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A SCHUR TYPE THEOREM FOR ALMOST $\alpha-$COSYMPLECTIC MANIFOLDS WITH KAEHLERIAN LEAVES

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Abstract. In this study, we give a Schur type theorem for almost $\alpha-$cosymplectic manifolds with Keahlerian leaves.

1. Introduction

Let M be a Riemannian manifold with curvature tensor R. The sectional curvature of a 2-plane α in a tangent space T_PM is defined by $K(\alpha, P) = R(X, Y, Y, X)$, where $\{X, Y\}$ is an orthonormal basis of T_PM. F. Schur's classical theorem remarks that if M is a connected manifold of dimension $n \geq 3$ and in any point $P \in M$, the curvature $K(\alpha, P)$ does not depend on $\alpha \in T_PM$ then it does not depend on the point P too, i.e. it is a global constant. Such a manifold is called a manifold of constant sectional curvature.

In following years, many authors has studied Schur's theorem for different structures ([6]-[10]).For instance,In 1989, Schur's theorem is improved by Nobuhiro and a new version for locally symmetric spaces is gotten. [10].In 2001, Kassabov regards connected $2n$-dimensional almost Hermitian manifold M to be of pointwise constant anti-holomorphic sectional curvature $\nu(p)$, $p \in M$ and proves that ν is a global constant [6].In 2006, Cho defines a contact strongly pseudo-convex CR space-form using the Tanaka-Webster connection in a method similar to the Sasakian space form. He studies the geometry of such spaces and introduces a Schur type theorem for such structures [7].And finally in 2013 a new version of Schur's lemma for almost cosymplectic manifolds with Kaehlerian leaves are given by Aktan et. al.[22]

The presence of an almost cosymplectic manifold was firstly introduced by Goldberg and Yano in 1969, [19]. The simplest examples of these manifolds are those being the products (possibly local) of almost Kaehlerian manifolds and the real line $\mathbb{R}$ or the circle S^1.

Later on, curvature properties of almost cosymplectic manifolds were studied mainly by Goldberg and Yano [12], Olszak [13], [14], Kirichenko [15], Endo [16] some other autors. We relate some of them in a historical order.

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A cosymplectic manifold of constant curvature is necessarily locally flat [17]. It is obvious that the locally flat cosymplectic manifolds exist. In fact, they are locally products of locally flat Kaehlerian manifolds and the real line (for instance, $C^n \times R$). If the curvature operator R of an almost cosymplectic manifold M commutes with the fundamental singular collineation φ, then M is normal, that is, it is a cosymplectic manifold [12]. In particular, an almost cosymplectic manifold of constant curvature is cosymplectic if and only if it is locally flat. Generalizing these,in [13], [14] , it is proved that almost cosymplectic manifolds of non-zero constant curvature do not exist. For a conformally flat almost cosymplectic manifold of dimension ≥ 5, the scalar curvature r is non-positive and the manifold is cosymplectic if and only if it is locally flat [13], [14]. If M is an almost cosymplectic manifold of constant φ-sectional curvature then the scalar curvature r and the φ-sectional curvature H satisfy the inequality $n(n+1)H \geq r$. This equality holds if and only if the manifold is cosymplectic [13].

In this paper, we focus on almost α-cosymplectic manifolds with Kaehlerian leaves and considering Schur's lemma on spaces of constant curvature and the paper [22]. We get a new version of Schur's lemma for almost α-cosymplectic manifolds with Kaehlerian leaves.

2. Almost $\alpha-$Cosymplectic Manifolds

We repeat the relevant material from Blair [4] without proofs.

Let M be a $(2n+1)$-dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a type of $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on M such that

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

which implies

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad rank(\varphi) = 2n. \tag{2.2}$$

If M admits a Riemannian metric g, such that

$$\begin{gathered} g(\varphi X, \varphi Y) = g(X,Y) - \eta\left(X\right)\eta\left(Y\right), \\ \eta\left(X\right) = g(X, \xi), \end{gathered} \tag{2.3}$$

then M is said to have an almost contact metric structure (φ, ξ, η, g) .On such a manifold, the fundamental 2-form Φ of M is defined by

$$\Phi(X,Y) = g(\varphi X, Y),$$

for any vector fields X, Y on M.for any vector fields X, Y on M.

An almost contact metric structure is almost cosymplectic if and only if both $d\eta$ and $d\Phi$ vanish. An almost contact manifold (M, φ, ξ, η) is said to be normal if the Nijenhuis torsion

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y] + 2d\eta(X,Y)\xi,$$

vanishes for any vector fields X, Y on M.

A normal almost cosymplectic manifold is called a cosymplectic manifold. A normal almost cosymplectic and almost Kenmotsu manifolds are called a cosymplectic manifold and Kenmotsu manifold, respectively. As it is known that an almost contact metric structure is cosymplectic if and only if both $\bigtriangledown\eta$ and $\bigtriangledown\Phi$ vanish and an almost contact metric structure is Kenmotsu [13] if and only if

$$(\nabla_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta\wedge\Phi$, α being a non-zero real constant. Geometrical properties and examples of almost α-Kenmotsu manifolds are studied in [1], [13], [24] and [5], . Given an almost Kenmotsu metric structure (φ, ξ, η, g), consider the deformed structure

$$\eta = \frac{1}{\alpha}\eta,\ \xi = \alpha\xi,\ \varphi = \varphi,\ g = \frac{1}{\alpha^2}g,\ \alpha \neq 0,\ \alpha \in \mathbb{R},$$

where α is a non-zero real constant. So we get an almost α-Kenmotsu structure (φ, ξ, η, g). This deformation is called a homothetic deformation. It is important to note that almost α-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures (see [5]).

If we join these two classes, we obtain a new notion of an almost α-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \quad d\Phi = 2\alpha\eta\wedge\Phi,$$

for any real number α (see [1]). Obviously, a normal almost α-cosymplectic manifold is an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu ($\alpha \neq 0$) for $\alpha \in \mathbb{R}$

Let M be an almost α-cosymplectic manifold with structure (φ, ξ, η, g) and $\mathcal{D}$ is the distribution of M defined by $\mathcal{D} = \ker\eta$. If the almost complex structure is Kaehlerian on every integral submanifold of the distribution $\mathcal{D}$, such manifold is said to be an almost α-cosymplectic manifold with Kaehlerian leaves. Suppose that M is an almost α-cosymplectic manifold. Denote by A the $(1,1)$-tensor field on M defined by

$$A = -\nabla\xi, \tag{2.4}$$

and by h the $(1,1)$-tensor field given by the following relation

$$h = \tfrac{1}{2}\mathcal{L}_\xi\varphi,$$

where $\mathcal{L}$ is the Lie derivative of g. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$.

Moreover, the tensor fields A and h are symmetric operators and satisfy the following relations [23]

Proposition 2.1.

$$\nabla_X\xi = -\alpha\varphi^2X - \varphi hX, \tag{2.5}$$

$$(\varphi\circ h)X + (h\circ\varphi)X = 0, \tag{2.6}$$

$$(\nabla_X\eta)Y = \alpha\,[g(X,Y) - \eta(X)\eta(Y)] + g(\varphi Y, hX), \tag{2.7}$$

$$\delta\eta = -2\alpha n, \quad tr(h) = 0, \tag{2.8}$$

$$tr(A) = -2\alpha n, \tag{2.9}$$

$$tr(\varphi A) = 0, \tag{2.10}$$

$$A\varphi + \varphi A = -2\alpha\varphi, \tag{2.11}$$

$$A\xi = 0, \tag{2.12}$$

$$(\nabla_X A)\,\xi = A^2 X, \tag{2.13}$$

for any vector fields X, Y *on* M. *We also remark that for any vector fields* X, Y *on* M. *We also remark that* $h = 0 \Leftrightarrow \nabla\xi = -\alpha\varphi^2$.

Proposition 2.2. *Let* M *be an almost* α*-cosymplectic manifold.* M *has Kaehlerian leaves if and only if it satisfies the condition*

$$(\nabla_X\varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX, \tag{2.14}$$

for any vector fields X, Y *on* M [23].

3. Basic Curvature Relations

Proposition 3.1. *Let* M *be an almost* α*-cosymplectic manifold. Then we have* [23]

$$\begin{aligned} R(X,Y)\xi &= \alpha^2\left[\eta(X)Y - \eta(Y)X\right] - \alpha\left[\eta(X)\varphi hY - \eta(Y)\varphi hX\right] \\ &\quad + (\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y, \end{aligned} \tag{3.1}$$

$$R(X,Y)\xi = -(\nabla_X A)Y + (\nabla_Y A)X, \tag{3.2}$$

$$R(X,\xi)\xi = \alpha^2\varphi^2 X + 2\alpha\varphi hX - h^2X + \varphi(\nabla_\xi h)X, \tag{3.3}$$

$$(\nabla_\xi h)X = -\varphi R(X,\xi)\xi - \alpha^2\varphi X - 2\alpha hX - \varphi h^2 X, \tag{3.4}$$

$$R(X,\xi)\xi - \varphi R(\varphi X,\xi)\xi = 2\left[\alpha^2\varphi^2 X - h^2 X\right], \tag{3.5}$$

$$S(X,\xi) = -2n\alpha^2\eta(X) -_{i=1}^{2n+1} g((\nabla_{e_i}\varphi h)\, e_i, X), \tag{3.6}$$

$$tr\,(l) = S(\xi,\xi) = -\left[2n\alpha^2 + tr(h^2)\right], \tag{3.7}$$

for any vector fields X, Y *on* M *.*

We have the following proposition that will be used in the next important result,by simple computations, .

Proposition 3.2.

Theorem 3.1. *For the curvature transformation of almost* α*-cosymplectic manifold with Kaehlerian leaves, we have* [22]

$$\begin{aligned} &R(X,Y)\varphi Z - \varphi R(X,Y)Z = g(AX,\varphi Z)AY - g(AY,\varphi Z)AX \\ &-g(AX,Z)\varphi AY + g(AY,Z)\varphi AX \\ &-\eta(Z)\varphi(R(X,Y)\xi) - g(R(X,Y)\xi,\varphi Z)\xi \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &R(\varphi X,\varphi Y)Z - R(X,Y)Z = \eta(Y)R(\xi,X,Z) + g(AZ,\varphi X)A\varphi Y \\ &-g(AZ,\varphi Y)A\varphi X - g(AZ,X)AY \\ &+g(AZ,Y)AX - \eta(X)R(\xi,Y,Z) + \eta(X)\eta(Y)R(\xi,\xi) \end{aligned} \tag{3.9}$$

Lemma 3.1. *Let* M *be an almost* α*-cosymplectic manifold with Kaehlerian leaves. If we denote*

$$P_\varphi\,(X,Y) = (\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y$$

and

$$P\,(X,Y) = (\nabla_Y h)X - (\nabla_X h)Y.$$

Then we satisfy following relations [22]*:*

$$P_{\varphi}(X,Y) = \varphi P(X,Y)$$

$$\varphi P_{\varphi}(X,Y) = -P(X,Y) + 2g(hX,\varphi hY)\xi$$

$$P_{\varphi}(X,Y) = -P_{\varphi}(Y,X)$$

Theorem 3.2. *Let M be an almost α-cosymplectic manifold with Kaehlerian leaves. The necessary and sufficient condition for M to have pointwise constant $\varphi-$sectional curvature H is*

$$\begin{aligned}
4R(X,Y,Z,W) = {} & (H+3\alpha^2)\left[g(X,W)g(Z,Y) - g(X,Z)g(W,Y)\right] \\
& -(H+\alpha^2)\left[\eta(X)\eta(W)g(Z,Y) + \eta(Y)\eta(Z)g(X,W)\right. \\
& \left. +2g(X,\varphi Y)g(Z,\varphi W) - \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(Z)g(W,Y\right] \\
& +(H-\alpha^2)\left[g(X,\varphi Z)g(W,\varphi Y) - g(X,\varphi W)g(Z,\varphi Y\right] \\
& -g(AX,\varphi Z)g(AY,\varphi W) + g(AW,\varphi X)g(AZ,\varphi Y) \\
& -g(AZ,\varphi X)g(AW,\varphi Y) + g(AX,\varphi W)g(AY,\varphi Z) \\
& +2g(AX,Z)g(AW,Y) - 2g(AX,W)g(AZ,Y) \\
& +\alpha\begin{bmatrix} -2g(AX,Z)g(W,Y) + 2\eta(Y)\eta(W)g(AX,Z) \\ +2g(AX,W)g(Z,Y) - 2\eta(Y)\eta(Z)g(AX,W) \\ +2g(AZ,Y)g(X,W) - 2\eta(X)\eta(W)g(AZ,Y) \\ -2g(X,Z)g(AW,Y) + 2\eta(X)\eta(Z)g(AW,Y) \end{bmatrix} \\
& +4\eta(X)P_{\varphi}(Z,W,Y) + 4\eta(Z)P_{\varphi}(X,Y,W) \\
& -4\eta(W)P_{\varphi}(X,Y,Z) - 4\eta(X)\eta(W)P_{\varphi}(Z,\xi,Y) \\
& -4\eta(X)\eta(Z)P_{\varphi}(\xi,W,Y) - 4\eta(X)\eta(Y)P_{\varphi}(Z,W,\xi) \\
& -4\eta(Y)P_{\varphi}(Z,W,X) + 4\eta(Y)\eta(W)P_{\varphi}(Z,\xi,X) \\
& +4\eta(Y)\eta(Z)P_{\varphi}(\xi,W,X) + 4\eta(X)\eta(Z)P_{\varphi}(\xi,Y,W) \\
& -4\eta(X)\eta(W)P_{\varphi}(\xi,Y,Z). \\
& -4\alpha\eta(X)\eta(Z)g(\varphi hY,W) + 4\alpha\eta(X)\eta(W)g(\varphi hY,Z) \\
& +4\alpha\eta(Y)\eta(Z)g(\varphi hX,W) - 4\alpha\eta(Y)\eta(W)g(\varphi hX,Z) \\
& -6\alpha^2\eta(X)\eta(W)g(Y,Z) - 6\alpha^2\eta(Y)\eta(Z)g(X,W) \\
& +6\alpha^2\eta(Y)\eta(W)g(X,Z) + 6\alpha^2\eta(X)\eta(Z)g(Y,W)
\end{aligned} \tag{3.10}$$

for all vector fields X,Y,Z,W in M.

Proof. For any vector fields X and $Y \in \mathcal{D}$,we have

$$g(R(X,\varphi X)X,\varphi X) = -Hg(X,X)^2 \tag{3.11}$$

By (3.8)we see

$$\begin{aligned}
R(X,\varphi Y,X,\varphi Y) = {} & R(X,\varphi Y,Y,\varphi X) + g(AX,\varphi X)g(AY,\varphi Y) \\
& -g(A\varphi Y,\varphi X)g(AX,Y) + g(A\varphi Y,\varphi Y)g(AX,X) \\
& +g(A\varphi Y,X)g(\varphi AX,Y)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
R(X,\varphi X,Y,\varphi X) = {} & R(X,\varphi X,X,\varphi Y) - 2\alpha g(X,X)g(AX,Y) \\
& +2\alpha g(AX,X)g(X,Y) - 2\alpha g(AX,\varphi X)g(X,\varphi Y)
\end{aligned} \tag{3.13}$$

for X,$Y \in \mathcal{D}$. Submitting $X+Y$ in (3.11), we get

$$
\begin{aligned}
&-H\left[2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right]\\
&=\tfrac{1}{2}\left(gR(X+Y,\varphi X+\varphi Y)\left(X+Y\right),\varphi X+\varphi Y\right)\\
&+\tfrac{1}{2}H\left(g(X,X)^2+g(Y,Y)^2\right)\\
&=R(X,\varphi X,Y,\varphi X)+R(X,\varphi X,X,\varphi Y)+R(X,\varphi X,Y,\varphi Y)\\
&+R(X,\varphi Y,Y,\varphi X)+R(Y,\varphi X,Y,\varphi Y)+R(X,\varphi Y,Y,\varphi Y)\\
&+\tfrac{1}{2}\left[R(Y,\varphi X,Y,\varphi X)+R(X,\varphi Y,X,\varphi Y)\right],
\end{aligned}
$$

using of (3.8)

$$
\begin{aligned}
&-H\left(2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right)\\
&=R(X,\varphi X,Y,\varphi X)+R(X,\varphi X,X,\varphi Y)+R(X,\varphi X,Y,\varphi Y)+R(X,\varphi Y,Y,\varphi X)\\
&+R(Y,\varphi X,Y,\varphi Y)+R(X,\varphi Y,Y,\varphi Y)+\tfrac{1}{2}\left[g(A\varphi X,\varphi X)g(AY,Y)-g(AY,\varphi X)^2\right.\\
&\left.g(A\varphi Y,\varphi Y)g(AX,X)+g(AX,\varphi Y)^2\right],
\end{aligned}
$$

Then using (3.13) and Bianchi identity

$$
\begin{aligned}
&-H\left(2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right)\\
&=2R(X,\varphi X,X,\varphi Y)-2\alpha g(X,X)g(AX,Y)+2\alpha g(AX,X)g(X,Y)\\
&-2\alpha g(AX,\varphi X)g(X,\varphi Y)-R(\varphi Y,X,Y,\varphi X)-R(X,Y,\varphi Y,\varphi X)\\
&+2R(Y,\varphi X,Y,\varphi Y)-2\alpha g(AX,Y)g(Y,Y)+2\alpha g(AY,Y)g(X,Y)\\
&-2\alpha g(AY,\varphi Y)g(Y,\varphi X)+R(X,\varphi Y,Y,\varphi X)+R(X,\varphi Y,X,\varphi Y)\\
&+\tfrac{1}{2}g(AY,Y)g(A\varphi X,\varphi X)-g(AY,\varphi X)^2-g(A\varphi Y,\varphi Y)g(AX,X)\\
&\left.+g(AX,\varphi Y)^2\right]
\end{aligned}
$$

It then turns to

$$
\begin{aligned}
&=2R(X,\varphi X,X,\varphi Y)+2R(X,\varphi Y,Y,\varphi X)+R(\varphi X,\varphi Y,X,Y)\\
&+2R(Y,\varphi X,Y,\varphi Y)+R(X,\varphi Y,X,\varphi Y)+\tfrac{1}{2}\left[-4\alpha g(X,X)g(AX,Y)\right.\\
&+4\alpha g(AX,X)g(X,Y)-4\alpha g(AX,\varphi X)g(X,\varphi Y)-4\alpha g(AX,Y)g(Y,Y)\\
&+4\alpha g(AY,Y)g(X,Y)-4\alpha g(AY,\varphi Y)g(Y,\varphi X)+g(AY,Y)g(A\varphi X,\varphi X)\\
&\left.-g(AY,\varphi X)^2-g(A\varphi Y,\varphi Y)g(AX,X)+g(AX,\varphi Y)^2\right]
\end{aligned}
$$

because of (3.9) and (3.12). Thus we get

(3.14)

$$
\begin{aligned}
&2R(X,\varphi X,X,\varphi Y)+2R(Y,\varphi X,Y,\varphi Y)+3R(X,\varphi Y,Y,\varphi X)\\
&+R(X,Y,X,Y)+\tfrac{1}{2}\left[2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)-2g\left(AX,\varphi Y\right)g\left(AY,\varphi X\right)\right.\\
&\left.-2g\left(AX,X\right)g\left(AY,Y\right)\right]+2g\left(AX,Y\right)^2-2g\left(AX,\varphi Y\right)^2\\
&+4\alpha g\left(AX,Y\right)g\left(X,Y\right)+2g\left(AX,Y\right)^2+2g\left(AX,\varphi X\right)g\left(AY,\varphi Y\right)\\
&-4\alpha g\left(X,X\right)g\left(AX,Y\right)+4\alpha g\left(AX,X\right)g\left(X,Y\right)-4\alpha g\left(AX,\varphi X\right)g\left(X,\varphi Y\right)\\
&-4\alpha g\left(Y,Y\right)g\left(AX,Y\right)+4\alpha g\left(AY,Y\right)g\left(X,Y\right)-4\alpha g\left(AY,\varphi Y\right)g\left(Y,\varphi X\right)\\
&+g\left(AY,Y\right)g\left(A\varphi X,\varphi X\right)-g\left(AY,\varphi X\right)^2-g\left(AX,X\right)g\left(A\varphi Y,\varphi Y\right)\\
&\left.+g\left(AX,\varphi Y\right)^2\right]\\
&=-H\left(2g\left(X,Y\right)^2+2g(X,X)g(X,Y)+2g(X,Y)g(Y,Y)+g(X,X)g(Y,Y)\right)
\end{aligned}
$$

Replacing Y by $-Y$ in (3.14) and summing it to (3.14) we have

(3.15)

$$\begin{aligned}
&3R(X,\varphi Y,Y,\varphi X)+R(X,Y,X,Y)=-H\left[2g(X,Y)^2+g(X,X)g(Y,Y)\right]\\
&-g(AX,\varphi X)g(AY,\varphi Y)+g(AX,\varphi Y)g(AY,\varphi X)+g(AX,X)g(AY,Y)\\
&-g(AX,Y)^2+g(AX,\varphi Y)^2+2\alpha g(AX,X)g(Y,Y)-2g(AX,Y)^2\\
&+g(AX,X)g(AY,Y)-2\alpha g(AX,Y)g(X,Y)-g(AX,Y)^2\\
&-g(AY,\varphi Y)g(AX,\varphi X)-\tfrac{1}{2}g(AY,Y)g(A\varphi X,\varphi X)-g(AY,\varphi X)^2\\
&-g(AX,X)g(A\varphi Y,\varphi Y)+g(AX,\varphi Y)^2\Big]
\end{aligned}$$

By virtue of (3.15) we see

$$\begin{aligned}
&-H\left[2g(X,\varphi Y)^2+g(X,X)g(\varphi Y,\varphi Y)\right]=-3R(X,Y,\varphi Y,\varphi X)\\
&+R(X,\varphi Y,X,\varphi Y)-g(AX,\varphi X)g(A\varphi Y,Y)\\
&+g(AX,Y)g(A\varphi Y,\varphi X)-g(AX,X)g(A\varphi Y,\varphi Y)\\
&+g(AX,\varphi Y)^2-g(AX,Y)^2-2\alpha g(AX,X)g(Y,Y)\\
&+2g(AX,\varphi Y)^2-g(AX,X)g(A\varphi Y,\varphi Y)\\
&-2\alpha g(AX,\varphi Y)g(X,\varphi Y)-g(AX,\varphi Y)^2\\
&+g(A\varphi Y,Y)g(AX,\varphi X)+\tfrac{1}{2}\left[g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X)\right.\\
&\left.-g(A\varphi Y,\varphi X)^2-g(AX,X)g(AY,Y)+g(AX,Y)^2\right]\\
&=3R(\varphi X,\varphi Y,X,Y)+R(X,\varphi Y,X,\varphi Y)\\
&-2g(AX,\varphi X)g(AY,\varphi Y)-g(AX,X)g(A\varphi Y,\varphi Y)\\
&+g(AX,Y)g(A\varphi Y,\varphi X)+2g(AX,\varphi Y)^2-2\alpha g(AX,X)g(Y,Y)\\
&+2\alpha g(AX,\varphi Y)g(X,\varphi Y)+\tfrac{1}{2}\left[g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X)\right.\\
&\left.-g(A\varphi Y,\varphi X)^2-g(AX,X)g(AY,Y)-g(AX,Y)^2\right]\\
&=3R(X,Y,X,Y)+R(X,\varphi Y,Y,\varphi X)\\
&+2g(AX,\varphi X)g(AY,\varphi Y)-3g(AX,\varphi Y)g(AY,\varphi X)\\
&-\tfrac{7}{2}g(AX,X)g(AY,Y)+\tfrac{5}{2}g(AX,Y)^2-g(AX,X)g(A\varphi Y,\varphi Y)\\
&+g(AX,\varphi Y)^2-2\alpha g(AX,X)g(Y,Y)+2\alpha g(AX,\varphi Y)g(X,\varphi Y)\\
&+\tfrac{1}{2}\left[g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X)-g(A\varphi Y,\varphi X)^2\right]
\end{aligned}$$

Because of (3.9), (3.15) and (3.12), After simplication (3.16) follows

$$\begin{aligned}
&=3R(X,Y,X,Y)-\tfrac{1}{3}R(X,Y,X,Y)-\tfrac{H}{3}\left[2g(X,Y)^2+g(X,X)g(Y,Y)\right]\\
&+\tfrac{4}{3}g(AX,\varphi X)g(AY,\varphi Y)-\tfrac{8}{3}g(AX,\varphi Y)g(AY,\varphi X)-\tfrac{17}{6}g(AX,X)g(AY,Y)\\
&+\tfrac{11}{6}g(AX,Y)^2+\tfrac{7}{6}g(AX,\varphi Y)^2-\tfrac{4}{3}\alpha g(AX,X)g(Y,Y)-\tfrac{2}{3}\alpha g(AX,Y)g(X,Y)\\
&-\tfrac{1}{6}g(AY,Y)g(A\varphi X,\varphi X)+\tfrac{1}{6}g(AY,\varphi X)^2-\tfrac{5}{6}g(AX,X)g(A\varphi Y,\varphi Y)\\
&+2\alpha g(AX,\varphi Y)g(X,\varphi Y)+\tfrac{1}{2}\left[g(A\varphi Y,\varphi Y)g(A\varphi X,\varphi X)-g(A\varphi Y,\varphi X)^2\right]
\end{aligned}$$

Therefore by a standard calculation we have

$$\begin{aligned}
&8R(X,Y,X,Y) = -3H\left[2g(X,\varphi Y)^2 + g(X,X)\,g(\varphi Y,\varphi Y)\right]\\
&+H\left[2g(X,Y)^2 + g(X,X)\,g(Y,Y)\right] - 4g(AX,\varphi X)\,g(AY,\varphi Y)\\
&+8g(AX,\varphi Y)\,g(AY,\varphi X) + \tfrac{17}{2}g(AX,X)\,g(AY,Y)\\
&-\tfrac{11}{2}g(AX,Y)^2 - \tfrac{7}{2}g(AX,\varphi Y)^2 + 4\alpha g(AX,X)\,g(Y,Y)\\
&+2\alpha g(AX,Y)\,g(X,Y) + \tfrac{1}{2}g(AY,Y)\,g(A\varphi X,\varphi X)\\
&-\tfrac{1}{2}g(AY,\varphi X)^2 + \tfrac{5}{2}g(AX,X)\,g(A\varphi Y,\varphi Y)\\
&-6\alpha g(AX,\varphi Y)\,g(X,\varphi Y) - \tfrac{3}{2}\left[g(A\varphi Y,\varphi Y)\,g(A\varphi X,\varphi X)\right.\\
&\left.-g(A\varphi Y,\varphi X)^2\right]
\end{aligned} \tag{3.16}$$

for any $X, Y \in \mathcal{D}$. Replacing $X = X + Z$ in (3.16), we obtain

$$\begin{aligned}
&16R(X,Y,X,Y) + 32R(Z,Y,X,Y) + 16R(Z,Y,Z,Y)\\
&= 2H\left[2g(X,Y)^2 + 4g(X,Y)\,g(Z,Y) + 2g(Z,Y)^2\right.\\
&\left.+g(X,X)\,g(Y,Y) + 2g(X,Z)\,g(Y,Y) + g(Z,Z)\,g(Y,Y)\right]\\
&-6H\left[2g(X,\varphi Y)^2 + 4g(X,\varphi Y)\,g(Z,\varphi Y)\right.\\
&+2g(Z,\varphi Y)^2 + g(X,X)\,g(Y,Y) + 2g(X,Z)\,g(Y,Y)\\
&\left.+g(Z,Z)\,g(Y,Y)\right] - 8g(AX,\varphi X)\,g(AY,\varphi Y)\\
&-8g(AX,\varphi Z)\,g(AY,\varphi Y) - 8g(AZ,\varphi X)\,g(AY,\varphi Y)\\
&-8g(AZ,\varphi Z)\,g(AY,\varphi Y) + 16g(AX,\varphi Y)\,g(AY,\varphi X)\\
&+16g(AX,\varphi Y)\,g(AY,\varphi Z) + 16g(AZ,\varphi Y)\,g(AY,\varphi X)\\
&+16g(AZ,\varphi Y)\,g(AY,\varphi Z) + 17g(AX,X)\,g(AY,Y)\\
&+34g(AZ,X)\,g(AY,Y) + 17g(AZ,Z)\,g(AY,Y)\\
&-11g(AX,Y)^2 - 22g(AX,Y)\,g(AZ,Y) - 11g(AZ,Y)^2\\
&-7g(AX,\varphi Y)^2 - 14g(AX,\varphi Y)\,g(AZ,\varphi Y) - 7g(AZ,\varphi Y)^2\\
&+g(AY,Y)\,g(A\varphi X,\varphi X) + 2g(AY,Y)\,g(A\varphi X,\varphi Z)\\
&+g(AY,Y)\,g(A\varphi Z,\varphi Z) - g(AY,\varphi X)^2\\
&-2g(AY,\varphi Z)\,g(AY,\varphi X) - g(AY,\varphi Z)^2\\
&+5g(AX,X)\,g(A\varphi Y,\varphi Y) + 10g(AZ,X)\,g(A\varphi Y,\varphi Y)\\
&+5g(AZ,Z)\,g(A\varphi Y,\varphi Y) - 3g(A\varphi Y,\varphi Y)\,g(A\varphi X,\varphi X)\\
&-6g(A\varphi Y,\varphi Y)\,g(A\varphi X,\varphi Z) - 3g(A\varphi Y,\varphi Y)\,g(A\varphi Z,\varphi Z)\\
&+3g(A\varphi Y,\varphi X)^2 + 6g(A\varphi Y,\varphi X)\,g(A\varphi Y,\varphi Z) + 3g(A\varphi Y,\varphi Z)^2\\
&+\alpha\left[8g(AX,X)\,g(Y,Y) + 16g(AX,Z)\,g(Y,Y) + 8g(AZ,Z)\,g(Y,Y)\right.\\
&+4g(AX,Y)\,g(X,Y) + 4g(AX,Y)\,g(Z,Y) + 4g(AZ,Y)\,g(X,Y)\\
&+4g(AZ,Y)\,g(Z,Y) - 12g(AX,\varphi Y)\,g(X,\varphi Y) - 12g(AX,\varphi Y)\,g(Z,\varphi Y)\\
&\left.-12g(AZ,\varphi Y)\,g(X,\varphi Y) - 12g(AZ,\varphi Y)\,g(Z,\varphi Y)\right]
\end{aligned} \tag{3.17}$$

If we replace $Y = Y + W$ in (3.17) again and use (2.6) ,then we obtain

$$
\begin{aligned}
&16R(X,Y,Z,W)+32R(X,W,Z,Y)+16R(X,Z,Y,W)\\
&=H12g(X,Y)\,g(Z,W)-12g(X,\varphi Y)\,g(Z,\varphi W)\\
&-24g(X,\varphi W)\,g(Z,\varphi Y)-12g(X,Z)\,g(Y,W)\\
&+12g(X,\varphi Z)\,g(Y,\varphi W)]+3g(AX,\varphi Z)\,g(AY,\varphi W)\\
&-3g(AX,\varphi Y)\,g(AZ,\varphi W)-12g(AX,\varphi Z)\,g(AW,\varphi Y)\\
&+12g(AX,\varphi Y)\,g(AW,\varphi Z)-12g(AZ,\varphi X)\,g(AY,\varphi W)\\
&+12g(AY,\varphi X)\,g(AZ,\varphi W)-3g(AZ,\varphi X)\,g(AW,\varphi Y)\\
&+3g(AY,\varphi X)\,g(AW,\varphi Z)+15g(AX,\varphi W)\,g(AY,\varphi Z)\\
&-15g(AX,\varphi W)\,g(AZ,\varphi Y)+9g(AZ,\varphi Y)\,g(AW,\varphi X)\\
&-9g(AY,\varphi Z)\,g(AW,\varphi X)+45g(AX,Z)\,g(AY,W)\\
&-45g(AX,Y)\,g(AZ,W)+2g(A\varphi X,\varphi Z)\,g(AY,W)\\
&-2g(A\varphi X,\varphi Y)\,g(AZ,W)+10g(AX,Z)\,g(A\varphi W,\varphi Y)\\
&-10g(AX,Y)\,g(A\varphi W,\varphi Z)-9g(A\varphi X,\varphi Z)\,g(A\varphi Y,\varphi W)\\
&+9g(A\varphi X,\varphi Y)\,g(A\varphi Z,\varphi W)+\alpha\,[14g(AX,Z)\,g(Y,W)\\
&-14g(AX,Y)\,g(Z,W)+2g(AZ,W)\,g(X,Y)\\
&-6g(AX,\varphi Y)\,g(Z,\varphi W)-12g(AX,\varphi W)\,g(Z,\varphi Y)\\
&-6g(AZ,\varphi Y)\,g(X,\varphi W)-6g(AZ,\varphi W)\,g(X,\varphi Y)\\
&-2g(AY,W)\,g(X,Z)+6g(AX,\varphi Z)\,g(Y,\varphi W)\\
&+6g(AY,\varphi Z)\,g(X,\varphi W)+6g(AY,\varphi W)\,g(X,\varphi Z)]
\end{aligned}
$$

and by using Bianchi identity and (2.6) we have

$$
\begin{aligned}
&48R(X,W,Z,Y)=H\,[12g(X,Y)\,g(Z,W)\\
&-12g(X,\varphi Y)\,g(Z,\varphi W)-24g(X,\varphi W)\,g(Z,\varphi Y)\\
&-12g(X,Z)g(Y,W)+12g(X,\varphi Z)g(Y,\varphi W)]\\
&+3g(AX,\varphi Z)\,g(AY,\varphi W)-3g(AX,\varphi Y)\,g(AZ,\varphi W)\\
&-12g(AX,\varphi Z)\,g(AW,\varphi Y)+12g(AX,\varphi Y)\,g(AW,\varphi Z)\\
&-12g(AZ,\varphi X)\,g(AY,\varphi W)+12g(AY,\varphi X)\,g(AZ,\varphi W)\\
&-3g(AZ,\varphi X)\,g(AW,\varphi Y)+3g(AY,\varphi X)\,g(AW,\varphi Z)\\
&+15g(AX,\varphi W)\,g(AY,\varphi Z)-15g(AX,\varphi W)\,g(AZ,\varphi Y)\\
&+9g(AZ,\varphi Y)\,g(AW,\varphi X)-9g(AY,\varphi Z)\,g(AW,\varphi X)\\
&+24g(AX,Z)\,g(AY,W)-24g(AX,Y)\,g(AZ,W)\\
&+36\alpha^2g(X,Y)\,g(Z,W)-36\alpha^2g(X,Z)\,g(Y,W)\\
&+\alpha\,[14g(AX,Z)\,g(Y,W)-14g(AX,Y)\,g(Z,W)\\
&+2g(AZ,W)\,g(X,Y)-6g(AX,\varphi Y)\,g(Z,\varphi W)\\
&-12g(AX,\varphi W)\,g(Z,\varphi Y)-6g(AZ,\varphi Y)\,g(X,\varphi W)\\
&-6g(AZ,\varphi W)\,g(X,\varphi Y)-2g(AY,W)\,g(X,Z)\\
&-4g(X,Z)g(AY,W)+6g(AX,\varphi Z)\,g(Y,\varphi W)\\
&+6g(AY,\varphi Z)\,g(X,\varphi W)+6g(AY,\varphi W)\,g(X,\varphi Z)\\
&+4g(X,Y)g(AZ,W)-20g(Y,W)g(AX,Z)\\
&+20g(W,Z)g(AX,Y)-18g(X,Z)g(AY,W)\\
&-18g(AX,Z)g(Y,W)+18g(AX,Y)g(Z,W)\\
&+18g(X,Y)g(AZ,W)]
\end{aligned}
\tag{3.18}
$$

where $X,Y,Z,W\in\mathcal{D}$. Here, X is an arbitrary vector field on M. Also we can write

$$X=X^T+\eta(X)\xi$$

where, X^T determines the horizontal part of X. We have all vector fields X,Y,Z,W on M.

$$\begin{aligned}&48R(X,Y,Z,W)=48R(X^T,Y^T,Z^T,W^T)\\&+48\eta(X)R(\xi,Y^T,Z^T,W^T)+48\eta(Y)R(X^T,\xi,Z^T,W^T)\\&+48\eta(Z)R(X^T,Y^T,\xi,W^T)+48\eta(W)R(X^T,Y^T,Z^T,\xi)\\&+48\eta(X)\eta(Z)R(\xi,Y^T,\xi,W^T)+48\eta(X)\eta(W)R(\xi,Y^T,Z^T,\xi)\\&+48\eta(Y)\eta(Z)R(X^T,\xi,\xi,W^T)+48\eta(Y)\eta(W)R(X^T,\xi,Z^T,\xi)\end{aligned}$$

Then if we use

$$\begin{aligned}&48R(X,Y,Z,W)=H\left[12g(X,W)g(Z,Y)-12\eta\left(X\right)\eta\left(W\right)g(Z,Y)\right.\\&-12\eta(Y)\eta\left(Z\right)g(X,W)-12g(X,\varphi W)g(Z,\varphi Y)-24g(X,\varphi Y)g(Z,\varphi W)\\&-12g(X,Z)g(W,Y)+12\eta\left(Y\right)\eta\left(W\right)g(X,Z)+12\eta(X)\eta\left(Z\right)g(W,Y)\\&\left.+12g(X,\varphi Z)g(W,\varphi Y)\right]+3g(\varphi X,Z)g(\varphi W,Y)-3g(\varphi X,W)g(\varphi Z,Y)\\&-12g(\varphi X,Z)g(\varphi Y,W)+12g(\varphi X,W)g(\varphi Y,Z)-12g(\varphi Z,X)g(\varphi W,Y)\\&+12g(\varphi W,X)g(\varphi Z,Y)-3g(\varphi Z,X)g(\varphi Y,W)+3g(\varphi W,X)g(\varphi Y,Z)\\&+15g(\varphi X,Y)g(\varphi W,Z)-15g(\varphi X,Y)g(\varphi Z,W)+9g(\varphi Z,W)g(\varphi Y,X)\\&-9g(\varphi W,Z)g(\varphi Y,X)+24g(X,Z)g(W,Y)-24\eta\left(W\right)\eta\left(Y\right)g(X,Z)\\&-24\eta\left(X\right)\eta\left(Z\right)g(W,Y)-24g(X,W)g(Z,Y)+24\eta\left(Z\right)\eta\left(Y\right)g(X,W)\\&+24\eta\left(X\right)\eta\left(W\right)g(Z,Y)+36g(X,W)g(Z,Y)-36\eta\left(X\right)\eta\left(W\right)g(Z,Y)\\&-36\eta(Y)\eta\left(Z\right)g(X,W)-36g(X,Z)g(W,Y)+36\eta\left(Y\right)\eta\left(W\right)g(X,Z)\\&+36\eta\left(Z\right)\eta\left(X\right)g(W,Y)+24g(X,Z)g(W,Y)-24\eta\left(X\right)\eta\left(Z\right)g(W,Y)\\&-24\eta\left(Y\right)\eta\left(W\right)g(X,Z)-24g(X,W)g(Z,Y)+24\eta\left(X\right)\eta\left(W\right)g(Z,Y)\\&+24\eta\left(Y\right)\eta\left(Z\right)g(X,W)-24g(Z,Y)g(X,W)+24\eta\left(Z\right)\eta\left(Y\right)g(X,W)\\&+24\eta\left(X\right)\eta\left(W\right)g(Z,Y)-6g(\varphi X,W)g(Z,\varphi Y)-12g(\varphi X,Y)g(Z,\varphi W)\\&-6g(\varphi Z,W)g(X,\varphi Y)-6g(\varphi Z,Y)g(X,\varphi W)+24g(X,Z)g(W,Y)\\&-24\eta\left(W\right)\eta\left(Y\right)g(X,Z)-24\eta\left(X\right)\eta\left(Z\right)g(W,Y)+6g(\varphi X,Z)g(W,\varphi Y)\\&+6g(\varphi W,Z)g(X,\varphi Y)+6g(\varphi W,Y)g(X,\varphi Z)]+48\eta(X)\eta\left(Z\right)g(Y,W)\\&-48\eta(X)\eta(W)g(Y,Z)-48\eta(Y)\eta(Z)g(X,W)+48\eta(Y)\eta(W)g(X,Z)\end{aligned}\tag{3.19}$$

from (3.10), we get

$$\begin{aligned}&2S\left(Y,Z\right)=\left[\left(n+1\right)H\right]\left\{g\left(Y,Z\right)-\eta\left(Y\right)\eta\left(Z\right)\right\}\\&+\alpha^2\left[1-3n\right]g\left(Y,Z\right)-\alpha^2\left[1+n\right]\eta\left(Y\right)\eta\left(Z\right)\\&+\alpha\left(2-4n\right)g\left(\varphi Y,hZ\right)+2\eta\left(Z\right)\sum P_\varphi\left(E_{\dot{I}},Y,E_{\dot{I}}\right)\\&-2\eta\left(Y\right)\sum P_\varphi\left(Z,E_{\dot{I}},E_{\dot{I}}\right)+2\eta\left(Y\right)\eta\left(Z\right)\sum P_\varphi\left(\xi,E_{\dot{I}},E_{\dot{I}}\right)\\&-4P_\varphi\left(\xi,Y,Z\right)\end{aligned}\tag{3.20}$$

for all vector fields X and Y in M where $\{e_i\}\,(i=1,2,...,2n+1)$ is an arbitrary local orthonormal frame field on M since the trace of h vanishes,from (3.20),we have for the scalar curvature

$$\tau=n\left(n+1\right)H-n\alpha^2\left[1-3n\right]-2Tr(h^2)$$

□

4. A class of almost cosymplectic manifolds $\mathfrak{D}$

There are two typical examples of contact manifolds;one is formed bye the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres)and the other is given by the unit tangent sphere bundles. The former admit a Riemannian metric which is Sasakian concerning the latter,in (15),it was proved that the associated CR-structure of a unit tangent sphere bundle

T_1M with standard contact Riemannian structure is integrable if and only if the base manifold is of constant curvature. Here,we note that the unit tangent sphere bundle of a space of constant curvature satisfies

$$g\left((\nabla_{X^T}h)Y^T, Z^T\right) = 0. \tag{4.1}$$

Now we consider a contact Riemannian manifold whose structure tensor h satisfies (4.1)and (3.4) simultaneously. Then we have,

$$\begin{aligned} 0 &= g\left((\nabla_{X^T}h)Y^T, Z^T\right) = g\left((\nabla_{X-\eta(X)\xi}h)(Y-\eta(Y)\xi, Z-\eta(Z)\xi\right) \\ &= g\left((\nabla_X h)Y, Z\right) - \eta(X)\, g\left((\nabla_\xi h)Y, Z\right) - \eta(Y)\, g\left((\nabla_X h)\xi, Z\right) \\ &\quad -\eta(Z)\, g\left((\nabla_X h)Y, \xi\right) + \eta(X)\eta(Y)\, g\left((\nabla_\xi h)\xi, Z\right) + \eta(Y)\eta(Z)\, g\left((\nabla_X h)\xi, \xi\right) \\ &\quad +\eta(Z)\eta(X)\, g\left((\nabla_\xi h)Y, \xi\right) - \eta(X)\eta(Y)\eta(Z)\, g\left((\nabla_\xi h)\xi, \xi\right) \end{aligned}$$

From the above equation ,by using(2.6),(2.7) and using(3.4),we have

$$\begin{aligned} (\nabla_X h)Y &= \eta(X)\left[-\varphi lY - \alpha^2\varphi Y - 2\alpha hY - \varphi h^2 Y\right] \\ &\quad -\eta(Y)\left[\alpha hX + \varphi h^2 X\right] - g\left(\alpha hX + \varphi h^2 X, Y\right)\xi \end{aligned} \tag{4.2}$$

moreover from (4.2) we have

$$\begin{aligned} P(X,Y) &= -\eta(X)\left[-\varphi lY - \alpha^2\varphi Y - \alpha hY\right] \\ &\quad +\eta(Y)\left[-\varphi lX - \alpha^2\varphi X - \alpha hX\right] - 2g\left(\varphi h^2 X, Y\right)\xi \end{aligned} \tag{4.3}$$

$$\begin{aligned} P_\varphi(X,Y) &= \eta(X)\left[-lY + \alpha^2\varphi^2 Y + \alpha\varphi hY\right] \\ &\quad -\eta(Y)\left[-lX + \alpha^2\varphi^2 X + \alpha\varphi hX\right] \end{aligned} \tag{4.4}$$

for any vector fields X and Y now we define a $(1,2)$ tensor field $Q_1(X,Y)$ by

$$\begin{aligned} Q_1(X,Y) &= (\nabla_X h)Y - \eta(X)\left[-\varphi lY - \alpha^2\varphi Y - 2\alpha hY - \varphi h^2 Y\right] \\ &\quad +\eta(Y)\left[\alpha hX - \varphi h^2 X\right] + g\left(-\alpha hX + \varphi h^2 X, Y\right)\xi \end{aligned}$$

Definition 4.1. The class $\mathfrak{D}$ is given by the spaces of almost α-cosymplectic manifold with Kaehlerian integral submanifolds satisfying $Q_1 = 0$, that is,

$$\mathfrak{D} = \left\{(M^{2n+1}, \phi, \xi, \eta, g) : Q_1 = 0\right\}$$

We can see that this class $\mathfrak{D}$ is invariant under D-homothetic deformations [21].

Lemma 4.1. *Let M be a space $\in \mathfrak{D}$ then the eigenvalues of h are constant.*

5. Schur Type Theorem

Theorem 5.1. *Let M be an almost α-cosymplectic manifold with Kaehlerian leaves belonging to the class $\mathfrak{D}$. If the $\varphi-$sectional curvature at any point of M is independent of the choice of $\varphi-$section, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned} 4R(X,Y,Z,W) &= (H+3\alpha^2)\left[g(X,W)g(Z,Y) - g(X,Z)g(W,Y)\right] \\ &\quad -(H+\alpha^2)\left[\eta(X)\eta(W)g(Z,Y) + \eta(Y)\eta(Z)g(X,W)\right. \\ &\quad \left. +2g(X,\varphi Y)g(Z,\varphi W) - \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(Z)g(W,Y\right] \\ &\quad +(H-\alpha^2)\left[g(X,\varphi Z)g(W,\varphi Y) - g(X,\varphi W)g(Z,\varphi Y\right] \\ &\quad -g(AX,\varphi Z)g(AY,\varphi W) + g(AW,\varphi X)g(AZ,\varphi Y) \\ &\quad -g(AZ,\varphi X)g(AW,\varphi Y) + g(AX,\varphi W)g(AY,\varphi Z) \\ &\quad +2g(AX,Z)g(AW,Y) - 2g(AX,W)g(AZ,Y) \end{aligned} \tag{5.1}$$

$$+\alpha\left[\begin{array}{c} -2g(AX,Z)g(W,Y)+2\eta(Y)\eta(W)g(AX,Z) \\ +2g(AX,W)g(Z,Y)-2\eta(Y)\eta(Z)g(AX,W) \\ +2g(AZ,Y)g(X,W)-2\eta(X)\eta(W)g(AZ,Y) \\ -2g(X,Z)g(AW,Y)+2\eta(X)\eta(Z)g(AW,Y) \end{array}\right]$$
$$-8\alpha\eta(Y)\eta(W)g(\varphi hX,Z)+4\alpha\eta(Y)g(\varphi hW,X)$$
$$-8\alpha\eta(X)g(\varphi hZ,Y)$$
$$-8\eta(X)\eta(Z)g(lW,Y)+8\eta(Y)\eta(W)g(lX,Z)$$
$$+8\eta(X)g(lZ,Y)-4\eta(Y)g(lX,Z)-4\eta(Y)g(lW,X)$$
$$+8\alpha^2\eta(X)\eta(Z)g(\varphi^2W,Y)$$
$$+8\alpha^2\eta(X)g(\varphi^2Z,Y)-8\alpha^2\eta(Y)\eta(W)g(\varphi^2X,Z)$$
$$+4\alpha^2\eta(Y)g(\varphi^2W,X)+4\alpha^2\eta(Y)g(\varphi^2Z,X)$$
$$-6\alpha^2\eta(X)\eta(W)g(Y,Z)-6\alpha^2\eta(Y)\eta(Z)g(X,W)$$
$$+6\alpha^2\eta(Y)\eta(W)g(X,Z)+6\alpha^2\eta(X)\eta(Z)g(Y,W)$$

for all vector fields X,Y,Z,W in M .

Corollary 5.1. *Let M be an almost α-cosymplectic manifold with Kaehlerian leaves belonging to the class $\mathfrak{D}$. If the $\varphi-$sectional curvature at any point of M is independent of the choice of $\varphi-$section, then Ricci and scalar curvature are given as following*

$$2S(Y,Z)=\left[(n+1)H+\alpha^2(5-3n)\right]\{g(Y,Z)-\eta(Y)\eta(Z)\} \\ +2Tr(l)\eta(Y)\eta(Z)+4g(lY,Z)+\alpha(4n-2)g(\varphi Y,hZ) \tag{5.2}$$

$$\tau=n(n+1)H+\alpha^2n(5-3n)+3Tr(l) \tag{5.3}$$

Proof. From (4.3) and by using (2.14) and Lemma 4.1, we have

$$2(\nabla_XS)(Y,Z)=[(n+1)X(H)]\{g(Y,Z)-\eta(Y)\eta(Z)\} \\ +[2Tr(l)-(n+1)H]\{\eta(Z)g(Y,\nabla_X\xi)-\eta(Y)g(Z,\nabla_X\xi)\} \\ +4g((\nabla_Xl)Y,Z),$$

which yields

(5.4)

$$\sum 2\left(\nabla_{E_i}S\right)(Y,E_i)=\sum[(n+1)E_i(H)]\{g(Y,E_i)-\eta(Y)\eta(E_i)\} \\ +\sum[2Tr(l)-(n+1)H]\left\{\eta(Y)g\left(E_i,\nabla_{E_i}\xi\right)-\eta(E_i)g\left(Y,\nabla_{E_i}\xi\right)\right\} \\ +\sum 4g\left(\left(\nabla_{E_i}l\right)Y,E_i\right) \\ =(n+1)\sum E_i(H)g(Y,E_i)-(n+1)\xi(H)\eta(Y)+\sum 4g\left(\left(\nabla_{E_i}l\right)Y,E_i\right).$$

by the well-known formula

$$(\nabla_X\tau)=2\sum\left(\nabla_{E_i}S\right)(X,E_i).$$

for any local orthonormal frame field $\{E_i\}$ $(i=1,2,...,2n+1)$ and by using (5.3), (5.4) and Lemma 4.1, we have

$$(n+1)\{XH-(\xi H)\eta(X)\}=2n(n+1)XH.$$

This says that $\xi H=0$ and $(n-1)XH=0$ .Since $n>1$,we see that H is constant,say c. by applying (4.2) (4.3) and (4.4) in Proposition 3.2, we obtain (5.1) □

Definition 5.1. A complete and simply connected almost α-cosymplectic manifold of class $\mathfrak{D}$ with constant $\varphi-$sectional curvature is said to be an almost α-cosymplectic space form.

And then, from the proof of Proposition 3.2 and Theorem 5.1, we have,

Theorem 5.2. *Let M be a complete and simply connected almost α-cosymplectic space belonging to the class $\mathfrak{D}$. Then M is an almost cosymplectic space form if and only if the curvature tensor R is given by (5.1).*

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