Konuralp Journal of Mathematics
Volume 4 No. 1 pp. 114–121 (2016) ©KJM

NONLOCAL INTEGRO-DIFFERENTIAL EQUATIONS WITH ARBITRARY FRACTIONAL ORDER

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Abstract. In this paper, we investigate the existence and uniqueness of some nonlocal boundary condition for fractional integro-differential equations with any order. The results are obtained by using fixed point theorems. An example is introduced to illustrate the theorem.

1. Introduction

In the last decades, there has been a great researches on the study of fractional differential equations. A variety of results on initial and boundary value problems of fractional order, ranging from the theoretical to the analytic and numerical methods for finding solutions, have appeared in the literature. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data (see [14]-[17] and references therein). The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part. The existence and uniqueness problems of fractional nonlinear differential equations as an analytical part are investigated by many authors (see [1]-[15]) and references therein). In [4], and [5], the authors obtained sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential equations of orders $\alpha \in (0,1]$ and $\alpha \in (1,2]$ respectively that involving the Caputo fractional derivative and nonlocal conditions. Whereas, The authors in [1], [2], and [13] considered the existence problem of solutions for a class of boundary value problems for fractional differential equations of higher orders involving the Caputo fractional derivative. The existence and uniqueness of initial value problems for some fractional differential equations are investigated by many

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 34A12, 34G20, 26A33.
Key words and phrases. Fractional integro-differential equations; Nonlocal condition; Existence and Uniqueness; Fixed point theorem.

authors (see for example [6], [8], and [9]). The fractional integro-differential equations in different orders are investigated by [10]-[12] using Banach and Krasnoleskii fixed point theorems. The existence of local solutions to initial value problem of Cauchy type for fractional differential equations involving Caputo definition are deeply investigated in the books ([14], [15], and references therein). In fact, the equivalent Volterra integral equation to Cauchy problem for nonlinear fractional differential equations introduced in the cited articles is essential to prove the existence of such systems. For arbitrary order case, there exists a gap that needs more investigations with various types of initial and boundary conditions. Recently, the existence and uniqueness problems of local solutions for arbitrary fractional differential equations is considered by the researchers in [7]. Motivated by these works, we study in this paper the existence and uniqueness of a local solution to nonlocal fractional integro-differential equations at any inner point of a finite interval involving the Caputo derivative. The results are obtained by applying the Banach fixed theorem on the corresponding Volterra integral equation.

2. Equivalence forms

In this section, for given fractional differential equations, we obtain equivalent integral forms in order to use it in the proof of the existence problems. Let us firstly introduce some basic definitions and properties of fractional calculus (see [14], and [15]) which will be used in this paper.

Definition 2.1. A function f is said to be fractional integrable of order $\alpha > 0$ if for all $t > t_0$,

$$I^{\alpha} f(t) = \left(I^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

exists and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Definition 2.2. The Caputo fractional derivative of x is defined as

$${}^{C}D_{t_0}^{\alpha} x(t) = I^{n-\alpha}\left(\frac{d^n x(t)}{dt^n}\right), t > t_0,$$

provided that $D^n x$ is fractional integrable of order $n - \alpha$.

In what follows, we assume that f, g are fractional integrable functions of any order less than or equal to n on their domains.

The compositions between the Caputo fractional derivative and fractional integrals are given by the following Lemma.

Lemma 2.1. *Let $t \in J$, and $c_k \in \mathbb{R}$, then*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}\left(I^{\alpha} x(t)\right) = x(t), \\ I^{\alpha}({}^{C}D_{t_0}^{\alpha}) = x(t) + c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}, \\ {}^{C}D_{t_0}^{\alpha} x(t) = 0, for\ x(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}. \end{cases}$$

We begin these forms by the following basic linear form:

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha} x(t) &= g(t) + \int_{t_0}^{t} f(s) ds, t \in J - \{t_*\} \\ x^{(k)}(t_*) &= b_k, k = 0, 1, 2, ..., n-1, t_* \in J \end{aligned} \tag{2.1}$$

where $g, f \in X$.

Theorem 2.1. *The fractional integro-differential system (2.1) is equivalent to the Volterra integral equation*

$$x(t)=\sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!}\left(b_k-I^{\alpha-k}F(t_*)\right)+I^{\alpha}F(t),t\in J, \tag{2.2}$$

where $F(t)=g(t)+\int\limits_{t_0}^{t}f(s)ds.$

Proof. See ([7]: Theorem 3.1). □

In accordance with Theorem (2.1), it is not hard to deduce some equivalent forms of different nonlinear integro-differential systems. In what follows, assume that $G(t,x(t))=g(t,x(t))+\int\limits_{t_0}^{t}f(s,x(t))ds.$

Corollary 2.1. *The nonlocal fractional integro-differential system*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}x(t)=g(t,x(t))+\int\limits_{t_0}^{t}f(s,x(t))ds,,t\in J-\{t_*\}\\ x^{(k)}(t_*)=h_k(x(t_*)),k=0,1,2,...,n-1,t_*\in J\end{cases} \tag{2.3}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!}h_k(x(t_*))+\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}G(s,x(s))ds\\ &\quad -\sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!\Gamma(\alpha-k)}\int\limits_{t_0}^{t_*}(t_*-s)^{\alpha-k-1}G(s,x(s))ds, \end{aligned}$$

for $t\in J$.

Corollary 2.2. *The nonlocal fractional integro-differential system*

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha}x(t) &= g(t,x(t))+\int\limits_{t_0}^{t}f(s,x(t))ds,t\in[t_0,T)\\ x^{(k)}(T) &= h_k(x(T)),k=0,1,2,...,n-1, \end{aligned} \tag{2.4}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1}(-1)^k\frac{(T-t)^k}{k!}h_k(x(T))+\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}G(s,x(s))ds\\ &\quad +\sum_{k=0}^{n-1}(-1)^{k+1}\frac{(T-t)^k}{k!\Gamma(\alpha-k)}\int\limits_{t_0}^{T}(T-s)^{\alpha-k-1}G(s,x(s))ds, \end{aligned}$$

for $t\in J$.

Corollary 2.3. *The nonlinear fractional integro-differential system*

$$(2.5)\qquad \begin{aligned} {}^{C}D^{\alpha}_{t_0}x(t) &= g(t,x(t)) + \int_{t_0}^{t} f(s,x(t))ds, t\in(t_0,T) \\ x(t_0) &= h_0(x(t_0)), x^{(k)}(T) = h_k(x(T)), k=1,2,...,n-1, \end{aligned}$$

is equivalent to the integral equation

$$(2.6)\qquad \begin{aligned} x(t) = h_0(x(t_0)) - \sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\left((T-t_0)^k-(T-t)^k\right)h_k(x(T)) \\ +\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}G(s,x(s))ds \\ +\sum_{k=1}^{n-1}\frac{(-1)^k}{k!\Gamma(\alpha-k)}\left((T-t_0)^k-(T-t)^k\right)\int_{t_0}^{T}(T-s)^{\alpha-k-1}G(s,x(s))ds, \end{aligned}$$

for $t\in J$.

3. Existence problems

We investigate in the following section the existence of solution for the fractional integrodifferential systems (2.3)-(2.5) by using Banach fixed point Theorem.

Let $J_1=[t_*-h,t_*+h]\subset(t_0,T)$, where $0<h<\min\{t_*-t_0,T-t_*\}$, and $X_1=C(J_1,\mathbb{R})$ be the space of all real valued continuous functions.

(H1): Let $f,g: J\times X\to\mathbb{R}$, and $h_k: X\to\mathbb{R}$ be jointly continuous Lipschitzian functions that is, there exist positive constants A_f, A_g,and A_k such that

$$\begin{cases} \|f(t,x)-f(t,y)\|\le A_f\|x-y\|, \\ \|g(t,x)-g(t,y)\|\le A_g\|x-y\|, \\ \|h_k(x)-h_k(y)\|\le A_k\|x-y\|, \end{cases}$$

for any $k=0,1,...,n-1$, $t\in J$, and $x,y\in X$. Moreover, let $B_f=\sup_{t\in J}\|f(t,0)\|$, $B_g=\sup_{t\in J}\|g(t,0)\|$, $B_k=|h_k(0)|$,and $L=\max\{A_f,A_g,A_k,B_f,B_g,B_k\}, k=0,1,2,...,n-1$.

Therefore, in accordance with Corollary 2.1, the nonlocal fractional system

$$(3.1)\qquad \begin{cases} {}^{C}D^{\alpha}_{t_*-h}x(t) = g(t,x(t)) + \int_{t_*-h}^{t} f(s,x(t))ds, t\in J_1-\{t_*\} \\ x^{(k)}(t_*) = h_k(x(t_*)), k=0,1,2,...,n-1, \end{cases}$$

is equivalent to the integral equation

$$(3.2)\qquad \begin{aligned} x(t) &= \sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!}h_k(x(t_*)) + \frac{1}{\Gamma(\alpha)}\int_{t_*-h}^{t}(t-s)^{\alpha-1}G(s,x(s))ds \\ &\quad -\sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!\Gamma(\alpha-k)}\int_{t_*-h}^{t_*}(t_*-s)^{\alpha-k-1}G(s,x(s))ds. \end{aligned}$$

Accordingly, we define the operator Ψ on X_1 as follows:

$$(3.3)\quad \Psi x(t) = \sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!}h_k(x(t_*)) + \frac{1}{\Gamma(\alpha)}\int_{t_*-h}^{t}(t-s)^{\alpha-1}G(s,x(s))ds - \sum_{k=0}^{n-1}\frac{(t-t_*)^k}{k!\Gamma(\alpha-k)}\int_{t_*-h}^{t_*}(t_*-s)^{\alpha-k-1}G(s,x(s))ds.$$

The next hypothesis is essential to state and prove the first main result in this section.

(H2): Let θ_1, and r_1 be positive real numbers such that

$$\begin{cases} \theta_1 = L\left(\sum_{k=0}^{n-1}\frac{h^k}{k!} + (1+2h)h^\alpha\left(\frac{2^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{1}{k!\Gamma(\alpha-k+1)}\right)\right) < 1, \\ r_1 \geq \frac{\theta_1}{1-\theta_1}. \end{cases}$$

Moreover, let $\Omega_1 = \{x \in X_1 : \|x\| \leq r_1\}$.

Theorem 3.1. *Let (H1) and (H2) be satisfied, then, there exists a unique solution for the nonlocal fractional integrodifferential system (3.1) in* X_1.

Proof. The Banach fixed point theorem is used to show that Ψ defined by (3.3) has a fixed point on the closed subspace Ω_1 of the Banach space X_1. This fixed point satisfies the integral equation (3.2), hence is a solution of (3.1). For any $t \in J_1$, the joint continuity of f, g, and h_k implies the joint continuity of $G(t,x)$ and hence the continuity of Ψx. By using (H1), we have

$$\begin{aligned} |\Psi x(t)| &\leq \sum_{k=0}^{n-1}\frac{|t-t_*|^k}{k!}(A_k\|x\|+B_k) \\ &+\sum_{k=0}^{n-1}\frac{|t-t_*|^k}{k!}\frac{(A_g\|x\|+B_g)+(A_f\|x\|+B_f)(t-t_*+h)}{\Gamma(\alpha-k+1)}h^{\alpha-k} \\ &+\frac{(A_g\|x\|+B_g)+(A_f\|x\|+B_f)(t-t_*+h)}{\Gamma(\alpha+1)}(t-t_*+h)^\alpha \\ &\leq L\sum_{k=0}^{n-1}\frac{h^k}{k!}(\|x\|+1) + \sum_{k=0}^{n-1}\frac{h^\alpha}{k!}\frac{L(\|x\|+1)(1+2h)}{\Gamma(\alpha-k+1)} \\ &+\frac{L(\|x\|+1)(1+2h)}{\Gamma(\alpha+1)}(2h)^\alpha \\ &\leq L\sum_{k=0}^{n-1}\frac{h^k}{k!} + L(1+2h)h^\alpha\left(\frac{2^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{1}{k!\Gamma(\alpha-k+1)}\right) \\ &+L\sum_{k=0}^{n-1}\frac{h^k}{k!} + L(1+2h)h^\alpha\left(\frac{2^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{1}{k!\Gamma(\alpha-k+1)}\right)\|x\|. \end{aligned}$$

Hence, if If $x \in \Omega_1$, it is obvious that $\Psi x \in \Omega_1$. Next, let $x, y \in \Omega_1$, then

$$\begin{aligned} |\Psi x(t) - \Psi y(t)| &\leq \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!} A_k \|x-y\| \\ &\sum_{k=0}^{n-1} \frac{|t-t_*|^k}{k!} \frac{(B_f + A_f (t - t_* + h)) \|x-y\|}{\Gamma(\alpha - k + 1)} h^{\alpha-k} \\ &+ \frac{(B_g + A_g (t - t_* + h)) \|x-y\|}{\Gamma(\alpha+1)} (t - t_* + h)^\alpha \\ &\leq L \left(\sum_{k=0}^{n-1} \frac{h^k}{k!} + (1+2h) h^\alpha \left(\frac{2^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) \right) \|x-y\| \\ &\leq \theta_1 \|x-y\|, \end{aligned}$$

since $\theta_1 < 1$, then Ψ is a contraction mapping on Ω_1. Hence, Ψ has a fixed point which is the unique solution to (3.1). □

Next result is getting the existence of a local solution for the Cauchy problem (2.4). Let $J_2 = [T-h, T] \subset (t_0, T]$, where $0 < h < T - t_0$, and $X_2 = C(J_2, \mathbb{R})$ be the space of all real valued continuous functions on J_2. The nonlocal system

$$(3.4) \qquad \begin{cases} {}^C D^\alpha_{T-h} x(t) = g(t, x(t)) + \int\limits_{T-h}^{t} f(s, x(t)) ds, t \in [T-h, T), \\ x^{(k)}(T) = h_k(x(T)), k = 0, 1, 2, ..., n-1 \end{cases}$$

is equivalent to the Fredholm-Volterra integral equation

$$\begin{aligned} x(t) &= \sum_{k=0}^{n-1} (-1)^k \frac{(T-t)^k}{k!} h_k(x(T)) + \frac{1}{\Gamma(\alpha)} \int\limits_{T-h}^{t} (t-s)^{\alpha-1} G(s, x(s)) ds \\ &\quad - \sum_{k=0}^{n-1} (-1)^k \frac{(T-t)^k}{k!\Gamma(\alpha-k)} \int\limits_{T-h}^{T} (T-s)^{\alpha-k-1} G(s, x(s)) ds \end{aligned}$$

for $x \in X_2$, and $t \in J_2$.

The modified version of (H2) can be given by the following:

(H3): Let θ_2, and r_2 be positive real numbers such that

$$\begin{cases} \theta_2 = L \left(\sum_{k=0}^{n-1} \frac{h^k}{k!} + (1+h) h^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) \right) < 1, \\ r_2 \geq \frac{\theta_2}{1-\theta_2}. \end{cases}$$

Moreover, let $\Omega_2 = \{x \in X_2 : \|x\| \leq r_2\}$.

The proof of the next result is similar to that one of Theorem (3.1), hence it is omitted.

Corollary 3.1. *Let (H1) and (H3) be satisfied, then, there exists a unique solution for the fractional integro-differential system (3.4) in* X_2.

Now, consider the fractional integro-differential system

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}x(t) = g(t,x(t)) + \int\limits_{t_*}^{t} f(s,x(t))ds, , t \in (t_0,T), \\ x^{(k)}(t_0) = h_k(x(t_0)), k = 0,1,2,...,n-1, \end{cases} \tag{3.5}$$

that has an equivalent Volterra integral equation given by

$$x(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} h_k(x(t_0)) + \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1} G(s,x(s))ds$$

for $t \in J_0 = [t_0, t_0 + h]$, $x \in X_0 = C(J_0, \mathbb{R})$.

To establish the existence and uniqueness results to the system (3.5), we replace the next hypothesis instead of (H2).

(H4): Let θ_0, and r_0 be positive real numbers such that

$$\theta_0 = L\left(\sum_{k=0}^{n-1} \frac{h^k}{k!} + \frac{h^\alpha(h+1)}{\Gamma(\alpha+1)}\right) < 1, \text{and } r_0 \geq \frac{\theta_0}{1-\theta_0}.$$

Moreover, let $\Omega_0 = \{x \in X_0 : \|x\| \leq r_0\}$.

Corollary 3.2. *Let (H1), and (H4) be satisfied, then, there exists a unique solution for the fractional integro-differential system (3.5) in* X_0.

The last result in this article is considering the system (2.5)-(2.6). For this, we search the existence and uniqueness of a solution in the interval J.

(H5): Let θ_3, and r_3 be positive real numbers such that

$$\begin{cases} \theta_3 = L\left(\sum_{k=0}^{n-1} \frac{(T-t_0)^k}{k!} + (1+T-t_0)(T-t_0)^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \sum_{k=1}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)}\right)\right) < 1, \\ r_3 \geq \frac{\theta_3}{1-\theta_3}. \end{cases}$$

Moreover, let $\Omega_3 = \{x \in X_3 : \|x\| \leq r_3\}$.

Corollary 3.3. *Let (H1), and (H5) be satisfied, then, there exists a unique solution for the fractional integro-differential system (2.5) in* X.

Example 3.1. Consider the following fractional system

$$\begin{cases} {}^{C}D_0^{\frac{5}{2}}x(t) = \frac{t|x(t)|}{3+3|x(t)|} + \int\limits_{0}^{t} s\sin\frac{x(s)}{3}ds, t \in (0,\frac{1}{2}) \cup (\frac{1}{2},1] \\ x(\frac{1}{2}) = x'\left(\frac{1}{2}\right) = x''\left(\frac{1}{2}\right) = 1. \end{cases} \tag{3.6}$$

The functions $f(t,x(t)) = t\sin\frac{x(t)}{3}$, and $g(t,x(t)) = \frac{t|x(t)|}{3+3|x(t)|}$are jointly continuous functions on $[0,1]\times[0,\infty)$.Moreover, the hypotheses (H1) and (H2) are satisfied such that $L = \frac{1}{3}, h < \frac{1}{2}$, and $\theta_1 < 1$. Hence for large r, there exist a unique solution for the fractional differential system (3.6) on $C[0,1]$.

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