A SCHUR TYPE THEOREM FOR ALMOST $\alpha$–COSYMPLECTIC MANIFOLDS WITH KAEHLERIAN LEAVES

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Abstract. In this study, we give a Schur type theorem for almost $\alpha$–cosymplectic manifolds with Kaehlerian leaves.

1. Introduction

Let $M$ be a Riemannian manifold with curvature tensor $R$. The sectional curvature of a 2-plane $\alpha$ in a tangent space $T_PM$ is defined by $K(\alpha, P) = R(X, Y, Y, X)$, where $\{X, Y\}$ is an orthonormal basis of $T_PM$. F. Schur’s classical theorem remarks that if $M$ is a connected manifold of dimension $n \geq 3$ and in any point $P \in M$, the curvature $K(\alpha, P)$ does not depend on $\alpha \in T_PM$ then it does not depend on the point $P$ too, i.e. it is a global constant. Such a manifold is called a manifold of constant sectional curvature.

In following years, many authors has studied Schur’s theorem for different structures ([6]–[10]). For instance, in 1989, Schur’s theorem is improved by Nobuhiro and a new version for locally symmetric spaces is gotten. [10]. In 2001, Kassabov regards connected $2n$-dimensional almost Hermitian manifold $M$ to be of pointwise constant anti-holomorphic sectional curvature $\nu(p), p \in M$ and proves that $\nu$ is a global constant [6]. In 2006, Cho defines a contact strongly pseudo-convex $CR$ space-form using the Tanaka-Webster connection in a method similar to the Sasakian space form. He studies the geometry of such spaces and introduces a Schur type theorem for such structures [7]. And finally in 2013 a new version of Schur’s lemma for almost cosymplectic manifolds with Kaehlerian leaves are given by Aktan et. al.[22]

The presence of an almost cosymplectic manifold was firstly introduced by Goldberg and Yano in 1969, [19]. The simplest examples of these manifolds are those being the products (possibly local) of almost Kaehlerian manifolds and the real line $\mathbb{R}$ or the circle $S^1$.

Later on, curvature properties of almost cosymplectic manifolds were studied mainly by Goldberg and Yano [12], Olszak [13], [14], Kirichenko [15], Endo [16] some other authors. We relate some of them in a historical order.

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A cosymplectic manifold of constant curvature is necessarily locally flat [17]. It is obvious that the locally flat cosymplectic manifolds exist. In fact, they are locally products of locally flat Kaehlerian manifolds and the real line (for instance, $C^n \times R$). If the curvature operator $R$ of an almost cosymplectic manifold $M$ commutes with the fundamental singular collineation $\varphi$, then $M$ is normal, that is, it is a cosymplectic manifold [12]. In particular, an almost cosymplectic manifold of constant curvature is cosymplectic if and only if it is locally flat. Generalizing these, in [13], [14], it is proved that almost cosymplectic manifolds of non-zero constant curvature do not exist. For a conformally flat almost cosymplectic manifold of dimension $\geq 5$, the scalar curvature $r$ is non-positive and the manifold is cosymplectic if and only if it is locally flat [13], [14]. If $M$ is an almost cosymplectic manifold of constant $\varphi$-sectional curvature then the scalar curvature $r$ and the $\varphi$-sectional curvature $H$ satisfy the inequality $n(n+1)H \geq r$. This equality holds if and only if the manifold is cosymplectic [13].

In this paper, we focus on almost $\alpha$-cosymplectic manifolds with Kaehlerian leaves and considering Schur's lemma on spaces of constant curvature and the paper [22]. We get a new version of Schur's lemma for almost $\alpha$-cosymplectic manifolds with Kaehlerian leaves.

2. Almost $\alpha-$Cosymplectic Manifolds

We repeat the relevant material from Blair [4] without proofs.

Let $M$ be a $(2n + 1)$-dimensional differentiable manifold equipped with a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a type of $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form on $M$ such that

\begin{equation}
\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi,
\end{equation}

which implies

\begin{equation}
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \text{rank}(\varphi) = 2n.
\end{equation}

If $M$ admits a Riemannian metric $g$, such that

\begin{equation}
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X) \eta(Y),
\end{equation}

\begin{equation}
\eta(X) = g(X, \xi),
\end{equation}

then $M$ is said to have an almost contact metric structure $(\varphi, \xi, \eta, g)$. On such a manifold, the fundamental 2-form $\Phi$ of $M$ is defined by

\begin{equation}
\Phi(X, Y) = g(\varphi X, Y),
\end{equation}

for any vector fields $X, Y$ on $M$. For any vector fields $X, Y$ on $M$.

An almost contact metric structure is almost cosymplectic if and only if both $d\eta$ and $d\Phi$ vanish. An almost contact manifold $(M, \varphi, \xi, \eta)$ is said to be normal if the Nijenhuis torsion

\begin{equation}
N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y] + 2d\eta(X, Y)\xi,
\end{equation}

vanishes for any vector fields $X, Y$ on $M$.

A normal almost cosymplectic manifold is called a cosymplectic manifold. A normal almost cosymplectic and almost Kenmotsu manifolds are called a cosymplectic manifold and Kenmotsu manifold, respectively. As it is known that an almost
contact metric structure is cosymplectic if and only if both $\nabla \eta$ and $\nabla \Phi$ vanish and an almost contact metric structure is Kenmotsu [13] if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$  

An almost contact metric manifold $M^{2n+1}$ is said to be almost Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha \eta \wedge \Phi$, $\alpha$ being a non-zero real constant. Geometrical properties and examples of almost Kenmotsu manifolds are studied in [1], [13], [24] and [5].

Given an almost Kenmotsu metric structure $(\varphi, \xi, \eta, g)$, consider the deformed structure

$\eta = \frac{1}{\alpha} \eta, \xi = \alpha \xi, \varphi = \varphi, g = \frac{1}{\alpha^2} g, \alpha \neq 0, \alpha \in \mathbb{R},$

where $\alpha$ is a non-zero real constant. So we get an almost $\alpha$-Kenmotsu structure $(\varphi, \xi, \eta, g)$. This deformation is called a homothetic deformation. It is important to note that almost $\alpha$-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures (see [5]).

If we join these two classes, we obtain a new notion of an almost $\alpha$-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \quad d\Phi = 2\alpha \eta \wedge \Phi,$$

for any real number $\alpha$ (see [1]). Obviously, a normal almost $\alpha$-cosymplectic manifold is an $\alpha$-cosymplectic manifold. An $\alpha$-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or $\alpha$-Kenmotsu ($\alpha \neq 0$) for $\alpha \in \mathbb{R}$.

Let $M$ be an almost $\alpha$-cosymplectic manifold with structure $(\varphi, \xi, \eta, g)$ and $D$ is the distribution of $M$ defined by $D = \ker \eta$. If the almost complex structure is Kaehlerian on every integral submanifold of the distribution $D$, such manifold is said to be an almost $\alpha$-cosymplectic manifold with Kaehlerian leaves. Suppose that $M$ is an almost $\alpha$-cosymplectic manifold. Denote by $A$ the $(1,1)$-tensor field on $M$ defined by

$$(2.4) \quad A = -\nabla \xi,$$

and by $h$ the $(1,1)$-tensor field given by the following relation

$$h = \frac{1}{2} \mathcal{L}_\xi \varphi,$$

where $\mathcal{L}$ is the Lie derivative of $g$. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$.

Moreover, the tensor fields $A$ and $h$ are symmetric operators and satisfy the following relations [23]

**Proposition 2.1.**

$$(2.5) \quad \nabla_X \xi = -\alpha \varphi^2 X - \varphi hX,$$

$$(2.6) \quad (\varphi \circ h)X + (h \circ \varphi)X = 0,$$

$$(2.7) \quad (\nabla_X \eta)Y = \alpha [g(X,Y) - \eta(X)\eta(Y)] + g(\varphi Y, hX),$$

$$(2.8) \quad \delta \eta = -2\alpha n, \quad tr(h) = 0,$$

$$(2.9) \quad tr(A) = -2\alpha n,$$

$$(2.10) \quad tr(\varphi A) = 0,$$

$$(2.11) \quad A \varphi + \varphi A = -2\alpha \varphi,$$

$$(2.12) \quad A \xi = 0,$$
(2.13) \((\nabla_X A) \xi = A^2 X\), for any vector fields \(X, Y\) on \(M\). We also remark that for any vector fields \(X, Y\) on \(M\). We also remark that \(h = 0 \iff \nabla \xi = -\alpha^2\).

**Proposition 2.2.** Let \(M\) be an almost \(\alpha\)-cosymplectic manifold. \(M\) has Kaehlerian leaves if and only if it satisfies the condition

\[
(\nabla_X \varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX,
\]

for any vector fields \(X, Y\) on \(M\) [22].

### 3. Basic Curvature Relations

**Proposition 3.1.** Let \(M\) be an almost \(\alpha\)-cosymplectic manifold. Then we have [23]

\[
(3.1) \quad R(X, Y) \xi = \alpha^2 [\eta(X)Y - \eta(Y)X] - \alpha [\eta(X)\varphi h Y - \eta(Y)\varphi h X] \\
+ (\nabla_Y \varphi h) X - (\nabla_X \varphi h) Y,
\]

\[
(3.2) \quad R(X, Y) \xi = -(\nabla_X A) Y + (\nabla_Y A) X,
\]

\[
(3.3) \quad R(X, \xi) \xi = \alpha^2 \varphi^2 X + 2\alpha \varphi h X - h^2 X + \varphi(\nabla \xi h) X,
\]

\[
(3.4) \quad (\nabla \xi h) X = -\varphi R(X, \xi) \xi - \alpha^2 \varphi X - 2\alpha h X - \varphi h^2 X,
\]

\[
(3.5) \quad R(X, \xi) \xi = \varphi R(\varphi X, \xi) \xi = 2 \left[\alpha^2 \varphi^2 X - h^2 X\right],
\]

\[
(3.6) \quad S(X, \xi) = -2\alpha^2 \eta(X) - \frac{2}{1+i} g((\nabla e_i \varphi h) e_i, X),
\]

\[
(3.7) \quad tr(l) = S(\xi, \xi) = -2\alpha^2 + tr(h^2),
\]

for any vector fields \(X, Y\) on \(M\).

We have the following proposition that will be used in the next important result, by simple computations,.

**Proposition 3.2.**

**Theorem 3.1.** For the curvature transformation of almost \(\alpha\)-cosymplectic manifold with Kaehlerian leaves, we have [22]

\[
R(\varphi X, \varphi Y) Z - \varphi R(X, Y) Z = g(AX, \varphi Z)A Y - g(AY, \varphi Z) A X
\]

\[
(3.8) \quad -g(AX, Z)\varphi A Y + g(AY, Z)\varphi A X \\
- \eta(Z) \varphi (R(X, Y) \xi) - g(R(X, Y) \xi, \varphi Z) \xi
\]

and

\[
R(\varphi X, \varphi Y) Z - R(X, Y) Z = \eta(Y)R(\xi, X, Z) + g(AZ, \varphi X)A \varphi Y
\]

\[
(3.9) \quad -g(AZ, \varphi Y)A \varphi X - g(AZ, X)AY
\]

\[
+ g(AZ, Y)AX - \eta(X) R(\xi, Y, Z) + \eta(X) \eta(Y) R(\xi, \xi)
\]

**Lemma 3.1.** Let \(M\) be an almost \(\alpha\)-cosymplectic manifold with Kaehlerian leaves. If we denote

\[
P(\varphi X, Y) = (\nabla_Y \varphi h) X - (\nabla X \varphi h) Y
\]

and

\[
P(X, Y) = (\nabla_Y h) X - (\nabla X h) Y.
\]

Then we satisfy following relations [22]:

\[
(\nabla_X A) \xi = A^2 X,
\]

for any vector fields \(X, Y\) on \(M\). We also remark that for any vector fields \(X, Y\) on \(M\). We also remark that \(h = 0 \iff \nabla \xi = -\alpha^2\).
Theorem 3.2. Let $M$ be an almost $\alpha$-cosymplectic manifold with Kaehlerian leaves. The necessary and sufficient condition for $M$ to have pointwise constant $\varphi$-sectional curvature $H$ is

\begin{align}
4R(X, Y, Z, W) &= (H + 3\alpha^2)[g(X, W)g(Z, Y) - g(X, Z)g(W, Y)] \\
&\quad - (H + \alpha^2)[\eta(X)\eta(W)g(Z, Y) + \eta(Y)\eta(Z)g(X, W)] \\
&\quad + 2g(X, \varphi Y)g(Z, \varphi W) - \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(Z)g(W, Y) \\
&\quad + (H - \alpha^2)[g(X, \varphi Z)g(W, \varphi Y) - g(X, \varphi W)g(Z, \varphi Y)] \\
&\quad - g(AX, \varphi Z)g(AY, \varphi W) + g(AW, \varphi X)g(AZ, \varphi Y) \\
&\quad - g(AZ, \varphi X)g(AW, \varphi Y) + g(AX, \varphi W)g(AY, \varphi Z) \\
&\quad + 2g(AX, Z)g(AW, Y) - 2g(AX, W)g(AZ, Y)
\end{align}

for all vector fields $X, Y, Z, W$ in $M$.

**Proof.** For any vector fields $X$ and $Y \in \mathcal{D}$, we have

\begin{equation}
4g(R(X, \varphi X)X, \varphi X) = -Hg(X, X)^2
\end{equation}

By (3.8) we see

\begin{align}
R(X, \varphi Y, X, \varphi Y) &= R(X, \varphi Y, X, \varphi Y) + g(AX, \varphi X)g(AY, \varphi Y) \\
&\quad - g(A\varphi Y, \varphi X)g(AX, Y) + g(A\varphi Y, \varphi Y)g(AX, X) \\
&\quad + g(A\varphi Y, X)g(\varphi AX, Y)
\end{align}

\begin{align}
R(X, \varphi X, Y, \varphi X) &= R(X, \varphi X, Y, \varphi X) - 2gX(AX, X)g(AY, Y) \\
&\quad + 2g(AX, X)g(AY, Y) - 2g(AX, \varphi X)g(X, \varphi Y)
\end{align}

for $X, Y \in \mathcal{D}$. Submitting $X + Y$ in (3.11), we get
\[-H \left[ 2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y) \right] \]
\[= \frac{1}{2} (gR(X + Y, \varphi X + \varphi Y)(X + Y), \varphi X + \varphi Y) \]
\[+ \frac{1}{2} H(g(X,X)^2 + g(Y,Y)^2) \]
\[= R(X, \varphi X, Y, \varphi X) + R(X, \varphi X, X, \varphi Y) + R(X, \varphi X, Y, \varphi Y) + R(Y, \varphi X, Y, \varphi Y) + R(X, \varphi Y, Y, \varphi Y) \]
\[+ \frac{1}{2} [R(Y, \varphi X, Y, \varphi X) + R(X, \varphi Y, X, \varphi Y)], \]
using of (3.8)

\[-H \left( 2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y) \right) \]
\[= R(X, \varphi X, Y, \varphi Y) + R(X, \varphi X, X, \varphi Y) + R(X, \varphi X, Y, \varphi Y) + R(X, \varphi Y, X, \varphi Y) + R(Y, \varphi Y, Y, \varphi Y) + \frac{1}{2} [g(A\varphi X, \varphi X)g(A\varphi Y, Y) - g(A\varphi Y, \varphi X)^2] \]
\[\text{Then using (3.13) and Bianchi identity} \]
\[= 2R(X, \varphi X, Y, \varphi Y) + 2R(X, \varphi Y, Y, \varphi X) + R(\varphi X, \varphi Y, X, Y) + R(Y, \varphi Y, X, \varphi Y) + \frac{1}{2} [\varphi g(AX, Y)^2 + g(AY, Y)^2] \]
\[\text{It then turns to} \]
\[= 2R(X, \varphi X, Y, \varphi Y) + 2R(X, \varphi Y, Y, \varphi X) + R(\varphi X, \varphi Y, X, Y) + R(Y, \varphi Y, X, \varphi Y) + \frac{1}{2} [\varphi g(AX, Y)^2 + g(AY, Y)^2] \]
\[\text{because of (3.9) and (3.12). Thus we get} \]

(3.14)
\[2R(X, \varphi X, Y, \varphi Y) + 2R(X, \varphi Y, Y, \varphi X) + 3R(X, \varphi Y, Y, \varphi X) \]
\[+ R(X, Y, X, Y) + \frac{1}{2} [2g(AX, Y)^2 - 2g(AX, Y)^2] \]
\[= -H \left( 2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y) \right) \]
Replacing $Y$ by $-Y$ in (3.14) and summing it to (3.14) we have
(3.15) 
\[ 3R(X, \varphi Y, Y, \varphi X) + R(X, Y, X, Y) = -H \left[ 2g(X, Y)^2 + g(X, X)g(Y, Y) \right] \]
\[ -g(AX, \varphi X)g(AY, \varphi Y) + g(AX, \varphi Y)g(AY, \varphi X) + g(AX, X)g(AY, Y) \]
\[ -g(AX, Y)^2 + g(AX, \varphi Y)^2 + 2\alpha g(AX, X)g(Y, Y) - 2g(AX, Y)^2 \]
\[ +g(AX, X)g(AY, Y) - 2\alpha g(AX, Y)g(X, Y) - g(AX, Y)^2 \]
\[ -g(AY, \varphi Y)g(AX, \varphi X) - \frac{1}{2}g(AY, Y)g(A\varphi X, \varphi X) - g(AY, \varphi X)^2 \]
\[ -g(AX, X)g(A\varphi Y, \varphi Y) + g(AX, \varphi Y)^2 \]

By virtue of (3.15) we see

\[ -H \left[ 2g(X, \varphi Y)^2 + g(X, X)g(\varphi Y, \varphi X) \right] = -3R(X, \varphi Y, \varphi X) \]
\[ +R(X, \varphi Y, X, \varphi Y) - g(AX, \varphi X)g(A\varphi Y, Y) \]
\[ +g(AX, Y)g(A\varphi Y, \varphi X) - g(AX, X)g(A\varphi Y, \varphi Y) \]
\[ +g(AX, \varphi Y)^2 - g(AX, Y)^2 - 2\alpha g(AX, X)g(Y, Y) \]
\[ +2g(AX, \varphi Y)^2 - g(AX, X)g(A\varphi Y, \varphi Y) \]
\[ -2\alpha g(AX, \varphi Y)g(X, \varphi Y) - g(AX, \varphi Y)^2 \]
\[ +g(A\varphi Y, Y)g(AX, \varphi X) + \frac{1}{2}g(A\varphi Y, \varphi Y)g(A\varphi X, \varphi X) \]
\[ -g(A\varphi Y, \varphi X)^2 - g(AX, X)g(AY, Y) + g(AX, Y)^2 \]
\[ = 3R(X, \varphi Y, X, Y) + R(X, \varphi Y, Y, X) \]
\[ +2g(AX, \varphi X)g(AY, \varphi Y) - 3g(AX, \varphi Y)g(AY, \varphi X) \]
\[ -\frac{2}{3}g(AX, X)g(AY, Y)^2 + \frac{5}{6}g(AX, \varphi Y)^2 - g(AX, X)g(A\varphi Y, \varphi Y) \]
\[ +g(AX, \varphi Y)^2 - 2\alpha g(AX, X)g(Y, Y) + 2\alpha g(AX, \varphi Y)g(X, \varphi Y) \]
\[ +\frac{1}{2}g(A\varphi Y, \varphi Y)g(A\varphi X, \varphi X) - g(A\varphi Y, \varphi X)^2 \]

Because of (3.9), (3.15) and (3.12), After simplification (3.16) follows

\[ = 3R(X, Y, X, Y) - \frac{1}{3}R(X, Y, X, Y) - \frac{4}{9} \left[ 2g(X, Y)^2 + g(X, X)g(Y, Y) \right] \]
\[ +\frac{4}{9}g(AX, \varphi X)g(AY, \varphi Y) - \frac{2}{9}g(AX, \varphi Y)g(AY, \varphi X) - \frac{1}{6}g(AX, X)g(AY, Y) \]
\[ +\frac{11}{9}g(AX, Y)^2 + \frac{7}{9}g(AX, \varphi Y)^2 - \frac{4}{9}g(AX, X)g(Y, Y) - \frac{2}{9}g(AX, Y)^2 \]
\[ -\frac{1}{2}g(AY, Y)g(A\varphi X, \varphi X) + \frac{1}{3}g(AY, \varphi X)^2 - \frac{2}{9}g(AX, X)g(A\varphi Y, \varphi Y) \]
\[ +2\alpha g(AX, \varphi Y)g(X, \varphi Y) + \frac{1}{2}g(A\varphi Y, \varphi Y)g(A\varphi X, \varphi X) - g(A\varphi Y, \varphi X)^2 \]

Therefore by a standard calculation we have
for any $X,Y \in \mathcal{D}$. Replacing $X = X + Z$ in (3.16), we obtain

\begin{equation}
8R(X,Y,X,Y) = -3H \left[ 2g(X,\varphi Y)^2 + g(X,X)g(\varphi Y,\varphi Y) \right] \\
+ H \left[ 2g(X,Y)^2 + g(X,X)g(Y,Y) \right] - 4g(AX,\varphi X)g(AY,\varphi Y) \\
+ 8g(AX,\varphi Y)g(AY,\varphi X) + \frac{17}{2}g(AX,X)g(AY,Y) \\
- \frac{11}{2}g(AX,Y)^2 - \frac{3}{2}g(AX,\varphi Y)^2 + 4ag(AX,X)g(Y,Y) \\
+ 2og(AX,Y)g(Y,Y) + \frac{1}{2}g(AY,Y)g(A\varphi X,\varphi X) \\
- \frac{1}{2}g(AY,\varphi X)^2 + \frac{2}{3}g(AX,X)g(\varphi Y,\varphi Y) \\
- 6og(AX,\varphi Y)g(X,\varphi Y) - \frac{3}{4}g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X) \\
- g(A\varphi Y,\varphi X)^2 \right] \\
\tag{3.16}
\end{equation}

(3.17)

\begin{align*}
16R(X,Y,X,Y) &+ 32R(Z,Y,X,Y) + 16R(Z,Y,Z,Y) \\
= 2H \left[ 2g(X,Y)^2 + 4g(X,Y)g(Z,Y) + 2g(Z,Y)^2 \right] \\
+ g(X,X)g(Y,Y) + 2g(X,Z)g(Y,Y) + g(Z,Z)g(Y,Y) \\
- 6H \left[ 2g(X,\varphi Y)^2 + 4g(X,\varphi Y)g(Z,\varphi Y) \\
+ 2g(Z,Y)^2 + 2g(X,Z)g(Y,Y) \right] - 8g(AX,\varphi X)g(AY,\varphi Y) \\
- 8g(AX,\varphi Z)g(AY,\varphi Y) - 8g(AZ,\varphi X)g(AY,\varphi Y) \\
- 8g(AZ,\varphi Z)g(AY,\varphi Y) + 16g(AX,\varphi Y)g(AY,\varphi X) \\
+ 16g(AX,\varphi Y)g(AY,\varphi Z) + 16g(AZ,\varphi Y)g(AY,\varphi X) \\
+ 16g(AZ,\varphi Y)g(AY,\varphi Z) + 17g(AZ,\varphi X)g(AY,Y) \\
+ 34g(AZ,\varphi Z)g(AY,Y) + 17g(AZ,Z)g(AY,Y) \\
- 11g(AX,Y)^2 - 22g(AX,Y)g(AZ,Y) - 11g(AZ,Y)^2 \\
- 7g(AX,\varphi Y)^2 - 14g(AX,\varphi Y)g(AZ,\varphi Y) - 7g(AZ,\varphi Y)^2 \\
+ g(AY,Y)g(A\varphi X,\varphi X) + 2g(AY,Y)g(A\varphi X,\varphi Z) \\
+ g(AY,Y)g(A\varphi Z,\varphi Z) - g(AY,Y)^2 \\
- 2g(AY,\varphi Z)g(AY,\varphi X) - g(AY,\varphi Z)^2 \\
+ 10g(AZ,X)g(A\varphi Y,\varphi Y) + 16g(AZ,Z)g(A\varphi X,\varphi Y) \\
+ 16g(AZ,\varphi Z)g(A\varphi X,\varphi Y) - 3g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X) \\
- 6g(A\varphi Y,\varphi X)g(A\varphi X,\varphi Z) - 3g(A\varphi Y,\varphi Y)g(A\varphi Z,\varphi Z) \\
+ 3g(A\varphi Y,\varphi X)^2 + 6g(A\varphi Y,\varphi X)g(A\varphi Y,\varphi Z) + 3g(A\varphi Y,\varphi Z)^2 \\
\end{align*}

If we replace $Y = Y + W$ in (3.17) again and use (2.6), then we obtain
by using Bianchi identity and (2.6) we have

\[ = H(12g(X, Y) g(Z, W) - 12g(X, \varphi Y) g(Z, \varphi W) \]
\[ - 24g(X, \varphi W) g(Z, \varphi Y) - 12g(X, Z) g(Y, W) \]
\[ + 12g(X, \varphi Z) g(Y, \varphi W) + 3g(AX, \varphi Z) g(AY, \varphi W) \]
\[ - 3g(AX, \varphi Y) g(AZ, \varphi W) - 12g(AX, \varphi Z) g(AW, \varphi Y) \]
\[ + 12g(AX, \varphi Y) g(AW, \varphi Z) - 12g(AZ, \varphi X) g(AY, \varphi W) \]
\[ + 12g(AY, \varphi X) g(AZ, \varphi W) - 3g(AZ, \varphi X) g(AW, \varphi Y) \]
\[ + 3g(AY, \varphi X) g(AW, \varphi Z) + 15g(AX, \varphi W) g(AY, \varphi Z) \]
\[ - 15g(AX, \varphi W) g(AZ, \varphi Y) + 9g(AZ, \varphi Y) g(AW, \varphi X) \]
\[ - 9g(AY, \varphi Z) g(AW, \varphi X) + 45g(AX, Z) g(AY, W) \]
\[ - 45g(AX, Y) g(AZ, W) + 2g(AX, Z) g(AY, W) \]
\[ - 2g(AX, \varphi Y) g(AZ, W) + 10g(AX, Z) g(AY, \varphi W) \]
\[ - 10g(AX, Y) g(AY, \varphi W) - 9g(AX, \varphi Z) g(AY, \varphi W) \]
\[ + 9g(AX, \varphi Y) g(AZ, \varphi W) + \alpha [14g(AX, Z) g(Y, W) \]
\[ - 14g(AX, Y) g(Z, W) + 2g(AX, Z) g(Y, W) \]
\[ - 6g(AX, \varphi Y) g(Z, \varphi W) - 12g(AX, \varphi W) g(Z, \varphi Y) \]
\[ - 6g(AZ, \varphi Y) g(X, \varphi W) - 6g(AZ, \varphi W) g(X, \varphi Y) \]
\[ - 2g(AY, W) g(X, Z) + 6g(AX, \varphi Z) g(Y, \varphi W) \]
\[ + 6g(AY, \varphi Z) g(X, \varphi W) + 6g(AY, \varphi W) g(X, \varphi Z) \]

and by using Bianchi identity and (2.6) we have

\[ 48R(X, W, Z, Y) = H(12g(X, Y) g(Z, W) \]
\[ - 12g(X, \varphi Y) g(Z, \varphi W) - 24g(X, \varphi W) g(Z, \varphi Y) \]
\[ - 12g(X, Z) g(Y, W) + 12g(X, \varphi Z) g(Y, \varphi W) \]
\[ + 3g(AX, \varphi Z) g(AY, \varphi W) - 3g(AX, \varphi Y) g(AZ, \varphi W) \]
\[ - 12g(AX, \varphi Z) g(AW, \varphi Y) + 12g(AX, \varphi Y) g(AW, \varphi Z) \]
\[ - 12g(AZ, \varphi X) g(AY, \varphi W) + 12g(AY, \varphi X) g(AZ, \varphi W) \]
\[ - 3g(AZ, \varphi X) g(AW, \varphi Y) + 3g(AY, \varphi X) g(AW, \varphi Z) \]
\[ + 15g(AX, \varphi W) g(AY, \varphi Z) - 15g(AX, \varphi W) g(AZ, \varphi Y) \]
\[ + 9g(AZ, \varphi Y) g(AW, \varphi X) - 9g(AY, \varphi Z) g(AW, \varphi X) \]
\[ + 24g(AX, Z) g(AY, W) - 24g(AX, Y) g(AZ, W) \]
\[ + 36g(AX, Y) g(Z, W) - 36g(AX, Z) g(Y, W) \]
\[ + \alpha [14g(AX, Z) g(Y, W) - 14g(AX, Y) g(Z, W) \]
\[ + 2g(AX, Z) g(Y, W) - 6g(AX, \varphi Y) g(Z, \varphi W) \]
\[ - 12g(AX, \varphi W) g(Z, \varphi Y) - 6g(AZ, \varphi Y) g(X, \varphi W) \]
\[ - 6g(AZ, \varphi W) g(X, \varphi Y) - 2g(AY, W) g(X, Z) \]
\[ - 4g(X, Z) g(AY, W) + 6g(AX, \varphi Z) g(Y, \varphi W) \]
\[ + 6g(AX, \varphi Z) g(X, \varphi W) + 6g(AY, \varphi W) g(X, \varphi Z) \]
\[ + 4g(X, Y) g(AZ, W) - 20g(Y, W) g(AX, Z) \]
\[ + 20g(W, Z) g(AX, Y) - 18g(X, Z) g(AY, W) \]
\[ - 18g(AX, Z) g(Y, W) + 18g(AX, Y) g(Z, W) \]
\[ + 18g(X, Y) g(AZ, W) \]

(3.18)

where \(X, Y, Z, W \in D\). Here, \(X\) is an arbitrary vector field on \(M\). Also we can write

\[ X = X^T + \eta(X) \xi \]

where, \(X^T\) determines the horizontal part of \(X\). We have all vector fields \(X, Y, Z, W\) on \(M\).
48R(X, Y, Z, W) = 48R(X^T, Y^T, Z^T, W^T)
+ 48\eta(Y)R(X, Y^T, Z, W^T) + 48\eta(Y)R(X^T, Y, Z, W^T)
+ 48\eta(Y)R(X^T, Y^T, Z^T, W^T) + 48\eta(Y)R(X, Y, Z, W^T)
+ 48\eta(Y)R(X, Y^T, Z^T, W^T) + 48\eta(Y)R(X^T, Y, Z^T, W^T)
+ 48\eta(Y)R(X^T, Y^T, Z, W^T)

Then if we use

(3.19)

from (3.10), we get

(3.20)

for all vector fields X and Y in M where \( \{ e_i \} \ (i = 1, 2, ..., 2n + 1) \) is an arbitrary local orthonormal frame field on M since the trace of \( h \) vanishes, from (3.20), we have for the scalar curvature

\[
\tau = n(n + 1)H - n\alpha^2[1 - 3n] - 2Tr(h^2)
\]
$T_1M$ with standard contact Riemannian structure is integrable if and only if the base manifold is of constant curvature. Here, we note that the unit tangent sphere bundle of a space of constant curvature satisfies

\[(4.1) \quad g\left( (\nabla_X h)Y^T, Z^T \right) = 0. \]

Now we consider a contact Riemannian manifold whose structure tensor $h$ satisfies (4.1) and (3.4) simultaneously. Then we have,

\[0 = g\left( (\nabla_X h)Y^T, Z^T \right) = g\left( (\nabla_X - \eta(X)\xi)h(Y - \eta(Y)\xi, Z - \eta(Z)\xi) \right) \]
\[= g\left( (\nabla_X h)Y, Z \right) - \eta(X)g\left( (\nabla_\xi h)Y, Z \right) - \eta(Y)g\left( (\nabla_X h)\xi, Z \right) \]
\[= -\eta(Z)g\left( (\nabla_X h)Y, \xi \right) + \eta(X)\eta(Y)g\left( (\nabla_\xi h)\xi, Z \right) + \eta(Y)\eta(Z)g\left( (\nabla_X h)\xi, \xi \right) \]
\[+ \eta(Z)\eta(X)g\left( (\nabla_\xi h)Y, \xi \right) - \eta(X)\eta(Y)\eta(Z)g\left( (\nabla_\xi h)\xi, \xi \right)\]

From the above equation, by using (2.6), (2.7) and using (3.4), we have

\[(4.2) \quad (\nabla_X h)Y = \eta(X)\left[ -\varphi lY - \alpha^2\varphi Y - 2\alpha hY - \varphi h^2Y \right] \]
\[+ \eta(Y)\left[ \alpha hX + \varphi h^2X \right] - g\left( \alpha hX + \varphi h^2X, Y \right)\eta\xi \]

moreover from (4.2) we have

\[(4.3) \quad P(X, Y) = -\eta(X)\left[ -\varphi lY - \alpha^2\varphi Y - \alpha hY \right] \]
\[+ \eta(Y)\left[ -\varphi lX - \alpha^2\varphi X - \alpha hX \right] - 2g\left( \varphi h^2X, Y \right)\eta\xi \]

\[(4.4) \quad P_\varphi(X, Y) = \eta(X)\left[ -lY + \alpha^2\varphi Y + \alpha \varphi hY \right] \]
\[+ \eta(Y)\left[ -lX + \alpha^2\varphi X + \alpha \varphi hX \right] \]

for any vector fields $X$ and $Y$ now we define a $(1, 2)$ tensor field $Q_1(X, Y)$ by

\[Q_1(X, Y) = (\nabla_X h)Y - \eta(X)\left[ -\varphi lY - \alpha^2\varphi Y - 2\alpha hY - \varphi h^2Y \right] \]
\[+ \eta(Y)\left[ \alpha hX + \varphi h^2X \right] + g\left( -\alpha hX + \varphi h^2X, Y \right)\eta\xi \]

**Definition 4.1.** The class $\mathcal{D}$ is given by the spaces of almost $\alpha$-cosymplectic manifold with Kaehlerian integral submanifolds satisfying $Q_1 = 0$, that is,

\[\mathcal{D} = \{ (M^{2n+1}, \phi, \xi, \eta, g) : Q_1 = 0 \} \]

We can see that this class $\mathcal{D}$ is invariant under $D$-homothetic deformations [21].

**Lemma 4.1.** Let $M$ be a space $\in \mathcal{D}$ then the eigenvalues of $h$ are constant.

5. Schur Type Theorem

**Theorem 5.1.** Let $M$ be an almost $\alpha$-cosymplectic manifold with Kaehlerian leaves belonging to the class $\mathcal{D}$. If the $\varphi$-sectional curvature at any point of $M$ is independent of the choice of $\varphi$-section, then it is constant on $M$ and the curvature tensor is given by

\[4R(X, Y, Z, W) = (H + 3\alpha^2)\left[ g(X, W)g(Z, Y) - g(X, Z)g(W, Y) \right] \]
\[- (H + \alpha^2)\eta(X)\eta(W)g(Z, Y) + \eta(Y)\eta(Z)g(X, W) \]
\[+ 2g(X, \varphi Y)g(Z, \varphi W) - \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(Z)g(W, Y) \]
\[+ (H + \alpha^2)\left[ g(X, \varphi Z)g(W, \varphi Y) - g(X, \varphi W)g(Z, \varphi Y) \right] \]
\[- g(AX, \varphi Z)g(AY, \varphi W) + g(AW, \varphi X)g(AZ, \varphi Y) \]
\[- g(AZ, \varphi X)g(AW, \varphi Y) + g(AX, \varphi W)g(AY, \varphi Z) \]
\[+ 2g(AX, Z)g(AW, Y) - 2g(AX, W)g(AZ, Y) \]

(5.1)
\[ + \alpha \left[ -2g(AX, Z)g(W, Y) + 2\eta(Y) \eta(W) g(AX, Z) + 2g(AX, W)g(Z, Y) - 2\eta(Y) \eta(Z) g(AX, W) + 2g(AZ, Y)g(X, W) - 2\eta(X) \eta(W) g(AZ, Y) - 2g(X, Z)g(AW, Y) + 2\eta(X) \eta(Z) g(AW, Y) \right] \\
- 8\alpha \eta(Y) \eta(W)g(\varphi h X, Z) + 4\alpha \eta(Y) g(\varphi h W, X) \\
- 8\alpha \eta(X) g(\varphi h Z, Y) \\
- 8\eta(X) \eta(Z) g(lW, Y) + 8\eta(Y) \eta(W) g(lX, Z) \\
+ 8\eta(X) g(lZ, Y) - 4\eta(Y) g(lX, Z) - 4\eta(Y) g(lW, X) \\
+ 8\alpha^2 \eta(X) \eta(Z) g(\varphi^2 W, Y) \\
+ 8\alpha^2 \eta(X) g(\varphi^2 Z, Y) - 8\alpha^2 \eta(Y) \eta(W) g(\varphi^2 X, Z) \\
+ 4\alpha^2 \eta(Y) g(\varphi^2 W, X) + 4\alpha^2 \eta(Y) g(\varphi^2 Z, X) \\
- 6\alpha^2 \eta(X) \eta(W) g(Y, Z) - 6\alpha^2 \eta(Y) \eta(Z) g(X, W) \\
+ 6\alpha^2 \eta(Y) \eta(W) g(X, Z) + 6\alpha^2 \eta(X) \eta(Z) g(Y, W) \]

for all vector fields \( X, Y, Z, W \) in \( M \).

**Corollary 5.1.** Let \( M \) be an almost \( \alpha \)-cosymplectic manifold with K"{a}hlerian leaves belonging to the class \( \mathcal{D} \). If the \( \varphi \)-sectional curvature at any point of \( M \) is independent of the choice of \( \varphi \)-section, then Ricci and scalar curvature are given as following

\[ 2S(Y, Z) = \left( (n + 1) H + \alpha^2 (5 - 3n) \right) \{ g(Y, Z) - \eta(Y) \eta(Z) \} + 2Tr(l) \eta(Y) \eta(Z) + 4g(lY, Z) + \alpha (4n - 2) g(\varphi Y, hZ) \]

\[ \tau = n (n + 1) H + \alpha^2 n (5 - 3n) + 3Tr(l) \]

**Proof.** From (4.3) and by using (2.14) and Lemma 4.1, we have

\[ 2(\nabla_X S)(Y, Z) = \left[ (n + 1) X(H) \left\{ g(Y, Z) - \eta(Y) \eta(Z) \right\} \right] \\
+ \left[ 2Tr(l) - (n + 1) H \right] \{ \eta(Y) g(X, \nabla_X \xi) - \eta(Y) g(Z, \nabla_X \xi) \} \\
+ 4g(\nabla_X l, Y, Z), \]

which yields

\[ \sum 2(\nabla_{E_i} S)(Y, E_i) = \sum \left( (n + 1) E_i(H) \left\{ g(Y, E_i) - \eta(Y) \eta(E_i) \right\} \right) \\
+ \sum \left[ 2Tr(l) - (n + 1) H \right] \{ \eta(Y) g(E_i, \nabla_{E_i} \xi) - \eta(E_i) g(Y, \nabla_{E_i} \xi) \} + 4g(\nabla_{E_i} l, Y, E_i) \]

\[ = (n + 1) \sum E_i(H) g(Y, E_i) - (n + 1) \xi(H) \eta(Y) + \sum 4g(\nabla_{E_i} l, Y, E_i). \]

by the well-known formula

\[ (\nabla_X \tau) = 2 \sum (\nabla_{E_i} S)(X, E_i). \]

for any local orthonormal frame field \( \{E_i\} (i = 1, 2, ..., 2n + 1) \) and by using (5.3), (5.4) and Lemma 4.1, we have

\[ (n + 1) \{ XH - \xi(H) \eta(X) \} = 2n (n + 1) XH. \]

This says that \( \xi(H) = 0 \) and \( (n - 1) XH = 0 \). Since \( n > 1 \), we see that \( H \) is constant, say \( c \). by applying (4.2) (4.3) and (4.4) in Proposition 3.2, we obtain (5.1) \( \Box \)

**Definition 5.1.** A complete and simply connected almost \( \alpha \)-cosymplectic manifold of class \( \mathcal{D} \) with constant \( \varphi \)-sectional curvature is said to be an almost \( \alpha \)-cosymplectic space form.
And then, from the proof of Proposition 3.2 and Theorem 5.1, we have,

**Theorem 5.2.** Let \( M \) be a complete and simply connected almost \( \alpha \)-cosymplectic space belonging to the class \( D \). Then \( M \) is an almost cosymplectic space form if and only if the curvature tensor \( R \) is given by (5.1).

**References**


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