THE (q, k)-EXTENSION OF SOME GAMMA FUNCTION INEQUALITIES

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ABSTRACT. In this paper, the authors establish some inequalities for the (q, k)-extension of the classical Gamma function. The procedure utilizes a monotonicity property of the (q, k)-extension of the psi function. As an application, some previous results are recovered as special cases of the results of this paper.

1. INTRODUCTION AND DISCUSSION

Let us begin by recalling the following basic definitions concerning our results.

The classical Euler's Gamma function, $\Gamma(x)$ and the classical psi or digamma function, $\psi(x)$ are usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dx = \lim_{n \to \infty} \frac{n! n^x}{x(x+1)(x+2) \dots (x+n)}$$

and

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The p-extension (also known as p-analogue, p-deformation or p-generalization) of the Gamma function, $\Gamma_p(x)$ is defined (see [2]) for $p \in \mathbb{N}$ and $x > 0$ by

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1) \dots (x+p)} = \frac{p^x}{x(1 + \frac{x}{1}) \dots (1 + \frac{x}{p})}$$

where $\lim_{p \to \infty} \Gamma_p(x) = \Gamma(x)$.

Also, the q-extension of the Gamma function, $\Gamma_q(x)$ is defined (see [7]) for $q \in (0,1)$ and $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}} = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+x}}$$

where $\lim_{q \to 1} \Gamma_q(x) = \Gamma(x)$.

The k-extension of the Gamma function, $\Gamma_k(t)$ is similarly defined (see [5]) for $k > 0$ and $x \in \mathbb{C} \backslash k\mathbb{Z}^-$ as

$$\Gamma_k(x) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{x-1}\, dx = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

where $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the k-Pochhammer symbol and $\lim_{k \to 1} \Gamma_k(x) = \Gamma(x)$.

Krasniqi and Merovci [8] also defined the (p,q)-extension of the Gamma for $p \in \mathbb{N}$, $q \in (0,1)$ and $x > 0$ by

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \dots [x+p]_q}$$

where $[p]_q = \frac{1-q^p}{1-q}$, and $\Gamma_{p,q}(x) \to \Gamma(x)$, as $p \to \infty$ and $q \to 1$.

Furthermore, Díaz and Teruel [6] also defined the (q,k)-extension of the Gamma for $q \in (0,1)$, $k > 0$ and $x > 0$ as follows.

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}(1-q)^{\frac{x}{k}-1}}$$

where
(i) $(x+y)_{q,k}^n := \prod_{i=0}^{n-1}(x+q^{ik}y)$ and
(ii) $(1+x)_{q,k}^t := \frac{(1+x)_{q,k}^{\infty}}{(1+q^{kt}x)_{q,k}^{\infty}}$, for $x,y,t \in \mathbb{R}$ and $n \in \mathbb{N}$.
By using the relations (i) and (ii), the function $\Gamma_{q,k}(x)$ takes the following form.

$$(1.1) \qquad \Gamma_{q,k}(x) = \frac{1}{(1-q)^{\frac{x}{k}-1}} \prod_{n=0}^{\infty} \frac{1-q^{(n+1)k}}{1-q^{nk+x}}$$

It satisfies the following identities.

$$(1.2) \qquad\qquad\qquad \Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x)$$
$$(1.3) \qquad\qquad\qquad \Gamma_{q,k}(k) = 1$$

Similarly, the (q,k)-extension of the psi fuction is defined as

$$\psi_{q,k}(x) = \frac{d}{dx}\ln\Gamma_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}$$

where $\Gamma_{q,k}(x) \to \Gamma(x)$, $\psi_{q,k}(x) \to \psi(x)$ as $q \to 1$, $k \to 1$.

The function $\psi_{q,k}(x)$ as defined above exhibits the following series representations
.

$$(1.4) \qquad \psi_{q,k}(x) = -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{nk+x}}{1-q^{nk+x}},$$

$$(1.5) \qquad = -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{(n+1)x}}{1-q^{(n+1)k}}$$

By taking the logarithmic derivative of (1.1), we obtain (1.4), and the relation (1.5) was established in [4].

In 2008, Shabani [14] proved the following inequality generalizing the previous results of [1], [3], [12] and [13].

$$(1.6) \qquad \frac{\Gamma(a+b)^{\mu}}{\Gamma(c+d)^{\lambda}} \leq \frac{\Gamma(a+bx)^{\mu}}{\Gamma(c+dx)^{\lambda}} \leq \frac{\Gamma(a)^{\mu}}{\Gamma(c)^{\lambda}}, \quad x \in [0,1]$$

where a, b, c, d, μ and λ are real numbers such that $a + bx > 0$, $c + dx > 0$, $a + bx \leq c + dx$, $0 < \mu b \leq \lambda d$ and $\psi(a+bx) > 0$ or $\psi(c+dx) > 0$.

Since then, this inequality has attracted the attention of several researchers. For instance, in [9], [15], [11] and [8], the authors respectively proved the results for the p-extension, q-extension, k-extension and (p,q)-extension of the gamma function.

Also, in 2009, Vinh and Ngoc [16] proved the following related results by using the Dirichlet's integral.

$$(1.7) \qquad \frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i)}{\Gamma(\beta + \sum_{i=1}^{n}\alpha_i)} \leq \frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i x)}{\Gamma(\beta + \sum_{i=1}^{n}\alpha_i x)} \leq \frac{1}{\Gamma(\beta)}$$

where $x \in [0,1]$, $\beta \geq 1$, $\alpha_i > 0$, $n \in \mathbb{N}$.

In addition, in the papers [11] and [17], the authors by using different procedures, separately proved the k-extension of (1.7) together with other results.

In this paper, the objective is to prove the (q,k)-extension of the inequalities (1.6) and (1.7) by using similar techniques as in [11] and [14]. We present our results in the following section.

2. RESULTS

In order to present our results, we need the following Lemmas.

Lemma 2.1. *Let $x > 0$, $y > 0$ with $x \leq y$, then for $q \in (0,1)$ and $k > 0$,*

$$(2.1) \qquad \psi_{q,k}(x) \leq \psi_{q,k}(y).$$

Proof. By differentiating the series representation (1.4), we obtain

$$\psi'_{q,k}(x) = (\ln q)^2 \sum_{n=0}^{\infty}\frac{q^{nk+x}}{(1-q^{nk+x})^2} \geq 0.$$

Thus, $\psi_{q,k}(x)$ is increasing for $x > 0$. Then, for $0 < x \leq y$, we have $\psi_{q,k}(x) \leq \psi_{q,k}(y)$ concluding the proof. See also [10]. $\qquad \square$

Remark 2.1. The fact that $\psi_{q,k}(x)$ is increasing for $x > 0$, can alternatively be established by using identity (1.2). That is, by taking the logarithmic derivative of (1.2), we obtain $\psi_{q,k}(x+k) - \psi_{q,k}(x) = -\frac{(\ln q)q^x}{1-q} \geq 0$.

Lemma 2.2. *Let a, b, c and d be real numbers such that $0 < a + bx \leq c + dx$. Then for $q \in (0,1)$ and $k > 0$, we have*

$$\psi_{q,k}(a + bx) \leq \psi_{q,k}(c + dx).$$

Proof. Follows directly from Lemma 2.1. □

Lemma 2.3. *Let a, b, c, d, μ and λ be positive real numbers such that $\mu b \leq \lambda d$ and $a + bx \leq c + dx$. For $q \in (0,1)$ and $k > 0$, if either:*
(i) $\psi_{q,k}(a + bx) > 0$ or
(ii) $\psi_{q,k}(c + dx) > 0$,
then,

$$\mu b \psi_{q,k}(a + bx) - \lambda d \psi_{q,k}(c + dx) \leq 0.$$

Lemma 2.4. *Let a, b, c, d, μ and λ be positive real numbers such that $\mu b \geq \lambda d$ and $a + bx \leq c + dx$. For $q \in (0,1)$ and $k > 0$, if either:*
(i) $\psi_{q,k}(a + bx) < 0$ or
(ii) $\psi_{q,k}(c + dx) < 0$,
then,

$$\mu b \psi_{q,k}(a + bx) - \lambda d \psi_{q,k}(c + dx) \leq 0.$$

The proofs of Lemmas 2.3 and 2.4 follow from Lemma 2.2, and are in a similar method as those in [14]. As a result, we omit them.

Lemma 2.5. *Let $x \geq 0$, $\alpha_i > 0$, $\beta \geq k > 0$ and $q \in (0,1)$. Then, $k + \alpha_i x \leq \beta + \sum_{i=1}^{n} \alpha_i x$, $n \in \mathbb{N}$ implies*

$$\psi_{q,k}(k + \alpha_i x) \leq \psi_{q,k}\left(\beta + \sum_{i=1}^{n} \alpha_i x\right).$$

Proof. Follows from Lemma 2.1. □

Theorem 2.1. *Define a function G for $x \geq 0$, $q \in (0,1)$ and $k > 0$ by*

$$G(x) = \frac{\Gamma_{q,k}(a + bx)^{\mu}}{\Gamma_{q,k}(c + dx)^{\lambda}}$$

where a, b, c, d, μ and λ are positive real numbers such that $a + bx \leq c + dx$, $\mu b \leq \lambda d$ and either $\psi_{q,k}(a + bx) > 0$ or $\psi_{q,k}(c + dx) > 0$. Then G is a decreasing function of x and the following inequalities hold true:

$$(2.2) \qquad \frac{\Gamma_{q,k}(a + b)^{\mu}}{\Gamma_{q,k}(c + d)^{\lambda}} \leq \frac{\Gamma_{q,k}(a + bx)^{\mu}}{\Gamma_{q,k}(c + dx)^{\lambda}} \leq \frac{\Gamma_{q,k}(a)^{\mu}}{\Gamma_{q,k}(c)^{\lambda}}$$

for $x \in [0, 1]$, and

$$(2.3) \qquad \frac{\Gamma_{q,k}(a + bx)^{\mu}}{\Gamma_{q,k}(c + dx)^{\lambda}} < \frac{\Gamma_{q,k}(a + b)^{\mu}}{\Gamma_{q,k}(c + d)^{\lambda}}$$

for $x \in (1, \infty)$.

Proof. Let $u(x) = \ln G(x)$. Then,

$$u(x) = \ln \frac{\Gamma_{q,k}(a+bx)^\mu}{\Gamma_{q,k}(c+dx)^\lambda} = \mu \ln \Gamma_{q,k}(a+bx) - \lambda \ln \Gamma_{q,k}(c+dx)$$

That implies,

$$\begin{aligned}
u'(x) &= \mu b \frac{\Gamma'_{q,k}(a+bx)}{\Gamma_{q,k}(a+bx)} - \lambda d \frac{\Gamma'_{q,k}c+dx)}{\Gamma_{q,k}(c+dx)} \\
&= \mu b \psi_{q,k}(a+bx) - \lambda d \psi_{q,k}(c+dx) \le 0. \quad \text{(i.e. by Lemma 2.3)}
\end{aligned}$$

That implies u is decreasing on $x \in [0, \infty)$. Consequently, G is also decreasing on $x \in [0, \infty)$. Then for $x \in [0, 1]$ we have,

$$G(1) \le G(x) \le G(0)$$

yielding the result (2.2). Also, for $x \in (1, \infty)$, we have $G(x) < G(1)$ yielding the result (2.3). $\qquad\square$

Remark 2.2. In Theorem 2.1, if $\mu b \ge \lambda d$ and either $\psi_{q,k}(a+bx) < 0$ or $\psi_{q,k}(c+dx) < 0$, then by Lemma 2.4, G is decreasing and the double inequalities (2.2) and (2.3) still hold true.

Theorem 2.2. *Define a function H for $x \ge 0$, $q \in (0,1)$ and $k > 0$ by*

$$H(x) = \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i x)}$$

where $\beta \ge k$, $\alpha_i > 0$, $n \in \mathbb{N}$. Then H is a decreasing function of x and the following inequalities are valid:

$$(2.4) \qquad \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i)} \le \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i x)} \le \frac{1}{\Gamma_{q,k}(\beta)}$$

for $x \in [0, 1]$, and

$$(2.5) \qquad \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i x)} < \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i)}$$

for $x \in (1, \infty)$.

Proof. Let $v(x) = \ln H(x)$. Then,

$$\begin{aligned}
v(x) &= \ln \frac{\prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i x)} \\
&= \ln \prod_{i=1}^n \Gamma_{q,k}(k+\alpha_i x) - \ln \Gamma_{q,k}(\beta + \sum_{i=1}^n \alpha_i x)
\end{aligned}$$

implying that,

$$v'(x) = \sum_{i=1}^{n} \left(\alpha_i \frac{\Gamma'_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(k + \alpha_i x)} \right) - \left(\sum_{i=1}^{n} \alpha_i \right) \frac{\Gamma'_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)}$$

$$= \sum_{i=1}^{n} (\alpha_i \psi_{q,k}(k + \alpha_i x)) - \left(\sum_{i=1}^{n} \alpha_i \right) \psi_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)$$

$$= \sum_{i=1}^{n} \alpha_i \left[\psi_{q,k}(k + \alpha_i x) - \psi_{q,k}(\beta + \sum_{j=1}^{n} \alpha_j x) \right] \leq 0. \quad \text{(by Lemma 2.5)}$$

That implies v and as a result H are decreasing on $x \in [0, \infty)$. Then for $x \in [0, 1]$ we obtain

$$H(1) \leq H(x) \leq H(0)$$

concluding the proof of (2.4). Also, for $x \in (1, \infty)$, we have $H(x) < H(1)$ establishing the result (2.5). $\qquad\square$

By a similar method, it is easy to prove the following results.

Theorem 2.3. *Let Q be defined for $x \geq 0$, $q \in (0, 1)$ and $k > 0$ by*

$$Q(x) = \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)}$$

where $\beta \geq k$, $\alpha_i < 0$, $n \in \mathbb{N}$ such that $0 < k + \alpha_i x \leq \beta + \sum_{i=1}^{n} \alpha_i x$. Then Q is increasing and the following inequalities are satisfied:

$$(2.6) \qquad \frac{1}{\Gamma_{q,k}(\beta)} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)}$$

for $x \in [0, 1]$, and

$$(2.7) \qquad \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)} > \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)}$$

for $x \in (1, \infty)$.

3. CONCLUDING REMARKS

Remark 3.1. By setting $k = 1$ in Theorem 2.1, we obtain the q-extension of (1.6) as presented in [15].

Remark 3.2. By allowing $q \to 1$ in Theorem 2.1, we obtain the k-extension of (1.6) as presented in [11].

Remark 3.3. By allowing $q \to 1$ whiles $k \to 1$ in Theorem 2.1, we obtain (1.6).

Remark 3.4. By allowing $q \to 1$ in Theorem 2.2, we obtain the k-extension of (1.7) as presented in [11] and [17].

Remark 3.5. By setting $k = 1$ in Theorem 2.2, we obtain the q-extension of (1.7).

Remark 3.6. By allowing $q \to 1$ whiles $k \to 1$ in Theorem 2.2, we obtain (1.7).

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