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COMMUTATIVITY OF WEIGHTED SLANT HANKEL OPERATORS

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Abstract. For a positive integer $k \geq 2$, the k^{th}-order weighted slant Hankel operator $D^{\beta}_{k,\phi}$ on $L^2(\beta)$ with $\phi \in L^{\infty}(\beta)$ is defined as $D^{\beta}_{k,\phi} = J^{\beta} W_k M^{\beta}_{\phi}$, where J^{β} is the reflection operator given by $J^{\beta} e_n = e_{-n}$ for each $n \in \mathbb{Z}$ and W_k is given by $W_k e_n(z) = \frac{\beta_m}{\beta_{km}} e_m(z)$ if $n = km, m \in \mathbb{Z}$ and $W_k e_n(z) = 0$ if $n \neq km$. The paper discusses the product and commutativity of k^{th}-order weighted slant Hankel operators of different order. Compactness and essential commutativity of these operators are also addressed and it is obtained that the commutativity of these operators coincides with the essential commutativity.

1. Introduction

Laurent operators or multiplication operators $M_{\phi}(f \mapsto \phi f)$ on $L^2(\mathbb{T})$ induced by $\phi \in L^{\infty}(\mathbb{T})$, $\mathbb{T}$ being the unit circle, play a vital role in the theory of operators with their tendency of inducing various classes of operators. The classes of Toeplitz and Hankel operators are some among them and form two of the most important classes of operators on Hardy spaces.

The notion of Toeplitz operators defined as $T_{\phi} = PM_{\phi}$, was introduced by O. Toeplitz in 1911, where P is an orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Later in 1964, Brown and Halmos studied algebraic properties of these operators. This class of operators was flourished to a great extent with the introduction of slant Toeplitz operators, k^{th}-order slant Toeplitz operators, λ-Toeplitz operators and essentially λ-Toeplitz operators (see [2],[9] and references therein). The study in this direction becomes very promising as spectral properties of the slant Toeplitz operators are well connected with the smoothness of wavelets and wavelet transforms are an alternative for the Fourier transforms, which have many applications in data compression and to solve the differential equations.

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However, the first appearance of Hankel operators is seen in terms of Hankel matrix in the dissertation submitted by H. Hankel in 1861, which is a square complex matrix (finite or infinite) that is constant on each diagonal orthogonal to the main diagonal. A Hankel operator, in abstract terms, is an operator on a Hilbert space that can be represented by a Hankel matrix with respect to an orthonormal basis. From the work of Nehari, it became apparent that a Hankel operator on Hardy spaces is always of the form H_ϕ given by $H_\phi = PJM_\phi$ for some $\phi \in L^\infty(\mathbb{T})$, where J denote the reflection operator on $L^2(\mathbb{T})$ given by $Jf(z) = f(\overline{z})$. This class of operators further resulted into the study and introduction of essentially Hankel operators, λ-Hankel operators, slant Hankel operators, k^{th}-order slant Hankel operators and (λ, μ)-Hankel operators (see [3],[5],[9] and references therein). A lot of applications of Hankel operators can be seen in different directions, which makes the study in this direction more demanding. For example : Hamburger's moment problem [10], interpolation problems [1], rational approximation and stationary processes.

Shields [12], during his study of multiplication operators and the weighted shift operators, discussed weighted sequence spaces which have the tendency to cover Hardy spaces, Bergman spaces and Dirichlet spaces. We begin with the following notational setup needed in the paper.

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ consists of all $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\, \overline{b}_n {\beta_n}^2,$$

for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum\limits_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. $H^\infty(\beta)$ denotes the set of formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. Throughout the paper, we consider these spaces under the assumption that $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual sequence (that is $\beta_n = \beta_{-n}$ for each n). We refer [12] as well as the references therein, for the details of the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and various properties of the weighted multiplication (Laurent) operator $M_\phi^\beta(f \mapsto \phi f)$ on $L^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$.

Let $P^\beta : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Lauric, in the year 2005, discussed the notion of weighted Toeplitz operator $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$. The operators of the kind $A_\phi^\beta = WM_\phi^\beta$ on $L^2(\beta)$, where W is the operator on $L^2(\beta)$ given by $We_{2n} = \frac{\beta_n}{\beta_{2n}} e_n$ and $We_{2n-1} = 0$ for $n \in \mathbb{Z}$, are discussed in [4] and are named as slant weighted Toeplitz operators. For a positive integer $k \geq 2$, let W_k be the operator on $L^2(\beta)$ given by

$$W_k e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m(z) & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

A k^{th}-order slant weighted Toeplitz operator $U_{k,\phi}^\beta$ on $L^2(\beta)$ with $\phi \in L^\infty(\beta)$ is defined as $U_{k,\phi}^\beta = W_k M_\phi^\beta$ (see [6]). A k^{th}-order weighted slant Hankel operator $D_{k,\phi}^\beta$ on $L^2(\beta)$ with $\phi \in L^\infty(\beta)$ is defined as $D_{k,\phi}^\beta = J^\beta W_k M_\phi^\beta$ (see [8]), where J^β is the reflection operator given by $J^\beta e_n = e_{-n}$ for each $n \in \mathbb{Z}$. If $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\beta_n = 1$ for each $n \in \mathbb{Z}$, then $U_{k,\phi}^\beta$ and $D_{k,\phi}^\beta$ on $L^2(\beta)$ become k^{th}-order slant Toeplitz operator $U_{k,\phi}$ ([2]) and k^{th}-order slant Hankel operator $D_{k,\phi}$ ([3]) on $L^2(\mathbb{T})$ respectively. In [11], Liu and Lu has derived relations among the inducing functions of two slant Toeplitz operators with different orders for them to commute or essentially commute. In this paper, we are interested to study the product, compactness and commutativity of weighted slant Hankel operators on $L^2(\beta)$ of different orders. If we assume $\beta_n = 1$ for each n then our results provide results for k^{th}-order slant Hankel operators. We derive a relation among the symbols of two k^{th}-order weighted slant Hankel operators on $L^2(\beta)$ so that their product is a k^{th}-order weighted slant Hankel operator on $L^2(\beta)$.

The algebra of all bounded operators on the Hilbert space $L^2(\beta)$ is denoted by $\mathfrak{B}(L^2(\beta))$.

2. Commutativity

Let $\phi \in L^\infty(\beta)$ be given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$. For an integer $k \geq 2$, the symbol $\phi(z^k)$ stands for $\phi(z^k) = \sum_{n=-\infty}^{\infty} a_n z^{kn}$ and by $\widetilde{\phi}$ we mean $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$. Under the assumption of semi-duality of the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$, we find that $\widetilde{\phi} \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$. We list here some simple facts which are used in the paper and follows using the definitions of respective operators (see [8]).

(1) $(J^\beta)^2 = I$, the identity operator on $L^2(\beta)$.
(2) $J^\beta M_\phi^\beta = M_{\widetilde{\phi}}^\beta J^\beta$.
(3) $J^\beta W_k = W_k J^\beta$, if $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual sequence (which is assumed throughout the paper).

(4) If $\phi \in L^\infty(\beta)$ is such that $\phi(z^k) \in L^\infty(\beta)$ then $M^\beta_\phi W_k = W_k M^\beta_{\phi(z^k)}$.

(5) $W_{k_1} W_{k_2} = W_{k_1 k_2}$ for k_1 and $k_2 \geq 2$. (see [6]).

It is evident, from the definition of reflection operator J^β and W_k, that the $k^{th}-$order weighted slant Hankel operator $D^\beta_{k,\phi}$ given by $D^\beta_{k,\phi} = J^\beta W_k M^\beta_\phi$ on $L^2(\beta)$ induced by the symbol $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ satisfies

$$D^\beta_{k,\phi} e_j = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-nk-j} \beta_{-n} e_n$$

for each integer j. We use the above mentioned facts to conclude that the product of two weighted slant Hankel operators of different order is a k^{th}-order slant weighted Toeplitz operator.

Theorem 2.1. *Let $k_1, k_2 \geq 2$ and $\phi \in L^\infty(\beta)$ be such that $\phi(z^{k_2}) \in L^\infty(\beta)$. Then for any $\psi \in L^\infty(\beta)$, $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi} = U^\beta_{k_1 k_2, \widetilde{\phi}(z^{k_2})\psi}$.*

Proof. A simple computation presents that

$$\begin{aligned} D^\beta_{k_1,\phi} D^\beta_{k_2,\psi} &= J^\beta W_{k_1} M^\beta_\phi J^\beta W_{k_2} M^\beta_\psi \\ &= J^\beta W_{k_1} J^\beta W_{k_2} M^\beta_{\widetilde{\phi}(z^{k_2})} M^\beta_\psi \\ &= W_{k_1 k_2} M^\beta_{\widetilde{\phi}(z^{k_2})\psi} = U^\beta_{k_1 k_2, \widetilde{\phi}(z^{k_2})\psi}. \end{aligned}$$ □

In [8], it is shown that, in case $S_k(\beta) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2(\beta) : f(z^k) = \sum_{n=-\infty}^{\infty} a_n z^{kn} \in L^2(\beta)\}$ is a closed subspace of $L^2(\beta)$ then $\phi(z^k) \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$. Thus an immediate corollary that follows from Theorem 2.1 is the following.

Corollary 2.1. *Let $k_2 \geq 2$ be such that $S_{k_2}(\beta)$ is a closed subspace of $L^2(\beta)$. Then for $\phi, \psi \in L^\infty(\beta)$ and $k_1 \geq 2$, $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi} = U^\beta_{k_1 k_2, \widetilde{\phi}(z^{k_2})\psi}$.*

In [8] it is shown that if the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded then $\phi(z^k) \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$. Thus, we have the following.

Corollary 2.2. *Let $k_2 \geq 2$ be such that $\{\frac{\beta_{nk_2}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then for $\phi, \psi \in L^\infty(\beta)$ and $k_1 \geq 2$, $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi} = U^\beta_{k_1 k_2, \widetilde{\phi}(z^{k_2})\psi}$.*

Now we can conclude from Theorem 2.1 that two k^{th}-order weighted slant Hankel operators, in general, do not commute. In fact, we have $D^\beta_{k,\phi} D^\beta_{k,\psi} = U^\beta_{k^2, \widetilde{\phi}(z^k)\psi}$ and $D^\beta_{k,\psi} D^\beta_{k,\phi} = U^\beta_{k^2, \widetilde{\psi}(z^k)\phi}$.

Theorem 2.1 of Liu and Lu [11], which is further extended for slant weighted Toeplitz operators and states that a k^{th}-order slant weighted Toeplitz operator is a m^{th}-order slant weighted Toeplitz operator ($k \neq m$) if and only if $\phi = 0$ [6] suggests the following.

Theorem 2.2. *The product $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi}$, $\phi, \psi \in L^\infty(\beta)$ is a k^{th}-order slant weighted Toeplitz operator if and only if one of the following holds:*

(1) $k = k_1k_2$.
(2) $\widetilde{\phi}(z^{k_2})\psi = 0$, *when* $k \neq k_1k_2$.

As a consequence of this, we have the following.

Theorem 2.3. *The product* $D_{k_1,\phi}D_{k_2,\psi}$, $\phi, \psi \in L^\infty(\mathbb{T})$, *of two slant Hankel operators of different orders is a* k^{th}*-order slant Toeplitz operator if and only if one of the following holds:*

(1) $k = k_1k_2$.
(2) $\widetilde{\phi}(z^{k_2})\psi = 0$, *when* $k \neq k_1k_2$.

Proof. If we take $\beta_n = 1$ for each $n \in \mathbb{Z}$ then Theorem 2.1 provides that $D_{k_1,\phi}D_{k_2,\psi} = U_{k_1k_2,\widetilde{\phi}(z^{k_2})\psi}$, a k_1k_2-order slant Toeplitz operator. Now result follows applying [Theorem 2.1, 11]. □

Now we check the feasibility for the product of two weighted slant Hankel operators of different orders to be a k^{th}-order weighted slant Hankel operator. It is easy to see that the product $W_{k_1}D^\beta_{k_2,\phi}$ is always a k_1k_2-order weighted slant Hankel operator. In fact it is $D^\beta_{k_1k_2,\phi}$. Our next result shows that this product is a k^{th}-order weighted slant Hankel operator, $k \neq k_1k_2$ only if it is zero operator.

Theorem 2.4. *Let* $k_1, k_2 \geq 2$ *and* $k \neq k_1k_2$. *The operator* $W_{k_1}D^\beta_{k_2,\phi}$, $\phi \in L^\infty(\beta)$ *is a* ${k_1}^{th}$*-order weighted slant Hankel operator if and only if* $\phi = 0$. *Further,* $J^\beta W_{k_1}D^\beta_{k_2,\phi}$ *is a* ${k_1}^{th}$*-order weighted slant Hankel operator if and only if* $\phi = 0$.

Proof. Let $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_nz^n \in L^\infty(\beta)$ and $W_{k_1}D^\beta_{k_2,\phi} = D^\beta_{k_1,\psi}$ for some $\psi(z) = \sum\limits_{n=-\infty}^{\infty} b_nz^n \in L^\infty(\beta)$. Then for each $i \in \mathbb{Z}$, $W_{k_1}D^\beta_{k_2,\phi}e_i = D^\beta_{k_1,\psi}e_i$, equivalently, $\frac{1}{\beta_i}\sum\limits_{n=-\infty}^{\infty} a_{-nk_1k_2-i}\beta_{-nk_1}\frac{\beta_n}{\beta_{nk_1}}e_n = \frac{1}{\beta_i}\sum\limits_{n=-\infty}^{\infty} b_{-nk_1-i}\beta_{-n}e_n$. This gives that $a_{-nk_1k_2-i} = b_{-nk_1-i}$ for each $i, n \in \mathbb{Z}$. This on taking $n = 0$ gives that $a_i = b_i$ for each i and hence we have $a_{k_1+i} = a_{k_1k_2+i} = a_{k_1k_2^2+i} = a_{k_1k_2^3+i} = \cdots$. Now $a_n \to 0$ as $n \to \infty$, we get that $a_{k_1+i} = 0$ for each i. Hence $\phi = 0$. The converse is obvious. Further, along the same lines of proof, we can show that $J^\beta W_{k_1}D^\beta_{k_2,\phi}$ is a k_1^{th}-order weighted slant Hankel operator if and only if $\phi = 0$. □

We are now in a position to obtain the following.

Theorem 2.5. *Let* $k_1, k_2 \geq 2$ *and* $k \neq k_1k_2$. *A necessary and sufficient condition for the product* $D^\beta_{k_1,\phi}D^\beta_{k_2,\psi}$ *of* $D^\beta_{k_1,\phi}$ *and* $D^\beta_{k_2,\psi}$, $\phi, \psi \in L^\infty(\beta)$ *to be a* k_1^{th}*-order weighted slant Hankel operator is that* $\widetilde{\phi}(z^{k_2})\psi = 0$.

Proof. Proof follows on applying Theorem 2.6 to the fact that $D^\beta_{k_1,\phi}D^\beta_{k_2,\psi} = J^\beta W_{k_1} D^\beta_{k_2,\widetilde{\phi}(z^{k_2})\psi}$. □

Now we extend the result [Theorem 2.1, 11] to the weighted slant Hankel operators. It is apparent to see that the doubly infinite matrix $[\lambda_{i,j}]_{i,j\in\mathbb{Z}}$ of a k^{th}-order slant weighted Hankel operator $D^\beta_{k,\phi}$, $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_nz^n \in L^\infty(\beta)$, with respect to the standard orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ of $L^2(\beta)$ always satisfies

$$\frac{\beta_j}{\beta_{-i}}\lambda_{i,j} = \frac{\beta_{j-k}}{\beta_{-i-1}}\lambda_{i+1,j-k},$$

where $\lambda_{i,j} = \langle D^\beta_{k,\phi}e_j, e_i\rangle = \frac{\beta_{-i}}{\beta_j}a_{-ki-j}$ for all $i, j \in \mathbb{Z}$. Now we have the following.

Theorem 2.6. *Let $\phi \in L^\infty(\beta)$ and $m \neq k$. Then $D^\beta_{k,\phi}$ is a m^{th}-order weighted slant Hankel operator on $L^2(\beta)$ if and only if $\phi = 0$.*

Proof. Let $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ be such that $D^\beta_{k,\phi}$ is a m^{th}-order weighted slant Hankel operator. Then for all $i, j \in \mathbb{Z}$, we have

$$\frac{\beta_j}{\beta_{-i}}\langle D^\beta_{k,\phi}e_j, e_i\rangle = \frac{\beta_{j-m}}{\beta_{-(i+1)}}\langle D^\beta_{k,\phi}e_{j-m}, e_{i+1}\rangle \quad (2.1)$$

As $D^\beta_{k,\phi}$ is a k^{th}-order weighted slant Hankel operator, we have $\langle D^\beta_{k,\phi}e_j, e_i\rangle = \frac{\beta_{-i}}{\beta_j}\frac{\beta_{j-k}}{\beta_{-(i+1)}}\langle D^\beta_{k,\phi}e_{j-k}, e_{i+1}\rangle$. Now (2.1) gives that

$$\langle D^\beta_{k,\phi}e_j, e_i\rangle = \frac{\beta_{-i}}{\beta_j}\frac{\beta_{j-mk}}{\beta_{-(i+k)}}\langle D^\beta_{k,\phi}e_{j-mk}, e_{i+k}\rangle,$$

equivalently,

$$\frac{\beta_{-i}}{\beta_j}a_{-ki-j} = \frac{\beta_{-i}}{\beta_j}a_{-k(i+k)-j+mk}$$

for each $i, j \in \mathbb{Z}$. This yields that

$$a_{-ki-j} = a_{-k(i+k)-j+mk} = a_{-k(i+k-m)-j} \quad (2.2)$$

for each $i, j \in \mathbb{Z}$. From (2.2), we get that $a_{-tk|k-m|} = a_0 = a_{tk|k-m|}$, $a_{-tk|k-m|+1} = a_1 = a_{tk|k-m|+1}, \cdots, a_{-tk|k-m|+k|k-m|-1} = a_{k|k-m|-1} = a_{tk|k-m|+k|k-m|-1}$, for each natural number t. But $\phi \in L^\infty(\beta) \subseteq L^2(\beta)$, so we have $\sum\limits_{n=-\infty}^{\infty}|a_n|^2 \leq \sum\limits_{n=-\infty}^{\infty}|a_n|^2{\beta_n}^2 < \infty$. Thus, $a_n \to 0$ as $n \to \infty$, and this helps us to conclude that $a_0 = a_1 = \cdots, a_{k|k-m|-1} = 0$. As a consequence of this, (2.2) helps to provide that $a_n = 0$ for each $n \in \mathbb{Z}$, which gives that $\phi = 0$.

The converse is straightforward. □

As the linear mapping $\phi \mapsto U^\beta_{k,\phi}$ is one-one between $L^\infty(\beta)$ and $\mathfrak{B}(L^2(\beta))$, we have the following.

Theorem 2.7. *Let $k_1, k_2 \geq 2$ and $\phi, \psi \in L^\infty(\beta)$.*

(1) *If $\phi(z^{k_2}) \in L^\infty(\beta)$ then $D^\beta_{k_1,\phi}D^\beta_{k_2,\psi} = 0$ if and only if $\widetilde{\phi}(z^{k_2})\psi = 0$.*
(2) *If $\phi(z^{k_2}), \psi(z^{k_1}) \in L^\infty(\beta)$ then $D^\beta_{k_1,\phi}$ and $D^\beta_{k_2,\psi}$ commute if and only if $\widetilde{\phi}(z^{k_2})\psi - \widetilde{\psi}(z^{k_1})\phi = 0$.*
(3) *If $k_1 \leq k_2$ and $\{\frac{\beta_{nk_2}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded then $D^\beta_{k_1,\phi}$ commutes with $D^\beta_{k_2,\psi}$ if and only if $\widetilde{\phi}(z^{k_2})\psi - \widetilde{\psi}(z^{k_1})\phi = 0$.*

Proof. Proof of (1) and (2) are straightforward. However, proof of (3) follows with the observation that boundedness of $\{\frac{\beta_{nk_1}}{\beta_n}\}_{n\in\mathbb{Z}}$ is obvious from the boundedness of $\{\frac{\beta_{nk_2}}{\beta_n}\}_{n\in\mathbb{Z}}$. Hence, $\phi(z^{k_2})$ and $\psi(z^{k_1})$ belong to the space $L^\infty(\beta)$. Rest of the proof follows using (2). □

3. Compact and Essentially commuting operators

We say that operators A and B essentially commute if $AB - BA$ is a compact operator. The aim of this section is to investigate essentially commuting k^{th}-order weighted slant Hankel operators on $L^2(\beta)$. To discuss the compactness of the operators, we consider the transformation V_k on $L^2(\beta)$ given by $V_k e_n = \frac{\beta_{nk}}{\beta_n} e_{kn}$ for each $n \in \mathbb{Z}$. It is a bounded operator if the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. For more properties and applications of this operator, we refer [9]. In [8], it is shown that a k^{th}-order weighted slant Hankel operator $D^\beta_{k,\phi}$, $\phi \in L^\infty(\beta)$ is compact if and only if $\phi = 0$ and this result is obtained for the k^{th}-order slant weighted Toeplitz operator $U^\beta_{k,\phi}$ in [6]. With the use of these results along with the linearity of $\phi \to D^\beta_{k,\phi}$, we prove the following without any extra efforts.

Theorem 3.1. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. For $\phi, \psi \in L^\infty(\beta)$, the product $S^\beta_\phi D^\beta_{k,\psi}$ of the weighted Hankel operator $S^\beta_\phi (= J^\beta M^\beta_\phi)$ and the k^{th}-order weighted slant Hankel operator $D^\beta_{k,\psi}$ is compact if only if $D^\beta_{k,\widetilde{\phi}(z^k)\psi}$ is compact.*

Proof. Proof follows as $S^\beta_\phi D^\beta_{k,\psi}$ is compact if and only if $J^\beta S^\beta_\phi D^\beta_{k,\psi}$ is compact (being J^β invertible) and $J^\beta S^\beta_\phi D^\beta_{k,\psi} = D^\beta_{k,\widetilde{\phi}(z^k)\psi}$. □

Theorem 3.2. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be such that $\{\frac{\beta_{k_2 n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Let $\phi, \psi \in L^\infty(\beta)$. Then the following are equivalent.*

(1) $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi}$ *is compact.*
(2) $D^\beta_{k_1,\phi} D^\beta_{k_2,\psi} = 0$.
(3) $\widetilde{\phi}(z^{k_2})\psi = 0$.
(4) $S^\beta_\phi D^\beta_{k_2,\psi}$ *is compact.*
(5) $S^\beta_\phi D^\beta_{k_2,\psi} = 0$.

We conclude this study with our next result which proves that the notion of essentially commuting and commuting coincides for k^{th}-order weighted slant Hankel operators on $L^2(\beta)$.

Theorem 3.3. *Let $\phi, \psi \in L^\infty(\beta)$ be such that $\phi(z^{k_2}), \psi(z^{k_1}) \in L^\infty(\beta)$. Then the following are equivalent.*

(1) $D^\beta_{k_1,\phi}$ *and* $D^\beta_{k_2,\psi}$ *essentially commute.*
(2) $D^\beta_{k_1,\phi}$ *and* $D^\beta_{k_2,\psi}$ *commute.*
(3) $\widetilde{\phi}(z^{k_2})\psi - \phi\widetilde{\psi}(z^{k_1}) = 0$.

Corollary 3.1. *If $2 \le k_1 \le k_2$ and $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\{\frac{\beta_{k_2 n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then the following are equivalent for $\phi, \psi \in L^\infty(\beta)$.*

(1) $D^{\beta}_{k_1,\phi}$ *and* $D^{\beta}_{k_2,\psi}$ *essentially commute.*
(2) $D^{\beta}_{k_1,\phi}$ *and* $D^{\beta}_{k_2,\psi}$ *commute.*
(3) $\widetilde{\phi}(z^{k_2})\psi - \phi\widetilde{\psi}(z^{k_1}) = 0$.

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