Konuralp Journal of Mathematics
Volume 4 No. 1 pp. 108–113 (2016) ©KJM

HADAMARD AND FEJÉR-HADAMARD INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRALS INVOLVING SPECIAL FUNCTIONS

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Abstract. Fractional calculus is as important as calculus. This paper is due to presentation of Hadamard and Fejér-Hadamard inequalities for fractional calculus. We prove Hadamard and Fejér-Hadamard inequalities for generalized fractional integral involving Mittag–Leffler function. Also, inequalities for special cases are obtained.

1. introduction

Definition 1.1. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{1.1}$$

holds, for all x,y $\in$ [a,b] and $\lambda \in [0, 1]$. The function f is called concave if reverse of inequality (1.1) holds.

For any convex function $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$, following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $a, b \in I$ and a<b.
Inequality (1.2) is well known in literature as Hadamard inequality. The Hadamard inequality got attention of many mathematicians and many generalizations, refinements have been found so far for example see, [7, 3, 4, 6, 14, 15, 16] and the references cited therein.
In [9] Fejér gave generalization of Hadamard inequality known as Fejér-Hadamard inequality.

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* Primary 26A51, 26A33; Secondary 33E15, 26D15.
Key words and phrases. Convex function, Hermite-Hadamard inequality, Fejér-Hadamard inequality, fractional integrals, Mittag-Leffer function.

For any convex function $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$, following inequality holds

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.3}$$

where g is a function which is inegrable, non-negative and symmetric about $\frac{a+b}{2}$. Fractional calculus refers to integration or differentiation of fractional order is as old as calculus. For a historical survey the reader may see [11, 12, 13].

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. Many researchers have explored certain extensions and generalizations of integral inequalities by involving fractional calculus (see, [1, 2, 5, 10, 17, 22, 8, 20]).
As we are going to give Hadamard and Fejér-Hadamard inequalities for generalized fractional integral operator containing Mittag–Leffler function [19]. We give two sided definition of this generalized fractional integral operator containing Mittag–Leffler function as follows:

Definition 1.2. Let $\alpha, \beta, k, l, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator containing Mittag–Leffler function $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ for a real-valued continuous function f is defined by:

$$(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f)(x) = \int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^\alpha)f(t)dt, \tag{1.4}$$

and

$$(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(x) = \int_x^b (t-x)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-x)^\alpha)f(t)dt, \tag{1.5}$$

where the function $E^{\gamma,\delta,k}_{\alpha,\beta,l}$ is generalized Mittag–Leffler function defined as

$$E^{\gamma,\delta,k}_{\alpha,\beta,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)}\frac{t^n}{(\delta)_{ln}}, \tag{1.6}$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)...(a+n-1), (a)_0 = 1$.

If $\delta = l = 1$ in (1.4), then integral operator $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ reduces to an integral operator containing generalized Mittag–Leffler function $E^{\gamma,1,k}_{\alpha,\beta,1}$ introduced by Srivastava, and Tomovski in [21]. Along $\delta = l = 1$ in addition if $k = 1$ (1.4) reduces to an integral operator defined by Prabhakar in [17] containing Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$. For $\omega = 0$ in (1.4), integral operator $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ would correspond essentially to the Riemann–Liouville fractional integral operator (see, [19]),

$$I^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}f(t)dt,\ \beta > 0.$$

$$I^{\beta}_{b-}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}f(t)dt,\ \beta > 0.$$

In[20], Sarikaya et al. proved the following result:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integral holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2(b-a)^\beta}\left[I^\beta_{a+}f(b) + I^\beta_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\beta > 0$.

In [18] Fejér-Hadamard inequality for Reimann-Liouville fractional integrals which appears as a generalization of Theorem 1.1 is given.
As fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations, in this paper we are interested to give versions of Hadamard and Fejér-Hadamard inequalities in fractional calculus. Also we show our results are more general than such results which already have been proved.

2. Hermite Hadamard inequality for generalized fractional integrals involving Mittag–Leffler function

In the following we give Hadamard and Fejér-Hadamard inequalities for generalized fractional integral containing generalized Mittag–Leffler function defined in (1.4). We also show that these inequalities are generalizations of Hadamard and Fejér-Hadamard inequalities for Reimann–Liouville fractional integrals given in [18] and [20].

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integral holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1)(b) &\le \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(a)}{2} \\ &\le \frac{f(a)+f(b)}{2}\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}1)(a), \end{aligned} \tag{2.1}$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

Proof. For $t \in [0,1]$; $ta + (1-t)b, (1-t)a + tb \in [a,b]$. As f is convex function on $[a,b]$, therefore we have

$$f\left(\frac{1}{2}(ta+(1-t)b) + \left(1-\frac{1}{2}\right)((1-t)a+tb)\right) \le \frac{f(ta+(1-t)b) + f((1-t)a+tb)}{2}$$

that gives after multiplying with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)$

$$2t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f\left(\frac{a+b}{2}\right) \le t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)\left(f(ta+(1-t)b) + f((1-t)a+tb)\right).$$

Integrating over t on $[0,1]$ we have

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)dt \\ &\le \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f(ta+(1-t)b)dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f((1-t)a+tb)dt. \end{aligned}$$

If $u = at + (1-t)b$, then $t = \frac{b-u}{b-a}$ and if $v = (1-t)a + tb$, then $t = \frac{v-a}{b-a}$. So one can have

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(a)}{2}. \tag{2.2}$$

On the other hand using that f is convex on $[a, b]$ we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq tf(a)+(1-t)f(b)+(1-t)f(a)+tf(b) = f(a)+f(b).$$

Now multiplying with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)$ and integrating over $[0, 1]$ we get,

$$\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f(ta + (1-t)b)dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f((1-t)a + tb)dt \leq [f(a) + f(b)] \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)dt$$

from which by using change of variables as for (2.2) we get

$$(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(a) \leq (f(a) + f(b))\, \epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}1)(a). \tag{2.3}$$

Combining equation (2.2) and equation (2.3) we get inequality in (2.1). □

Remark 2.1. If $\delta = l = 1$ in (2.1), then we have fractional Hadamard inequality for integral operator introduced by Srivastava, and Tomovski in [21]. Along $\delta = l = 1$ in addition if $k = 1$ in (2.1), then we have fractional Hadamard inequality for integral operator defined by Prabhakar in [17].

Remark 2.2. If we take $\omega = 0$, the above theorem gives inequality in Theorem 1.1. Moreover if along $\omega = 0$ we take $\alpha = 1$, then we get (1.2).

In the following we give Fejér-Hadamard inequality for generalized fractional integral operator defined in (1.4).

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, let $g : [a, b] \to \mathbb{R}$ be a function which is non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality for generalized fractional integral holds*

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}g)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}fg)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}fg)(a)}{2} \leq \frac{f(a) + f(b)}{2}\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}g)(a), \tag{2.4}$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

Proof. For $t \in [0, 1]$; $ta + (1-t)b, (1-t)a + tb \in [a, b]$. As f is convex function, therefore we have

$$f\left(\frac{1}{2}(ta + (1-t)b) + \left(1 - \frac{1}{2}\right)((1-t)a + tb)\right) \leq \frac{f(ta + (1-t)b) + f((1-t)a + tb)}{2}$$

that gives after multiplying with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g(tb + (1-t)a)$

$$2t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f\left(\frac{a+b}{2}\right)g(tb + (1-t)a) \leq t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)\left(f(ta + (1-t)b) + f((1-t)a + tb)\right)g(tb + (1-t)a).$$

Integrating over t on $[0,1]$

$$2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})g(tb+(1-t)a)dt$$
$$\leq \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f(ta+(1-t)b)g(tb+(1-t)a)dt$$
$$+\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f((1-t)a+tb)g(tb+(1-t)a)dt.$$

If $u=at+(1-t)b$, then $t=\frac{b-u}{b-a}$ and if $v=(1-t)a+tb$, then $t=\frac{v-a}{b-a}$. So one can have

$$2f\left(\frac{a+b}{2}\right)\int_a^b (b-u)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}\left(\omega\left(\frac{b-u}{b-a}\right)^{\alpha}\right)g(a+b-u)du$$
$$\leq \int_a^b (b-u)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}\left(\omega\left(\frac{b-u}{b-a}\right)^{\alpha}\right)f(u)g(a+b-u)du$$
$$+\int_b^a (v-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}\left(\omega\left(\frac{v-a}{b-a}\right)^{\alpha}\right)f(v)g(a+b-v)dv.$$

From which by symmetry of function g about $\frac{a+b}{2}$ one can have

$$f\left(\frac{a+b}{2}\right)\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}g)(b)\leq\frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}fg)(b)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}fg)(a)}{2}. \tag{2.5}$$

On the other hand using that f is convex on $[a,b]$ we have

$$f(ta+(1-t)b)+f((1-t)a+tb)\leq tf(a)+(1-t)f(b)+(1-t)f(a)+tf(b)=f(a)+f(b).$$

Now multiplying with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})g(ta+(1-t)b)$ and integrating over $[0,1]$ we get,

$$\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f(ta+(1-t)b)g(ta+(1-t)b)dt$$
$$+\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f((1-t)a+tb)g(ta+(1-t)b)dt$$
$$\leq (f(a)+f(b))\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})g(ta+(1-t)b)dt.$$

From which by change of variables it can be seen

$$(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}fg)(b)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}fg)(a)\leq (f(a)+f(b))\,\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}g)(a). \tag{2.6}$$

Combining equation (2.5) and equation (2.6) we get inequality in (2.4). □

Remark 2.3. If we take $g=1$, then we get Theorem 2.1.

Remark 2.4. If $\delta=l=1$ in (2.4), then we have fractional Fejér-Hadamard inequality for integral operator introduced by Srivastava, and Tomovski in [21]. Along $\delta=l=1$ in addition if $k=1$ in (2.4), then we have fractional Fejér-Hadamard inequality for integral operator defined by Prabhakar in [17].

Remark 2.5. If we take $\omega=0$, the above theorem gives Fejér-Hadamard inequality given in [18]. Moreover if along $\omega=0$ we take $\alpha=1$, then we get (1.3).

References

[1] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London 1995.

[2] G. A. Anastassiou, *Advanced inequalities*, **11**, World Scientific, 2011.

[3] A. G. Azpeitia, *Convex functions and the Hadamard inequality*, Revista Colombina Mat. **28** (1994)7-12.

[4] M. K. Bakula, J. Pecaric, *Note on some Hadamard type inequalities*, J. ineq. Pure Appl. Math. **5** (3)(2004) Art. 74.

[5] L. Curiel, L. Galué, *A generalization of the integral operators involving the Gauss hypergeometric function*, Revista Técnica de la Facultad de Ingenieria Universidad del Zulia, **19** (1) (1996), 17–22.

[6] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett. **11** (5)(1998), 91-95.

[7] S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000. Math. Sic. Marh. Roum., **47** (2004), 3-14.

[8] G. Farid, J. Pečarić and Ž. Tomovski, *Opial-type inequalities for fractional integral operator involving Mittag-Leffler function*, Fractional Differ. Calc., Vol. **5** , No. 1 (2015), 93-106.

[9] L. Fejér, *Uberdie Fourierreihen*, II, Math. Naturwise. Anz Ungar. Akad., Wiss, **24** (1906), 369-390, (in Hungarian).

[10] A. A. Kilbas, M. Saigo, R.K. Saxena, *Generalized MittagLeffler function and generalized fractional calculus operators*, Integral Transform. Spec. Funct. **15** (2004) 3149.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of fractional derivatial Equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[12] K. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential Equations*, John Wiley and Sons Inc., New York, 1993.

[13] K. Oldham and J. Spanier *The fractional calculus*, Academic Press, New York - London, 1974.

[14] M. E. Ozdemir, M Avci, E. Set, *On some inequalities of Harmite-Hadamard type via m-convexity*, Appl. Math. Lett. **23** (9)(2010), 1065-1070.

[15] E. Set, M. E. Ozdemir, S. S. Dragomir, *On the Harmite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl. (2010) 9. Article ID 148102.

[16] E. Set, M. E. Ozdemir, S. S. Dragomir, *On Hadamard-type inequalities involving several kinds of convexity*, J. Inequal. Appl. (2010) 12. Article ID 286845.

[17] T. R. Prabhakar, *A singular integral equation with a generalized Mittag–Leffler function in the kernel*, Yokohama Math. J. **19** (1971) 715.

[18] I. Şcan, *Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals*, arXiv preprint arXiv: 1404. 7722 (2014).

[19] T. O. Salim, and A. W. Faraj, *A Generalization of Mittag–Leffler function and integral operator associated with fractional calculus*, J. Fract. Calc. Appl. Vol. **3** July 2012, No. 5, pp. 1 - 13.

[20] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math., Comp., Modelling, **57** (2013), 2403-2407.

[21] H. M. Srivastava, and Ž. Tomovski, *Fractional calculus with an integral operator containing generalized Mittag–Leffler function in the kernal*, Appl. Math. Comput. (2009), doi:10.1016/j.amc.2009.01.055

[22] D. V. Widder, *The Laplace transform*, Princeton Uni. Press, New Jersey, 1941.

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