2} f)(0^+) - (\overline{\delta}_x^{k-1} f)(0^+) \tag{2.1}$$

for $k \geq 1$, k is a positive integer.

Proof. Suppose that $f(x)$ is a continuous function with a piecewise continuous derivative $f'(x)$ on the interval $[0,\infty)$. Also, suppose that f and f' are of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$ where α is a constant. With using the definitions of $\mathcal{L}_n$-transform and the $\overline{\delta}$ derivative and integration by parts, we obtain

$$\mathcal{L}_n\{\overline{\delta}_x f(x); y\} = \int_0^\infty \exp(-y^n x^n) f'(x) dx, \tag{2.2}$$

$$\int_0^\infty \exp(-y^n x^n) f'(x) dx = \lim_{b\to\infty} f(x) \exp(-y^n x^n)|_0^b$$

$$+ny^n \int_0^\infty x^{n-1} \exp(-y^n x^n) f(x) dx. \tag{2.3}$$

Since f is of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$, it follows

$$\lim_{x\to\infty} \exp(-y^n x^n) f(x) = 0 \tag{2.4}$$

and consequently,

$$\mathcal{L}_n\{\overline{\delta}_x f(x); y\} = ny^n \mathcal{L}_n\{f(x); y\} - f(0^+). \tag{2.5}$$

Similarly, if f and f' are continuous functions with a piecewise continuous derivative f'' on the interval $[0, \infty)$. If all three functions are of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$, we can use (1.8) to obtain

$$\mathcal{L}_n\{\overline{\delta}_x^2 f(x); y\} = n^2 y^{2n} \mathcal{L}_n\{f(x); y\} - ny^n f(0^+) - \overline{\delta}_x f(0^+). \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\mathcal{L}_n\{\overline{\delta}_x^3 f(x); y\} = n^3 y^{3n} \mathcal{L}_n\{f(x); y\} - n^2 y^{2n} f(0^+)$$

$$-ny^n \overline{\delta}_x f(0^+) - \overline{\delta}_x^2 f(0^+). \tag{2.7}$$

With repeated application of (2.5) and (2.7), we obtain the identity (2.1) of Theorem 1.

Theorem 2.2. *If f is piecewise continuous on the interval $[0, \infty)$ and is of exponential order $\exp(\alpha^n x^n)$ as $x \to \infty$, then the following relation holds true:*

$$\mathcal{L}_n\{x^{kn} f(x); y\} = \frac{(-1)^k}{n^k} \overline{\delta}_y^k \mathcal{L}_n\{f(x); y\} \tag{2.8}$$

for $k \geq 1$, k is a positive integer.

Proof. The $\mathcal{L}_n\{f(x); y\}$ defined by (1.4) is an analytic function in the half plane $Re(y) > \alpha$. It has derivatives of all orders and the derivatives can be formally obtained by differentiating (1.4). Applying the $\overline{\delta}$ with respect to the variable y, we obtain

$$\overline{\delta}_y \mathcal{L}_n\{f(x); y\} = \frac{1}{y^{n-1}} \frac{d}{dy} \int_0^\infty x^{n-1} \exp(-y^n x^n) f(x) dx$$

$$= \frac{1}{y^{n-1}} \int_0^\infty x^{n-1}(-x^n n y^{n-1} \exp(-y^n x^n)) f(x) dx = -n\mathcal{L}_n\{x^n f(x); y\}. \tag{2.9}$$

If we keep taking the $\overline{\delta}$-derivative of (1.4) with respect to the variable y, then we deduce

$$\overline{\delta}_y^k \mathcal{L}_n\{f(x);y\} = \int_0^\infty x^{n-1} \overline{\delta}_y^k \exp(-y^n x^n) f(x) dx \tag{2.10}$$

for $k \in \mathbb{N}$. Where

$$\int_0^\infty x^{n-1}\overline{\delta}_y^k \exp(-y^n x^n) f(x)dx = \int_0^\infty x^{n-1}\overline{\delta}_y^{k-1}[(-n)x^n \exp(-y^n x^n)]f(x)dx$$

$$= \int_0^\infty x^{n-1}\overline{\delta}_y^{k-2}[(-n)^2 x^{2n} \exp(-y^n x^n)]f(x)dx$$

$$\cdots$$

$$= \int_0^\infty x^{n-1}[(-n)^k x^{kn} \exp(-y^n x^n)]f(x)dx = (-n)^k \mathcal{L}_n\{x^{kn} f(x);y\}. \tag{2.11}$$

Thus we obtain the relation (2.8).

Theorem 2.3. *Let $\mathcal{L}_n\{f(x);y^{1/n}\}$ be an analytic function of y except at singular points each of which lies to the left of the vertical line Re $y = a$ and they are finite numbers. Suppose that $y = 0$ is not a branch point and $\lim_{y\to\infty} \mathcal{L}_n\{f(x);y^{1/n}\} = 0$ in the left plane Re $y \le a$ then, the following identity*

$$\mathcal{L}_n^{-1}\{\mathcal{L}_n\{f(x);y\}\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} n\mathcal{L}_n\{f(x);y^{1/n}\}\exp(yx^n)dy$$

$$= \sum_{k=1}^{m}[Res\{n\mathcal{L}_n\{f(x);y^{1/n}\}\exp(yx^n);y=y_k\}] \tag{2.12}$$

holds true for m singular points.

Proof. We take a vertical closed semi-circle as contour of integration. Using residues theorem and boundedness of $\mathcal{L}_n\{f(x);y^{1/n}\}$, we show that the identity (2.12) of Theorem 3 is valid. When $y = 0$ is a branch point we take key-hole contour instead of simple vertical semi-circle.

We assume that $\mathcal{L}_n\{f(x),y^{1/n}\}$ has a finite number of singularities in the left half plane $Rey \le a$. Let $\gamma = \gamma_1 + \gamma_2$ be the closed contour consisting of the vertical line segment γ_1, which is defined from $a - iR$ to $a + iR$ and vertical semi-circle γ_2, that is defined as $|y - a| = R$. Let γ_2 lie to the left of vertical line γ_1. The radius R can be taken large enough so that γ encloses all the singularities of the $\mathcal{L}_n\{f(x);y^{1/n}\}$. Hence, by the residues theorem we have

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} n\mathcal{L}_n\{f(x);y^{1/n}\}\exp(yx^n)dy$$

$$= \frac{1}{2\pi i}\int_{\gamma_1} n\mathcal{L}_n\{f(x);y^{1/n}\}\exp(yx^n)dy - \frac{1}{2\pi i}\int_{\gamma_2} n\mathcal{L}_n\{f(x);y^{1/n}\}\exp(yx^n)dy$$

$$= \sum_{k=1}^{m} [Res\{n\mathcal{L}_n\{f(x); y^{1/n}\} \exp(yx^n); y = y_k\}]$$

$$- \frac{1}{2\pi i} \int_{\gamma_2} n\mathcal{L}_n\{f(x); y^{1/n}\} \exp(yx^n) dy \tag{2.13}$$

where $y_1, y_2, \ldots, y_m$ are all the singularities of $\mathcal{L}_n\{f(x); y^{1/n}\}$. Taking the limit from both sides of the relation (2.13) as R tends to $+\infty$, because of the Jordan's Lemma, the second integral in the right tends to zero.

Even $\mathcal{L}_n\{f(x); y^{1/n}\}$ has one branch point at $y = 0$, we can use the identity (2.12). The proof of the proposition is similar to the proof of the Main Theorem in the paper [1], where we take $n = 2$.

If the number of singularities is infinite, we take the semi-circles γ_m which is centered at point a, with radius $R_m = \pi^2 m^2, m \in \mathbb{N}$.

We illustrate the above Theorem with showing the following examples.

Example 2.1. We show

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^{2n} + a^{2n}}; x\Big\} = \frac{n}{a^n} \sin(a^n x^n) \tag{2.14}$$

where $Re\ a > 0$.

Using the assertion (2.12) of Theorem 3, we obtain

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^{2n} + a^{2n}}; x\Big\} = \sum_{k=1}^{2} Res\Big[n\frac{1}{y^2 + a^{2n}} \exp(yx^n); y = y_k\Big] \tag{2.15}$$

where the singular points are $y_k = \mp ia^n$, $k = 1, 2$. Then we have

$$Res\Big[\frac{n \exp(yx^n)}{y^2 + a^{2n}}; ia^n\Big] = \lim_{y \to ia^n} \frac{n(y - ia^n) \exp(yx^n)}{y^2 + a^{2n}} = \frac{n \exp(ia^n x^n)}{2ia^n} \tag{2.16}$$

and similarly we have

$$Res\Big[n\frac{1}{y^2 + a^{2n}} \exp(yx^n); -ia^n\Big] = -n\frac{\exp(-ia^n x^n)}{2ia^n}. \tag{2.17}$$

Using the relations (2.16) and (2.17), we find the formula (2.14) from (2.15) as follows:

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^{2n} + a^{2n}}; x\Big\} = \frac{n}{a^n} \frac{\exp(ia^n x^n) - \exp(-ia^n x^n)}{2i}$$

$$= \frac{n}{a^n} \sin(a^n x^n). \tag{2.18}$$

Example 2.2. We show

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^n} \exp\Big(-\frac{a^n}{y^n}\Big); x\Big\} = nJ_0(2a^{n/2} x^{n/2}) \tag{2.19}$$

where J_0 is the Bessel function of order zero.

Using the assertion (2.12) of Theorem 3, we have

$$\mathcal{L}_n^{-1}\{\frac{1}{y^n} \exp\Big(-\frac{a^n}{y^n}\Big); x\} = Res\Big[n\frac{1}{y} \exp\Big(-\frac{a^n}{y}\Big) \exp(yx^n), y = y_k\Big]. \tag{2.20}$$

From the following Taylor expansions of the exponential functions in (2.20),

$$n\frac{1}{y}\exp\Big(-\frac{a^n}{y}\Big)\exp(yx^n) = \frac{n}{y}\sum_{m=0}^{\infty}(-1)^m\frac{a^{mn}}{m!y^m}\sum_{k=0}^{\infty}\frac{y^k x^{nk}}{k!}$$

$$= \frac{n}{y}\Big[1-\frac{a^n}{1!y}+\frac{a^{2n}}{2!y^2}-\frac{a^{3n}}{3!y^3}+...\Big]\Big[1+\frac{x^n y}{1!}+\frac{x^{2n}y^2}{2!}+\frac{x^{3n}}{3!}+...\Big], \tag{2.21}$$

we find $Res[n\frac{1}{y}\exp(-\frac{a^n}{y})\exp(yx^n)]$ as the coefficient of the term $\frac{1}{y}$ as follows

$$Res\Big[n\frac{1}{y}\exp\Big(-\frac{a^n}{y}\Big)\exp(yx^n)\Big] = n\Big[1-\frac{a^n x^n}{(1!)^2}+\frac{a^{2n}x^{2n}}{(2!)^2}-\frac{a^{3n}x^{3n}}{(3!)^2}+...\Big]$$

$$= n\sum_{m=0}^{\infty}(-1)^m\frac{(ax)^{mn}}{(m!)^2} = nJ_0(2a^{n/2}x^{n/2}). \tag{2.22}$$

Thus, we obtain from (2.22) and the formula (2.20), the assertion (2.19) of Example 2.

Theorem 2.4. ***(Convolution Theorem)***
If $\mathcal{L}_n\{f(x);y\} = F(y)$ and $\mathcal{L}_n\{g(x);y\} = G(y)$, then we have

$$\mathcal{L}_n\{f(x)*g(x);y\} = \mathcal{L}_n\{f(x);y\}\mathcal{L}_n\{g(x);y\} = F(y)G(y). \tag{2.23}$$

Or equivalently,

$$\mathcal{L}_n^{-1}\{F(y)G(y);x\} = f(x)*g(x), \tag{2.24}$$

where $f(x)*g(x)$ is called the convolution of $f(x)$ and $g(x)$ and it is defined by the relation (1.9).

Proof. We have, by definitions (1.4) and (1.9),

$$\mathcal{L}_n\{f(x)*g(x);y\} = \int_0^{\infty} x^{n-1}\exp(-x^n y^n)\int_0^{x}\tau^{n-1}g(\tau)f((x^n-\tau^n)^{1/n})d\tau dx. \tag{2.25}$$

The integration in (2.25) is first performed with respect to τ from $\tau = 0$ to $\tau = x$ of the vertical strip and then from $x = 0$ to ∞ by moving the vertical strip from $x = 0$ outwards to cover the whole region under the line $\tau = x$. We now change the order of integration so that we integrate first along the horizontal strip from $t = \tau$ to ∞ and then from $\tau = 0$ to ∞ by moving the horizontal strip vertically from $\tau = 0$ upwards. Evidently, (2.25) becomes

$$\mathcal{L}_n\{f(x)*g(x);y\}$$

$$= \int_0^{\infty}\tau^{n-1}g(\tau)\int_{\tau=x}^{\infty}x^{n-1}\exp(-x^n y^n)f((x^n-\tau^n)^{1/n})dxd\tau, \tag{2.26}$$

which is, by the change of variable $x^n - \tau^n = u^n$,

$$\mathcal{L}_n\{f(x)*g(x);y\} = \int_0^{\infty}\tau^{n-1}g(\tau)\int_0^{\infty}u^{n-1}\exp(-(u^n+\tau^n)y^n)f(u)dud\tau$$

$$= \Big(\int_0^\infty \tau^{n-1}\exp(-\tau^n y^n) g(\tau)d\tau\Big)\Big(\int_0^\infty u^{n-1}\exp(-u^n y^n) f(u)du\Big)$$

$$= G(y)F(y). \tag{2.27}$$

3. Application of the $\mathcal{L}_n$-transform to ordinary differential equations

Example 3.1. We solve the following ordinary differential equation

$$xz'' - (2v+n-3)z' + x^{n-1}z = 0,\ k\in\mathbb{N},\ v\in\mathbb{N}. \tag{3.1}$$

solution: Dividing (3.1) by x^{n-1}, adding and subtracting the term $\frac{n-1}{x^{n-1}}z'$ we obtain

$$x^n\Big(\frac{1}{x^{2n-2}}z'' - \frac{n-1}{x^{2n-1}}z'\Big) + \frac{n-1}{x^{n-1}}z' - \frac{2v+n-3}{x^{n-1}}z' + z = 0. \tag{3.2}$$

Using the definition of the $\overline{\delta}$-derivative given in (1.7) and (1.8), we can express (3.2) as

$$x^n\overline{\delta}_x^2 z(x) - 2(v-1)\overline{\delta}_x z(x) + z(x) = 0. \tag{3.3}$$

Applying the $\mathcal{L}_n$-transform to (3.3), we find

$$\mathcal{L}_n\{x^n\overline{\delta}_x^2 z;y\} - 2(v-1)\mathcal{L}_n\{\overline{\delta}_x z;y\} + \mathcal{L}_n\{z(x);y\} = 0. \tag{3.4}$$

Using Theorem 1 for $k=1$ and $k=2$ in (3.4) and performing necessary calculations we obtain

$$-\frac{1}{n}\overline{\delta}_y\mathcal{L}_n\{\overline{\delta}_x^2 z;y\} - 2(v-1)\mathcal{L}_n\{\overline{\delta}_x z;y\} + \mathcal{L}_n\{z;y\} = 0, \tag{3.5}$$

$$-\frac{1}{n}\frac{1}{y^{n-1}}\frac{d}{dy}(n^2y^{2n}\overline{z}(y) - ny^n z(0^+) - \overline{\delta}_x z(0^+))$$

$$-2(v-1)(ny^n\overline{z}(y) - z(0^+)) + \overline{z}(y) = 0 \tag{3.6}$$

where $\overline{z}(y) = \mathcal{L}_n\{z(x);y\}$. We assume that $z(0^+) = 0$. Thus, we obtain the following first order differential equation:

$$\overline{z}'(y) + \Big(2(n+v-1)\frac{1}{y} - \frac{1}{ny^{n+1}}\Big)\overline{z}(y) = 0. \tag{3.7}$$

Solving the first order differential equation (3.7), we have

$$\overline{z}(y) = C\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!n^{2m}y^{mn+2n+2v-2}}. \tag{3.8}$$

Applying the $\mathcal{L}_n^{-1}$ transform, we obtain

$$z(x) = C\sum_{m=0}^{\infty}(-1)^m\frac{x^{mn+n+2v-2}}{m!\Gamma(m+\frac{n+2v-2}{n}+1)n^{2m-1}} \tag{3.9}$$

where we use the following relations

$$\mathcal{L}_n\{x^k;y\} = \frac{\Gamma(\frac{k}{n}+1)}{ny^{n+k}}\ ,\ k = mn+n+2v-2 \tag{3.10}$$

and

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^{mn+n+2v-2+n}};x\Big\} = \frac{nx^{mn+n+2v-2}}{\Gamma(m+1+\frac{2v-2}{n}+1)}. \tag{3.11}$$

Setting $\alpha = \frac{2v+n-2}{n}$, $C = n^{-\frac{2v-2}{n}-2}$, we obtain the solution of the ordinary differential equation (3.1),

$$z(x) = x^{\frac{n\alpha}{2}} J_\alpha\Big(\frac{2}{n} x^{\frac{n}{2}}\Big), \tag{3.12}$$

where $\alpha \in \mathbb{Z}$ because of the inequality $v > n$ ($v, n \in \mathbb{N}$) and J_α is the Bessel function of the first kind of order α.

Example 3.2. We solve the following ordinary differential equation

$$xz'' - (n^2-1)z' + x^{n-1}z = 0, \ n = 0, 1, 2, ..., \tag{3.13}$$

solution: Dividing (3.13) by x^{n-1}, adding and subtracting the term $\frac{n-1}{x^{n-1}}z'$ we obtain

$$x^n\Big(\frac{1}{x^{2n-2}}z''(x) - \frac{n-1}{x^{2n-1}}z'(x)\Big) + \frac{n-1}{x^{n-1}}z'(x)$$

$$-(n^2-1)\frac{1}{x^{n-1}}z'(x) + z(x) = 0. \tag{3.14}$$

Using the definition of the $\overline{\delta}_x$-derivative (1.7) and (1.8), we can express (3.14) as

$$x^n \overline{\delta}_x^2 z(x) - n(n-1)\overline{\delta}_x z(x) + z(x) = 0. \tag{3.15}$$

Considering the following relations;
(3.16)

$$\mathcal{L}_n\{x^n \overline{\delta}_x^2 z(x); y\} = -\frac{1}{n}\overline{\delta}_y \mathcal{L}_n\{\overline{\delta}_x^2 z(x); y\} = -2n^2 y^n \overline{z}(y) - ny^{n+1}\overline{z}'(y) + nz(0^+),$$

$$n(n-1)\mathcal{L}_n\{\overline{\delta}_x z(x); y\} = n(n-1)(ny^n \overline{z}(y) - z(0^+))$$

$$= n^2(n-1)y^n\overline{z}(y) - n(n-1)z(0^+), \tag{3.17}$$

and applying the $\mathcal{L}_n$-transform to (3.15), we obtain

$$\mathcal{L}_n\{x^n \overline{\delta}_x^2 z(x); y\} - n(n-1)\mathcal{L}_n\{\overline{\delta}_x z(x); y\} + \mathcal{L}_n\{z(x); y\} = 0 \tag{3.18}$$

$$ny^{n+1}\overline{z}'(y) + [n^2(n+1)y^n - 1]\overline{z}(y) - n^2 z(0^+) = 0 \tag{3.19}$$

where $\overline{z}(y) = \mathcal{L}_n\{z(x); y\}$.

We may assume

$$z(0^+) = 0. \tag{3.20}$$

Solving the first order differential equation after substituting (3.20) into (3.19), we get

$$\overline{z}(y) = Cy^{-n^2-n}\exp\Big(-\frac{1}{n^2 y^n}\Big). \tag{3.21}$$

Calculating the Taylor expansion of the exponential function in (3.21), we have

$$\overline{z}(y) = C\sum_{m=0}^{\infty}\frac{(-1)^m}{m!n^{2m}}\frac{1}{y^{n+nm+n^2}}. \tag{3.22}$$

Using the following relation,

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{y^{n+nm+n^2}}; x\Big\} = \frac{nx^{nm+n^2}}{\Gamma(m+n+1)}, \tag{3.23}$$

and applying the $\mathcal{L}_n^{-1}$ transform to (3.22), we find

$$z(x) = Cn^{n+1}x^{\frac{n^2}{2}} \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m+n+1)}\Big(\frac{2x^{n/2}}{2n}\Big)^{2m+n}. \tag{3.24}$$

Setting $C = n^{-n-1}$ in (3.24), we obtain the solution of the equation (3.13)

$$z(x) = x^{\frac{n^2}{2}} J_n\Big\{\frac{2}{n}x^{\frac{n}{2}}\Big\} \tag{3.25}$$

where J_n is the Bessel function of the first kind of order n.

Example 3.3. We solve the following initial-value problem:

$$u_{xx} - (n-1)\frac{1}{x}u_x - x^{n-1}u_x = x^{2n-2}f(x),\ x > 0, \tag{3.26}$$

$$u(0^+) = 0,\ u_x(0^+) = 0. \tag{3.27}$$

solution: Dividing both sides of (3.26) by x^{2n-2}, we get

$$x^{-2n+2}u_{xx} - (n-1)x^{-2n+1}u_x - x^{-n+1}u_x = f(x). \tag{3.28}$$

We use the definitions (1.7) and (1.8), the equation (3.28) becomes

$$\overline{\delta}_x^2 u - \overline{\delta}_x u = f(x). \tag{3.29}$$

Applying the $\mathcal{L}_n$-transform on both sides of (3.29), we have

$$\mathcal{L}_n\{\overline{\delta}_x^2 u; y\} - \mathcal{L}_n\{\overline{\delta}_x u; y\} = \mathcal{L}_n\{f(x); y\}. \tag{3.30}$$

Using the definitions (1.7) and (1.8), we get

$$n^2y^{2n}U - ny^n u(0^+) - (\overline{\delta}_x u)(0^+) - ny^nU + u(0^+) = F(y) \tag{3.31}$$

where $\mathcal{L}_n\{u(x); y\} = U(y)$, $\mathcal{L}_n\{f(x); y\} = F(y)$.
Applying the initial conditions (3.27), we get the following equation:

$$U(y) = \frac{1}{ny^n - 1}F(y) - \frac{1}{ny^n}F(y). \tag{3.32}$$

The inverse generalized Laplace transform (1.5) together with the Convolution Theorem (2.24) leads to the solution:

$$u(x) = \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n-1}; x\Big\} * f(x) - \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n}; x\Big\} * f(x), \tag{3.33}$$

where

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n-1}; x\Big\} = \lim_{y\to\frac{1}{n}} \Big(y - \frac{1}{n}\Big)\frac{n}{ny-1}\exp(yx^n) = \exp(x^n/n), \tag{3.34}$$

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n}; x\Big\} = 1 \tag{3.35}$$

and

$$u(x) = (\exp(x^n/n) - 1) * f(x). \tag{3.36}$$

By the definition of convolution for the $\mathcal{L}_n$-transform, we get the following formal solution:

$$u(x) = \int_0^x \tau^{n-1}\Big[\exp\Big(\frac{1}{n}(x^n - \tau^n)\Big) - 1\Big]f(\tau)d\tau. \tag{3.37}$$

In particular, if we take $f(x) = A_0$ =constant then the solution (3.37) is reduced to

$$u(x) = A_0\Big(\exp(x^n/n) - \frac{x^n}{n} - 1\Big). \tag{3.38}$$

Example 3.4. We solve the following initial-value problem:

$$u_{xx} - \frac{n-1}{x}u_x + x^{n-1}u_x = x^{2n-2}f(x),\ \ x > 0 \tag{3.39}$$

$$u(0^+) = 0,\ \ u_x(0^+) = 0. \tag{3.40}$$

solution: Dividing both sides of (3.39) by x^{2n-2}, we have

$$\frac{1}{x^{2n-2}}u_{xx} - \frac{n-1}{x^{2n-1}}u_x + \frac{1}{x^{n-1}}u_x = f(x). \tag{3.41}$$

Using the definitions of $\overline{\delta}_x$ and $\overline{\delta}^2_x$-derivatives (1.7,1.8), we get

$$\overline{\delta}^2_x u + \overline{\delta}_x u = f(x) \tag{3.42}$$

Applying the $\mathcal{L}_n$-transform to both sides of (3.42), we obtain

$$\mathcal{L}_n\{\overline{\delta}^2_x u; y\} + \mathcal{L}_n\{\overline{\delta}_x u; y\} = \mathcal{L}_n\{f(x); y\}. \tag{3.43}$$

Using the formulas (2.5) and (2.6) of Theorem 1 and the initial conditions (3.40), we find the following equation:

$$U(y) = \frac{1}{ny^n}F(y) - \frac{1}{ny^n+1}F(y). \tag{3.44}$$

Applying the $\mathcal{L}_n^{-1}$-inverse transform to both sides of (3.44) and using the Convolution Theorem, we get

$$u(x) = \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n}F(y); x\Big\} - \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n+1}F(y); x\Big\}, \tag{3.45}$$

$$u(x) = \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n}; x\Big\} * f(x) - \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n+1}; x\Big\} * f(x), \tag{3.46}$$

where

$$\mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n}; x\Big\} = 1 \ \textit{and}\ \mathcal{L}_n^{-1}\Big\{\frac{1}{ny^n+1}; x\Big\} = \exp(-x^n/n). \tag{3.47}$$

Substituting the relations in (3.47) into (3.46), we find

$$u(x) = (1 - \exp(-x^n/n)) * f(x). \tag{3.48}$$

From the definition (1.9) of convolution for the $\mathcal{L}_n$-transform, we have the following formal solutions:

$$u(x) = \int_0^x \tau^{n-1}(1 - \exp(-\tau^n/n))f((x^n - \tau^n)^{1/n})d\tau \tag{3.49}$$

or

$$u(x) = \int_0^x \tau^{n-1}\Big(1 - \exp\Big(-\frac{1}{n}(x^n - \tau^n)\Big)\Big)f(\tau)d\tau. \tag{3.50}$$

In particular if $f(x) = A_0$ =constant, then the solution (3.49) reduces to

$$u(x) = A_0\Big(\exp(-x^n/n) + \frac{x^n}{n} - 1\Big). \tag{3.51}$$

Example 3.5. We solve the following initial-value problem:

$$x^2 u_{xx} - nxu_x = f(x),\ x > 0 \tag{3.52}$$

$$u(0^+) = 0,\ u_x(0^+) = 0. \tag{3.53}$$

solution: We can write the non-homogenous equation (3.52) the following form:

$$x^{2n}\Big(\frac{1}{x^{2n-2}}u_{xx} - \frac{n-1}{x^{2n-1}}u_x\Big) - x^n\frac{1}{x^{n-1}}u_x = f(x) \tag{3.54}$$

Using the definitions $\overline{\delta}_x$ and $\overline{\delta}_x^2$ differential operators (1.7,1.8), we have

$$x^{2n}\overline{\delta}_x^2 u - x^n\overline{\delta}_x u = f(x). \tag{3.55}$$

Taking the $\mathcal{L}_n$-transform yields

$$\mathcal{L}_n\{x^{2n}\overline{\delta}_x^2 u; y\} - \mathcal{L}_n\{x^n\overline{\delta}_x u; y\} = \mathcal{L}_n\{f(x); y\}. \tag{3.56}$$

Using the relation 2.8 of Theorem 2 and the relation 2.1 of Theorem 1, we find

$$\frac{1}{n^2}\overline{\delta}_y^2\mathcal{L}_n\{\overline{\delta}_x^2 u; y\} + \frac{1}{n}\overline{\delta}_y\mathcal{L}_n\{\overline{\delta}_x u; y\} = F(y) \tag{3.57}$$

$$\frac{1}{n^2}\Big(\frac{1}{y^{2n-2}}\frac{d^2}{dy^2} - \frac{n-1}{y^{2n-1}}\frac{d}{dy}\Big)[n^2y^{2n}U - ny^n u(0^+) - (\overline{\delta}_x u)(0^+)]$$

$$+\frac{1}{ny^{n-1}}\frac{d}{dy}[ny^nU - u(0^+)] = F(y) \tag{3.58}$$

Using the given initial conditions 3.53, we obtain the following differential equations:

$$y^2U_{yy} + (3n+2)yU_y + n(2n+1)U = F(y). \tag{3.59}$$

Multiplying to y^{2n} of (3.59), we get

$$d(y^{2n+2}U_y) + nd(y^{2n+1}U) = y^{2n}F(y). \tag{3.60}$$

Integrating both sides of (3.60) and multiplying by y^{-n-2} both sides of the result, we have

$$y^nU_y + ny^{n-1}U = y^{-n-2}\int y^{2n}F(y)dy + c_1y^{-n-2} \tag{3.61}$$

and then,

$$d(y^nU) = y^{-n-2}\int y^{2n}F(y)dy + c_1y^{-n-2} \tag{3.62}$$

where c_1 is an arbitrary constant.
Integrating both sides of (3.62) and multiplying y^{-n} both sides of the result, we obtain

$$U(y) = y^{-n}\int y^{-n-2}[\int y^{2n}F(y)dy]dy - c_1\frac{y^{-2n-1}}{n+1} + c_2y^{-n} \tag{3.63}$$

where c_2 is an arbitrary constant. If we take $f(x) = 0$, then $\mathcal{L}_n\{f(x); y\} = F(y) = 0$. Making use the following relation:

$$\mathcal{L}_n\{x^{kn}; y\} = \frac{\Gamma(k+1)}{ny^{n(k+1)}}, \tag{3.64}$$

the solution of the problem becomes

$$u(x) = nc_2 - c_1 \frac{n}{n+1} \frac{x^{n+1}}{\Gamma(2+\frac{1}{n})}. \tag{3.65}$$

Conclusion: We conclude this investigation by remarking that many other available initial-boundary value problems can be solved in this manner by applying the above theorems. In some problems, this method is useful than the other methods.

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