GENERALIZED FOURIER-DUNKL TRANSFORM OF
\((\delta, \gamma)\)-GENERALIZED DUNKL LIPSCHITZ FUNCTIONS

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ABSTRACT. Using a generalized translation operator, we obtain an analog of Younis Theorem 5.2 in [5] for the generalized Fourier-Dunkl transform for functions satisfying the \((\delta, \gamma)\)-generalized Dunkl Lipschitz condition in the space \(L_{2,n}^2\).

1. Introduction and Preliminaries

Younis Theorem 5.2 [5] characterized the set of functions in \(L^2(\mathbb{R})\) satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

**Theorem 1.1.** ([5]) Let \(f \in L^2(\mathbb{R})\). Then the following are equivalents

(a) \(\|f(x+h) - f(x)\| = O \left( \frac{h^{\delta}}{(\log h)^{\gamma}} \right)\), as \(h \to 0, 0 < \delta < 1, \gamma \geq 0\),

(b) \(\int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = O \left( \frac{r^{-2\delta}}{(\log r)^{2\gamma}} \right)\), as \(r \to \infty\),

where \(\hat{f}\) stands for the Fourier transform of \(f\).

In this paper, we consider a first-order singular differential-difference operator \(\Lambda\) on \(\mathbb{R}\) which generalizes the Dunkl operator \(\Lambda_\alpha\). We prove an analog of Theorem 1.1 in the generalized Fourier-Dunkl transform associated to \(\Lambda\) in \(L_{2,n}^2\). For this purpose, we use a generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator \(\Lambda\). Further details can be found in [1] and [6]. In all that follows assume where \(\alpha > -1/2\) and \(n\) a non-negative integer.

Consider the first-order singular differential-difference operator on \(\mathbb{R}\)

\[ \Lambda f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x}. \]

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For \( n = 0 \), we regain the differential-difference operator
\[
\Lambda_\alpha f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x},
\]
which is referred to as the Dunkl operator of index \( \alpha + 1/2 \) associated with the reflection group \( \mathbb{Z}_2 \) on \( \mathbb{R} \). Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics.

Let \( M \) be the map defined by
\[
Mf(x) = x^{2n}f(x), \quad n = 0, 1, \ldots
\]

Let \( L_{\alpha,n}^p, 1 \leq p < \infty \), be the class of measurable functions \( f \) on \( \mathbb{R} \) for which
\[
\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,
\]
where
\[
\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p |x|^{\alpha+1} dx \right)^{1/p}.
\]
If \( p = 2 \), then we have \( L_{\alpha,n}^2 = L^2(\mathbb{R}, |x|^{2\alpha+1}) \).

The one-dimensional Dunkl kernel is defined by
\[
e_{\alpha}(z) = j_{\alpha}(iz) + \frac{z}{2(\alpha + 1)}j_{\alpha+1}(iz), \quad z \in \mathbb{C},
\]
where
\[
j_{\alpha}(z) = \Gamma(\alpha + 1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\Gamma(m + \alpha + 1)}, \quad z \in \mathbb{C},
\]
is the normalized spherical Bessel function of index \( \alpha \). It is well-known that the functions \( e_{\alpha}(\lambda z) \), \( \lambda \in \mathbb{C} \), are solutions of the differential-difference equation
\[
\Lambda_{\alpha} u = \lambda u, \quad u(0) = 1.
\]

**Lemma 1.1.** ([2]) For \( x \in \mathbb{R} \) the following inequalities are fulfilled
i) \( |e_{\alpha}(ix)| \leq 1 \),
ii) \( |1 - e_{\alpha}(ix)| \leq |x| \),
iii) \( |1 - e_{\alpha}(ix)| \geq c \) with \( |x| \geq 1 \), where \( c > 0 \) is a certain constant which depends only on \( \alpha \).

For \( \lambda \in \mathbb{C} \), and \( x \in \mathbb{R} \), put
\[
\varphi_{\lambda}(x) = x^{2n}e_{\alpha+2n}(i\lambda x),
\]
where \( e_{\alpha+2n} \) is the Dunkl kernel of index \( \alpha + 2n \) given by (1.1).

**Proposition 1.1.** i) \( \varphi_{\lambda} \) satisfies the differential equation
\[
\Lambda_{\alpha} \varphi_{\lambda} = i\lambda \varphi_{\lambda}.
\]

ii) For all \( \lambda \in \mathbb{C} \), and \( x \in \mathbb{R} \)
\[
|\varphi_{\lambda}(x)| \leq |x|^{2n}e^{\text{Im} \lambda |x|}.
\]
The generalized Fourier-Dunkl transform we call the integral transform
\[ \mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x) \varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}. \]

Let \( f \in L^1_{\alpha,n} \) such that \( \mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx) \). Then the inverse generalized Fourier-Dunkl transform is given by the formula
\[ f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda) \varphi_{\lambda}(x)d\mu_{\alpha+2n}(\lambda), \]
where
\[ d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}. \]

**Proposition 1.2.** i) For every \( f \in L^2_{\alpha,n} \),
\[ \mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda \mathcal{F}_\Lambda(f)(\lambda). \]

ii) For every \( f \in L^1_{\alpha,n} \cap L^2_{\alpha,n} \) we have the Plancherel formula
\[ \int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda). \]

iii) The generalized Fourier-Dunkl transform \( \mathcal{F}_\Lambda \) extends uniquely to an isometric isomorphism from \( L^2_{\alpha,n} \) onto \( L^2(\mathbb{R}, \mu_{\alpha+2n}) \).

The generalized translation operators \( \tau^x, x \in \mathbb{R} \), tied to \( \Lambda \) are defined by
\[
\tau^x f(y) = \frac{(xy)^{2n}}{2} \int_{-1}^{1} f\left(\frac{\sqrt{x^2 + y^2 - 2xyt}}{x^2 + y^2 - 2xyt}\right) \left(1 + \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}}\right) A(t)dt \\
+ \frac{(xy)^{2n}}{2} \int_{-1}^{1} f\left(\frac{-\sqrt{x^2 + y^2 - 2xyt}}{x^2 + y^2 - 2xyt}\right) \left(1 - \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}}\right) A(t)dt,
\]
where
\[ A(t) = \frac{\Gamma(\alpha + 2n + 1)}{\sqrt{\pi} \Gamma(\alpha + 2n + 1/2)} (1 + t)(1 - t^{2})^{\alpha+2n-1/2}. \]

**Proposition 1.3.** Let \( x \in \mathbb{R} \) and \( f \in L^2_{\alpha,n} \). Then \( \tau^x f \in L^2_{\alpha,n} \) and
\[ \|\tau^x f\|_{2,\alpha,n} \leq 2x^{2n}\|f\|_{2,\alpha,n}. \]

Furthermore,
\[ \mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n}e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \]

2. **Main Results**

In this section we give the main result of this paper. We need first to define \((\delta, \gamma)\)-generalized Dunkl Lipschitz class.

**Definition 2.1.** Let \( \delta \in (0,1) \) and \( \gamma \geq 0 \). A function \( f \in L^2_{\alpha,n} \) is said to be in the \((\delta, \gamma)\)-generalized Dunkl Lipschitz class, denoted by \( DLip(\delta, \gamma, \alpha, \gamma) \), if
\[ \|\tau^h f(x) - h^{2n} f(x)\|_{2,\alpha,n} = O\left(\frac{h^{\delta + 2n}}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \to 0. \]
Theorem 2.1. Let $f \in L^2_{\alpha, n}$. Then the following are equivalents

(a) $f \in \text{DLip}(\delta, \gamma, 2)$

(b) $\int_{|\lambda| \geq r} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$, as $r \to \infty$.

Proof.

(a) $\Rightarrow$ (b). Let $f \in \text{DLip}(\delta, \gamma, 2)$. Then we have

$$\|\tau^h f(x) - h^{2n} f(x)\|_{2, \alpha, n} = O\left(\frac{h^{\delta+2n}}{(\log \frac{1}{h})^\gamma}\right) \quad \text{as} \quad h \to 0.$$ 

Formula (1.2) and Plancherel equality give

$$\|\tau^h f(x) - h^{2n} f(x)\|_{2, \alpha, n}^2 = h^{4n} \int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

If $|\lambda| \in \left[\frac{1}{h}, \frac{2}{h}\right]$, then $|\lambda h| \geq 1$ and (iii) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c^2} |e_{\alpha+2n}(i\lambda h) - 1|^2.$$ 

Then

$$\int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leq \frac{1}{c^2} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$\leq \frac{1}{c^2} \int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$\leq \frac{h^{-4n}}{4c^2} \|\tau^h f(x) - h^{2n} f(x)\|_{2, \alpha, n}^2$$

$$= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}}\right).$$

We obtain

$$\int_{|\lambda| \leq 2r} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leq C \frac{r^{-2\delta}}{(\log r)^{2\gamma}}, \quad r \to \infty,$$

where $C$ is a positive constant. Now,

$$\int_{|\lambda| \geq r} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \sum_{i=0}^{\infty} \int_{2^ir \leq |\lambda| \leq 2^{i+1}r} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$$

$$\leq C \left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}} + \frac{(2r)^{-2\delta}}{(\log 2r)^{2\gamma}} + \frac{(4r)^{-2\delta}}{(\log 4r)^{2\gamma}} + \cdots\right)$$

$$\leq C \frac{r^{-2\delta}}{(\log r)^{2\gamma}} (1 + 2^{-2\delta} + (2^{-2\delta})^2 + (2^{-2\delta})^3 + \cdots)$$

$$\leq K_\delta \frac{r^{-2\delta}}{(\log r)^{2\gamma}},$$

where $K_\delta = C(1 - 2^{-2\delta})^{-1}$ since $2^{-2\delta} < 1$.

Consequently

$$\int_{|\lambda| \geq r} |F_{\lambda}f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right), \quad \text{as} \quad r \to \infty.$$
(b) ⇒ (a). Suppose now that
\[ \int_{|\lambda| \geq r} |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left( \frac{r^{-2\delta}}{(\log r)^{2\gamma}} \right), \quad \text{as} \quad r \to \infty. \]
and write
\[ \|\tau^h f(x) - h^{2n} f(x)\|_{2,\alpha,n}^2 = h^{4n}(I_1 + I_2), \]
where
\[ I_1 = \int_{|\lambda| < \frac{1}{h}} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda), \]
and
\[ I_2 = \int_{|\lambda| \geq \frac{1}{h}} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda). \]
Firstly, we use the formulas \(|e_{\alpha+2n}(i\lambda h)| \leq 1\) and
\[ I_2 \leq 4 \int_{|\lambda| \geq \frac{1}{h}} |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right), \quad \text{as} \quad h \to 0. \]
Set
\[ \phi(x) = \int_{-\infty}^{+\infty} |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda). \]
Integrating by parts we obtain
\[ \int_{0}^{x} \lambda^2 |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \int_{0}^{x} -\lambda^2 \phi'(\lambda) d\lambda = -x^2 \phi(x) + 2 \int_{0}^{x} \lambda \phi(\lambda) d\lambda \leq C_1 \int_{0}^{x} \lambda \lambda^{-2\delta} (\log \lambda)^{-2\gamma} d\lambda = O(x^{2-2\delta}(\log x)^{-2\gamma}), \]
where \(C_1\) is a positive constant.
We use the formula (ii) of Lemma 1.1
\[ \int_{-\infty}^{+\infty} |e_{\alpha+2n}(i\lambda h) - 1|^2 |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left( h^2 \int_{|\lambda| < \frac{1}{h}} \lambda^2 |F_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \right) + \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right) + O \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right) + O \left( \frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}} \right), \]
and this ends the proof. □

**References**


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