Konuralp Journal of Mathematics
Volume 4 No. 1 pp. 130–139 (2016) ©KJM

ON NEW INEQUALITIES OF HERMITE-HADAMARD-FEJER TYPE FOR GA-s CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, some Hermite-Hadamard-Fejer type integral inequalities for GA-s convex functions in fractional integral forms are obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative,integrable and symmetric to $(a + b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [3, 12, 13, 15].

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.
Key words and phrases. Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Hadamard fractional integral, GA-s convex function.

Definition 1.1. [10, 11]. A function $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. [14]. A function $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be GA-s convex on I if

$$f(x^t y^{1-t}) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [9], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows:

Theorem 1.2. *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let g:$[a, b] \rightarrow [0, \infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then*

$$f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx \leq \int_a^b \frac{f(x) g(x)}{x} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx. \tag{1.3}$$

In [8], the authors established new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms and new Hermite-Hadamard-Fejer inequality for GA-convex function in fractional integral forms as follows:

Theorem 1.3. *Let $f : [a, b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right] \leq \frac{f(a) + f(b)}{2}. \tag{1.4}$$

Theorem 1.4. *Let $f : [a, b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative,integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\sqrt{ab}\right) \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] &\leq \left[J^{\alpha}_{\sqrt{ab}-} (fg)(a) + J^{\alpha}_{\sqrt{ab}+} (fg)(b)\right] \\ &\leq \frac{f(a) + f(b)}{2} \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] \end{aligned} \tag{1.5}$$

with $\alpha > 0$.

Lemma 1.1. [8]. *Let $f : [a, b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $f' \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ then the following equality for fractional integrals holds:*

$$\begin{aligned} & f\left(\sqrt{ab}\right) \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-} (fg)(a) + J^{\alpha}_{\sqrt{ab}+} (fg)(b)\right] \\ = & \frac{1}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f'(t)\, dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}\right) f'(t)\, dt \end{array} \right] \end{aligned} \tag{1.6}$$

with $\alpha > 0$.

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 1.3. [7]. Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [4, 5, 6, 16, 17].

In this paper, we obtained some Hermite-Hadamard-Fejer type integral inequalities for GA-s convex functions in fractional integral forms.

2. Main results

Throughout this section, let $\|g\|_{\infty} = \sup_{t\in[a,b]}|g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Theorem 2.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is GA-s convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left| f\left(\sqrt{ab}\right)\left[J_{\sqrt{ab}-}^{\alpha}g(a) + J_{\sqrt{ab}+}^{\alpha}g(b)\right] - \left[J_{\sqrt{ab}-}^{\alpha}(fg)(a) + J_{\sqrt{ab}+}^{\alpha}(fg)(b)\right]\right| \\ \leq\ & \frac{\|g\|_{\infty}\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_1(\alpha)|f'(a)| + C_2(\alpha)|f'(b)|\right] \end{aligned}$$

where

$$\begin{aligned} C_1(\alpha) &= \begin{pmatrix} \int_0^{\frac{1}{2}} u^{\alpha}(1-u)^s\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1 (1-u)^{\alpha+s}\left(a^{1-u}b^u\right)du \end{pmatrix} \\ C_2(\alpha) &= \begin{pmatrix} \int_0^{\frac{1}{2}} u^{\alpha+s}\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1 (1-u)^{\alpha}u^s\left(a^{1-u}b^u\right)du \end{pmatrix} \end{aligned}$$

with $\alpha > 0$.

Proof. From Lemma 1.1 we have

$$\begin{aligned} &\left| f\left(\sqrt{ab}\right)\left[J_{\sqrt{ab}-}^{\alpha}g(a) + J_{\sqrt{ab}+}^{\alpha}g(b)\right] - \left[J_{\sqrt{ab}-}^{\alpha}(fg)(a) + J_{\sqrt{ab}+}^{\alpha}(fg)(b)\right]\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\begin{bmatrix} \int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt \end{bmatrix} \\ \leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\begin{bmatrix} \int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt \end{bmatrix}. \end{aligned}$$

Setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right) du$ gives

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \tag{2.1}$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right]$$

$$= \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\left.\frac{\left(\ln\frac{s}{a}\right)^{\alpha}}{\alpha}\right|_a^{a^{1-u}b^u}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\left.\frac{-\left(\ln\frac{b}{s}\right)^{\alpha}}{\alpha}\right|_{a^{1-u}b^u}^{b}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right]$$

$$= \frac{\|g\|_\infty\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c} \int_0^{\frac{1}{2}} u^{\alpha}\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \end{array}\right]$$

Since $|f'|$ is GA-s convex on $[a,b]$, we know that for $u \in [0,1]$

$$\left|f'(a^{1-u}b^u)\right| \leq (1-u)^s\left|f'(a)\right| + u^s\left|f'(b)\right|, \tag{2.2}$$

Hence, if we use (2.2) in (2.1), we obtain

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{\|g\|_\infty\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{c} \int_0^{\frac{1}{2}} u^{\alpha}\left[(1-u)^s\left|f'(a)\right| + u^s\left|f'(b)\right|\right]\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha}\left[(1-u)^s\left|f'(a)\right| + u^s\left|f'(b)\right|\right]\left(a^{1-u}b^u\right)du \end{array}\right]$$

$$= \frac{\|g\|_\infty\left(\ln\frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)}\left\{\left(\begin{array}{c} \int_0^{\frac{1}{2}} u^{\alpha}(1-u)^s\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha+s}\left(a^{1-u}b^u\right)du \end{array}\right)\left|f'(a)\right|\right.$$

$$\left.\left(\begin{array}{c} \int_0^{\frac{1}{2}} u^{\alpha+s}\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1(1-u)^{\alpha}u^s\left(a^{1-u}b^u\right)du \end{array}\right)\left|f'(b)\right|\right\}$$

This completes the proof. □

Corollary 2.1. *In Theorem 2.1;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-s convex function which is related the left-hand side of (1.3)*:*

$$\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx - \int_a^b\frac{f(x)g(x)}{x}dx\right|$$

$$\leq \|g\|_\infty\ln^2\left(\frac{b}{a}\right)\left[C_1(1)\left|f'(a)\right| + C_2(1)\left|f'(b)\right|\right],$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for GA-s convex function in fractional integral forms which is related the left-hand side of (1.4)*:*

$$\left|f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right]\right|$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{1-\alpha}}\left[C_1(\alpha)\left|f'(a)\right| + C_2(\alpha)\left|f'(b)\right|\right],$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for GA-s convex function

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \\ \leq & \ln\left(\frac{b}{a}\right) \left[C_1(1) \left| f'(a) \right| + C_2(1) \left| f'(b) \right| \right]. \end{aligned}$$

Theorem 2.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q, q \geq 1$, is GA-s convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right) \left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b) \right] - \left[J^\alpha_{\sqrt{ab}-} (fg)(a) + J^\alpha_{\sqrt{ab}+} (fg)(b) \right] \right| \\ \leq & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)} (\alpha+1)^{1-\frac{1}{q}} \Gamma(\alpha+1)} \\ & \times \left[\begin{array}{c} \left[C_3(\alpha) \left| f'(a) \right|^q + C_4(\alpha) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \\ + \left[C_5(\alpha) \left| f'(a) \right|^q + C_6(\alpha) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \end{array} \right] \end{aligned}$$

where

$$\begin{aligned} C_3(\alpha) &= \int_0^{\frac{1}{2}} u^\alpha (1-u)^s \left(a^{1-u} b^u\right)^q du, \\ C_4(\alpha) &= \int_0^{\frac{1}{2}} u^{\alpha+s} \left(a^{1-u} b^u\right)^q du, \\ C_5(\alpha) &= \int_{\frac{1}{2}}^1 (1-u)^{\alpha+s} \left(a^{1-u} b^u\right)^q du, \\ C_6(\alpha) &= \int_{\frac{1}{2}}^1 (1-u)^\alpha u^s \left(a^{1-u} b^u\right)^q du, \end{aligned}$$

and $\alpha > 0$.

Proof. Using Lemma 1.1, we have

$$\begin{aligned} & \left| f\left(\sqrt{ab}\right) \left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b) \right] - \left[J^\alpha_{\sqrt{ab}-} (fg)(a) + J^\alpha_{\sqrt{ab}+} (fg)(b) \right] \right| \\ \leq & \frac{1}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| dt \end{array} \right] \\ \leq & \frac{\|g\|_\infty}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a}\right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s}\right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| dt \end{array} \right]. \end{aligned}$$

Setting $t=a^{1-u}b^{u}$ and $dt=a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du$ gives

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right|\\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\\ +\int_{\frac{1}{2}}^{1}\left(\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\end{array}\right]\\
=\ & \frac{\|g\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du\\ +\int_{\frac{1}{2}}^{1}\left(\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du\end{array}\right].
\end{aligned}$$

Using power mean inequality we have

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \qquad (2.3)\\
\leq\ & \frac{\|g\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}}\\ \times\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\\ +\left[\int_{\frac{1}{2}}^{1}\left(\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}}\\ \times\left[\int_{\frac{1}{2}}^{1}\left(\int_{a^{1-u}b^{u}}^{b}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\end{array}\right]\\
=\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\\
&\times\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}u^{\alpha}\left|f'\left(a^{1-u}b^{u}\right)\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\\ +\left[\int_{\frac{1}{2}}^{1}\left(1-u\right)^{\alpha}\left|f'\left(a^{1-u}b^{u}\right)\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Since $|f'|^q$ is GA-s convex on $[a,b]$, we know that for $u\in[0,1]$

$$\left|f'(a^{1-u}b^{u})\right|^{q}\leq\left[\left(1-u\right)^{s}\left|f'\left(a\right)\right|^{q}+u^{s}\left|f'\left(b\right)\right|^{q}\right]. \qquad (2.4)$$

Hence, if we use (2.4) in (2.3), we obtain

$$\begin{aligned}
&\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right|\\
\leq\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\\
&\times\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}u^{\alpha}\left[\left(1-u\right)^{s}\left|f'\left(a\right)\right|^{q}+u^{s}\left|f'\left(b\right)\right|^{q}\right]\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\\ +\left[\int_{\frac{1}{2}}^{1}\left(1-u\right)^{\alpha}\left[\left(1-u\right)^{s}\left|f'\left(a\right)\right|^{q}+u^{s}\left|f'\left(b\right)\right|^{q}\right]\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}\end{array}\right]\\
=\ & \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\\
&\left[\begin{array}{c}\left[\left(\int_0^{\frac{1}{2}}u^{\alpha}\left(1-u\right)^{s}\left(a^{1-u}b^{u}\right)^{q}du\right)\left|f'\left(a\right)\right|^{q}+\left(\int_0^{\frac{1}{2}}u^{\alpha+s}\left(a^{1-u}b^{u}\right)^{q}du\right)\left|f'\left(b\right)\right|^{q}\right]^{\frac{1}{q}}\\ +\left[\left(\int_{\frac{1}{2}}^{1}\left(1-u\right)^{\alpha+s}\left(a^{1-u}b^{u}\right)^{q}du\right)\left|f'\left(a\right)\right|^{q}+\left(\int_{\frac{1}{2}}^{1}\left(1-u\right)^{\alpha}u^{s}\left(a^{1-u}b^{u}\right)^{q}du\right)\left|f'\left(b\right)\right|^{q}\right]^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

This completes the proof. □

Corollary 2.2. *In Theorem 2.2;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-s convex function which is related the left-hand side of (1.3)*:*

$$\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \leq \frac{\|g\|_\infty \ln^2\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}}$$
$$\times \left[\begin{array}{c} \left[C_3(1) \left|f'(a)\right|^q + C_4(1) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[C_5(1) \left|f'(a)\right|^q + C_6(1) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right],$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for GA-s convex function in fractional integral forms which is related the left-hand side of (1.4)*:*

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^\alpha} \left[J^\alpha_{\sqrt{ab}-} f(a) + J^\alpha_{\sqrt{ab}+} f(b)\right] \right|$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{2-\frac{\alpha+1}{q}} (\alpha+1)^{1-\frac{1}{q}}} \left[\begin{array}{c} \left[C_3(\alpha) \left|f'(a)\right|^q + C_4(\alpha) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[C_5(\alpha) \left|f'(a)\right|^q + C_6(\alpha) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right],$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for GA-s convex function

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln\frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}}$$
$$\left[\begin{array}{c} \left[C_3(1) \left|f'(a)\right|^q + C_4(1) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[C_5(1) \left|f'(a)\right|^q + C_6(1) \left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right].$$

Theorem 2.3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q, q > 1$, is GA-s convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\left| f\left(\sqrt{ab}\right) \left[J^\alpha_{\sqrt{ab}-} g(a) + J^\alpha_{\sqrt{ab}+} g(b)\right] - \left[J^\alpha_{\sqrt{ab}-} (fg)(a) + J^\alpha_{\sqrt{ab}+} (fg)(b)\right] \right|$$
$$\leq \frac{\|g\|_\infty \, a \ln^{\alpha+1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}} \Gamma(\alpha+1)} \left[\begin{array}{c} \left[C_7 \left|f'(a)\right|^q + C_8 \left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[C_9 \left|f'(a)\right|^q + C_{10} \left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right],$$

where

$$\begin{aligned} C_7 &= \int_0^{\frac{1}{2}} (1-u)^s \left(a^{1-u} b^u\right)^q du \\ C_8 &= \int_0^{\frac{1}{2}} u^s \left(a^{1-u} b^u\right)^q du \\ C_9 &= \int_{\frac{1}{2}}^1 (1-u)^s \left(a^{1-u} b^u\right)^q du \\ C_{10} &= \int_{\frac{1}{2}}^1 u^s \left(a^{1-u} b^u\right)^q du \end{aligned}$$

with $\alpha > 0$ and $1/p + 1/q = 1$.

Proof. Using Lemma 1.1, setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right) du$, Hölder's inequality and (2.4) we have

$$\begin{aligned}
&\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|\,dt \end{array}\right] \\
\leq\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|\,dt \end{array}\right] \\
=\ & \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \end{array}\right] \\
=\ & \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \end{array}\right] \\
\leq\ & \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \end{array}\right] \\
=\ & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \end{array}\right]
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\left[(1-u)^s|f'(a)|^q + u^s|f'(b)|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left[(1-u)^s|f'(a)|^q + u^s|f'(b)|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \end{array}\right] \\
=\ & \frac{\|g\|_\infty\, a \ln^{\alpha+1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}}\Gamma(\alpha+1)} \\
&\times\left[\begin{array}{c} \left[\left(\int_0^{\frac{1}{2}}(1-u)^s\left(a^{1-u}b^u\right)^q du\right)|f'(a)|^q + \left(\int_0^{\frac{1}{2}}u^s\left(a^{1-u}b^u\right)^q du\right)|f'(b)|^q\right]^{\frac{1}{q}} \\ +\left[\left(\int_{\frac{1}{2}}^1(1-u)^s\left(a^{1-u}b^u\right)^q du\right)|f'(a)|^q + \left(\int_{\frac{1}{2}}^1 u^s\left(a^{1-u}b^u\right)^q du\right)|f'(b)|^q\right]^{\frac{1}{q}} \end{array}\right]
\end{aligned}$$

This completes the proof. □

Corollary 2.3. *In Theorem 2.3;*

(1) If we take $\alpha = 1$ we have the following Hermite-Hadamard-Fejer type inequality for GA-s convex function which is related the left-hand side of (1.3)*:*

$$\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)g(x)}{x} dx \right|$$
$$\leq \frac{\|g\|_\infty a \ln^{2-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{p+1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\begin{array}{c} \left[C_7 |f'(a)|^q + C_8 |f'(b)|^q\right]^{\frac{1}{q}} \\ + \left[C_9 |f'(a)|^q + C_{10} |f'(b)|^q\right]^{\frac{1}{q}} \end{array} \right],$$

(2) If we take $g(x) = 1$ we have following Hermite-Hadamard type inequality for GA-s convex function in fractional integral forms which is related the left-hand side of (1.4)*:*

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^\alpha} \left[J^\alpha_{\sqrt{ab}-} f(a) + J^\alpha_{\sqrt{ab}+} f(b) \right] \right|$$
$$\leq \frac{a \ln^{1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}+1-\alpha} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\begin{array}{c} \left[C_7 |f'(a)|^q + C_8 |f'(b)|^q\right]^{\frac{1}{q}} \\ + \left[C_9 |f'(a)|^q + C_{10} |f'(b)|^q\right]^{\frac{1}{q}} \end{array} \right],$$

(3) If we take $\alpha = 1$ and $g(x) = 1$ we have the following Hermite-Hadamard type inequality for GA-s convex function

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right|$$
$$\leq \frac{a \ln^{1-\frac{1}{q}}\left(\frac{b}{a}\right)}{2^{\frac{p+1}{p}} (p+1)^{\frac{1}{p}} q^{\frac{1}{q}}} \left[\begin{array}{c} \left[C_7 |f'(a)|^q + C_8 |f'(b)|^q\right]^{\frac{1}{q}} \\ + \left[C_9 |f'(a)|^q + C_{10} |f'(b)|^q\right]^{\frac{1}{q}} \end{array} \right].$$

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