Konuralp Journal of Mathematics
Volume 4 No. 1 pp. 239–245 (2016) ©KJM

MERIDIAN SURFACES OF WEINGARTEN TYPE IN 4-DIMENSIONAL EUCLIDEAN SPACE $\mathbb{E}^4$

GÜNAY ÖZTÜRK, BETÜL BULCA, BENGÜ K. BAYRAM, AND KADRI ARSLAN

Abstract. In this paper, we study meridian surfaces of Weingarten type in Euclidean 4-space $\mathbb{E}^4$. We give the necessary and sufficient conditions for a meridian surface in $\mathbb{E}^4$ to become Weingarten type.

1. Introduction

A surface M in $\mathbb{E}^n$ is called Weingarten surface if there exist a non-trivial function

$$\Psi(K,H)=0 \tag{1.1}$$

between the Gauss curvature K and mean curvature H of the surface M. The existence of a non-trivial functional relation $\Psi(K,H)=0$ on a surface M parametrized by a patch $X(u,v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely

$$\left|\frac{\partial(K,H)}{\partial(u,v)}\right|=0. \tag{1.2}$$

The condition (1.2) that must be satisfied for the Weingarten surface M leads to

$$K_uH_v-K_vH_u=0 \tag{1.3}$$

with subscripts denoting partial derivatives.

These surfaces were introduced by Weingarten [16, 17] in the context of the problem of finding all surfaces isometric to a given surface of revolution. For the study of these surfaces, W. Kühnel [12] investigated ruled Weingarten surface in a Euclidean 3-space $\mathbb{E}^3$. Further, D. W. Yoon [18] classified ruled linear Weingarten surface in $\mathbb{E}^3$. Meanwhile, F. Dillen and W. Kühnel [5] and Y. H. Kim and D. W. Yoon [11] gave a classification of ruled Weingarten surfaces in a Minkowski 3-space $\mathbb{E}^3_1$. Also, linear Weingarten surfaces were studied by Galvez et. all. [6]. Recently, M. I. Munteanu and I. Nistor [15], R. Lopez [13, 14] and D.W. Yoon [19] studied

Date: January 1, 2013 and, in revised form, February 2, 2013.
2010 *Mathematics Subject Classification.* 53C40, 53C42.
Key words and phrases. meridian surface, Weingarten surface, second fundamental form.

polynomial translation Weingarten surfaces in a Euclidean 3-space. W. Kühnel and M. Steller classified the closed Weingarten surfaces [10].

The study of meridian surfaces in $\mathbb{E}^4$ was first introduced by G. Ganchev and V. Milousheva (See, [7], [8] and [9]). Basic source of examples of surfaces in 4-dimensional Euclidean or pseudo-Euclidean space are the standard rotational surfaces and the general rotational surfaces. Further, Ganchev and Milousheva defined another class of surfaces of rotational type which are one-parameter system of meridians of a rotational hypersurface. They constructed a family of surfaces with flat normal connection lying on a standard rotational hypersurface in $\mathbb{R}^4$ as a meridian surfaces. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two dimensional axis in $\mathbb{R}^4$. So, they constructed a surface M^2 in $\mathbb{E}^4$ in the following way:

$$M^2 : X(u,v) = f(u)\ r(v) + g(u)\ e_4, \quad u \in I,\ v \in J \tag{1.4}$$

where $f = f(u),\ g = g(u)$ are non-zero smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $(f'(u))^2 + (g'(u))^2 = 1, \quad u \in I$ and $r = r(v)$ $(v \in J \subset \mathbb{R})$ is a curve on $S^2(1)$ parameterized by the arc-length and e_4 is the fourth vector of the standard orthonormal frame in $\mathbb{E}^4$. See also [2] and [1] for the classification of meridian surfaces in 4-dimensional Euclidean space and 4-dimensional Minkowski space which have pointwise 1-type Gauss map.

In this paper, we study meridian surfaces of Weingarten type in 4-dimensional Euclidean space $\mathbb{E}^4$. We proved the following main theorem:

Let M^2 be a meridian surface given with the parametrization (3.2). Then M^2 is a Weingarten surface if and only if M^2 is one of the following surfaces;

i) a planar surface lying in the constant 3-dimensional space spanned by $\{x, y, n_2\}$,

ii) a developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$,

iii) a developable ruled surface in a 4-dimensional Euclidean space $\mathbb{E}^4$,

iv) a surface given with the surface patch

$$\begin{aligned} X(u,v) &= \left(\frac{\cos(au + ac_1)}{a} + c_2\right)\ r(v) + \\ &\quad + \left(\frac{2\left(\sin(au + ac_1) - 1\right)\sqrt{1 + \sin(au + ac_1)}}{\cos(au + ac_1)}\right) e_4, \end{aligned}$$

v) a surface given with the surface patch

$$X(u,v) = (c_1 \cos u + c_2 \sin u)\ r(v) + \sqrt{1 - (c_2 \cos u - c_1 \sin u)^2} e_4,$$

vi) a surface given with the surface patch

$$\begin{aligned} X(u,v) &= \pm \frac{a}{2}\left(e^{\frac{u+c}{b}} + e^{-\frac{u+c}{b}}\right)\ r(v) \\ &\quad \pm \frac{1}{2b}\sqrt{\left(2b - a(e^{\frac{u+c}{b}} - e^{-\frac{u+c}{b}})\right)\left(2b + a(e^{\frac{u+c}{b}} - e^{-\frac{u+c}{b}})\right)} e_4 \end{aligned}$$

where a, b, c, c_1, c_2 are real constants.

2. Basic Concepts

Let M be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$.

In the chart (u, v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Given any local vector fields X_1, X_2 tangent to M, consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2. \tag{2.2}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal frame field $\{N_1, N_2, ..., N_{n-2}\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_k} X_j = -(\widetilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M). \tag{2.3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \ 1 \leq i, j \leq 2; \ 1 \leq k \leq n-2 \tag{2.4}$$

where c_{ij}^k are the coefficients of the second fundamental form.

The equation (2.2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} c_{ij}^k N_k, \quad 1 \leq i, j \leq 2. \tag{2.5}$$

Then the Gauss curvature K of a regular patch $X(u, v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{n-2} (c_{11}^k c_{22}^k - (c_{12}^k)^2). \tag{2.6}$$

Further, the mean curvature vector of a regular patch $X(u, v)$ is given by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \tag{2.7}$$

where E, F, G are the coefficients of the first fundamental form and c_{ij}^k are the coefficients of the second fundamental form.

The norm of the mean curvature vector $H = \left\| \overrightarrow{H} \right\|$ is called the mean curvature of M. The mean curvature H and the Gauss curvature K play the most important roles in differential geometry for surfaces [4]. Recall that a surface M is said to be *flat* (resp. *minimal*) if its Gauss curvature (resp. mean curvature vector) vanishes identically [3].

3. Meridian Surfaces in $\mathbb{E}^4$

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$, and $S^2(1)$ be a 2-dimensional sphere in $\mathbb{E}^3 = span\{e_1, e_2, e_3\}$, centered at the origin O. We consider a smooth curve $c : r = r(v),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length $({r'}^2(v) = 1)$. We denote $t(v) = r'(v)$ and consider the moving frame field $\{t(v), n(v), r(v)\}$ of the curve c on $S^2(1)$. With respect to this orthonormal frame field the following Frenet formulas hold good:

$$\begin{aligned} r'(v) &= t(v); \\ t'(v) &= \kappa(v)\ n(v) - r(v); \\ n'(v) &= -\kappa\ (v)t(v), \end{aligned} \tag{3.1}$$

where κ is the spherical curvature of c.

Let $f = f(u),\ g = g(u)$ be non-zero smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $(f'(u))^2 + (g'(u))^2 = 1,\ u \in I$. Now we construct a surface M^2 in $\mathbb{E}^4$ in the following way:

$$M^2 : X(u, v) = f(u)\ r(v) + g(u)\ e_4, \quad u \in I,\ v \in J \tag{3.2}$$

The surface M^2 lies on the rotational hypersurface M^3 in $\mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \to (f(u), g(u))$ around the Oe_4-axis in $\mathbb{E}^4$. Since M^2 consists of meridians of M^3, we call M^2 a *meridian surface* (see, [7]).

The tangent space of M^2 is spanned by the vector fields:

$$\begin{aligned} X_u(u, v) &= f'(u)r(v) + g'(u)e_4; \\ X_v(u, v) &= f(u)\ t(v), \end{aligned} \tag{3.3}$$

and hence the coefficients of the first fundamental form of M^2 are $E = 1;\ F = 0;\ G = f^2(u)$. Without lose of generality we can take $g'(u) \neq 0$. Taking into account (3.1), we calculate the second partial derivatives of $X(u, v)$:

$$\begin{aligned} X_{uu}(u, v) &= f''(u)r(v) + g''\ (u)e_4; \\ X_{uv}(u, v) &= f'(u)t(v); \\ X_{vv}(u, v) &= f(u)\kappa(v)\ n(v) - f(u)\ r(v). \end{aligned} \tag{3.4}$$

Let us denote $X = X_u,\quad Y = \frac{X_v}{f} = t$ and consider the following orthonormal normal frame field of M^2:

$$N_1 = n(v); \quad N_2 = -g'(u)\ r(v) + f'(u)\ e_4. \tag{3.5}$$

Thus we obtain a positive orthonormal frame field $\{X, Y, N_1, N_2\}$ of M^2. If we denote by $\kappa_\alpha(u)$ the curvature of the meridian curve $\alpha(u)$, i.e.

$$\kappa_\alpha(u) = f'(u)\ g''(u) - g'(u)f''(u) = \frac{-f''(u)}{\sqrt{1 - f'^2(u)}}. \tag{3.6}$$

Using (3.4) and (3.5) we can calculate the coefficients of the second fundamental form of $X(u,v)$ as follows;

$$\begin{aligned} c_{11}^1 &= 0, c_{22}^1 = f(u)\kappa(v), \\ c_{12}^1 &= c_{12}^2 = 0, \\ c_{11}^2 &= \kappa_\alpha(u), \\ c_{22}^2 &= f(u)g'(u). \end{aligned} \tag{3.7}$$

Lemma 3.1. *Let M^2 be a meridian surface given with the surface patch (3.2) then*

$$A_{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\kappa(v)}{f(u)} \end{bmatrix}, \; A_{N_2} = \begin{bmatrix} \kappa_\alpha(u) & 0 \\ 0 & \frac{g'(u)}{f(u)} \end{bmatrix}. \tag{3.8}$$

Further by the use of (2.6) and (2.7) with (3.7), the Gauss curvature is given by

$$K = \frac{\kappa_\alpha(u)g'(u)}{f(u)}. \tag{3.9}$$

and the mean curvature vector field of M^2 becomes

$$\vec{H} = \frac{\kappa(v)}{2f(u)}N_1 + \frac{\kappa_\alpha(u)f(u) + g'(u)}{2f(u)}N_2. \tag{3.10}$$

From the equation (3.10), we get the mean curvature of M^2

$$H = \frac{1}{2f(u)}\sqrt{\kappa(v)^2 + (\kappa_\alpha(u)f(u) + g'(u))^2}. \tag{3.11}$$

4. Proof of the Main Theorem

Let M^2 be meridian surface given with the surface patch (3.2). Then differentiating K and H with respect to u and v one can get

$$\begin{aligned} K_v &= 0, \; K_u = -\frac{\left(f(u)f'''(u) - f'(u)f''(u)\right)}{f(u)^2}, \\ H_v &= \frac{\kappa(v)\kappa'(v)}{2f(u)\sqrt{\kappa(v)^2 + (\kappa_\alpha(u)f(u) + g'(u))^2}}. \end{aligned}$$

Suppose that M^2 is a Weingarten surface then by the use of equation (1.3), we get,

$$\frac{-\kappa(v)\kappa'(v)\left(f(u)f'''(u) - f'(u)f''(u)\right)}{2f(u)^3\sqrt{\kappa(v)^2 + (\kappa_\alpha(u)f(u) + g'(u))^2}} = 0. \tag{4.1}$$

Thus we distinguish the following cases:
Case I: $\kappa(v) = 0$;
Case II: $\kappa'(v) = 0$;
Case III: $f(u)f'''(u) - f'(u)f''(u) = 0$.

Let us consider these in turn;

Case I: Suppose $\kappa(v) = 0$, i.e. the curve c is a great circle on $S^2(1)$. In this case $N_1 = \text{const}$, and M^2 is a planar surface lying in the constant 3-dimensional space spanned by $\{X, Y, N_2\}$. Particularly, if in addition $\kappa_\alpha(u) = 0$, i.e. the meridian curve lies on a straight line, then M^2 is a developable surface in the 3-dimensional space span $\{X, Y, N_2\}$ [7].

Case II: Suppose $\kappa^{'}(v) = 0$. This implies that $\kappa(v)$ is nonzero constant. Then we have the following subcases;

Case II(a): $\kappa_\alpha(u) = 0$. In this case c is a circle on $S^2(1)$, then M^2 is a developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$.

Case II(b): $\kappa_\alpha(u)$ is nonzero constant. In this case we obtain the following ordinary differential equation.

$$\frac{-f''(u)}{\sqrt{1 - f'^2(u)}} = a. \tag{4.2}$$

Thus, the following expression is obtained from the solution of the differential equation (4.2)

$$f(u) = \frac{\cos(au + ac_1)}{a} + c_2.$$

Further, using the condition $(f'(u))^2 + (g'(u))^2 = 1$ we get

$$g(u) = \frac{2(\sin(au + ac_1) - 1)\sqrt{1 + \sin(au + ac_1)}}{\cos(au + ac_1)}.$$

Case III: Suppose $f(u)f^{'''}(u) - f^{'}(u)f^{''}(u) = 0$. Then we have the following subcases;

Case III(a): $f^{''}(u) = 0$. This implies that $\kappa_\alpha(u) = K = 0$, i.e. the meridian curve is part of a straight line and M^2 is a developable ruled surface. If in addition $\kappa(v) \neq$ const, i.e. c is not a circle on $S^2(1)$, then M^2 is a developable ruled surface in $\mathbb{E}^4$ [7].

Case III(b): $f^{''}(u) \neq 0$. In this case we obtain the following ordinary differential equation.

$$f(u)f^{'''}(u) - f^{'}(u)f^{''}(u) = 0 \tag{4.3}$$

An easy calculation shows that

$$f(u) = c_1 \cos u + c_2 \sin u$$

is a non-trivial solution of (4.3). Furthermore, the following expression is obtained from the general solution of the differential equation (4.3)

$$f(u) = \pm\frac{a}{2}\left(e^{\frac{u+c}{b}} + e^{-\frac{u+c}{b}}\right).$$

Further, using the condition $(f'(u))^2 + (g'(u))^2 = 1$ one can get

$$g(u) = \pm\frac{1}{2b}\sqrt{\left(2b - a(e^{\frac{u+c}{b}} - e^{-\frac{u+c}{b}})\right)\left(2b + a(e^{\frac{u+c}{b}} - e^{-\frac{u+c}{b}})\right)}$$

where a, b, c, c_1, c_2 are real constants. This completes the proof of the theorem.

References

[1] K. Arslan and V. Milousheva, Meridian Surfaces of Elliptic or Hyperbolic Type with Pointwise 1-type Gauss map in Minkowski 4-Space, accepted Taiwanese Journal of Mathematics.

[2] B. Bulca, K. Arslan and V. Milousheva, Meridian Surfaces in $\mathbb{E}^4$ with Pointwise 1-type Gauss Map, Bull. Korean Math. Soc., 51 (2014), 911–922.

[3] B. Y. Chen, *Geometry of Submanifolds* , Dekker, New York, 1973.

[4] B. Y. Chen, Pseudo-umbilical surfaces with constant Gauss curvature, Proceedings of the Edinburgh Mathematical Society (Series 2), 18(2) (1972), 143-148.
[5] F. Dillen and W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, Manuscripta Math., 98 (1999), 307-320.
[6] J. A. Galvez, A. Martinez and F. Milan, Linear Weingarten Surfaces in $\mathbb{R}^3$, Monatsh. Math., 138 (2003), 133-144.
[7] G. Ganchev and V. Milousheva, Invariants and Bonnet-type theorem for surfaces in $\mathbb{R}^4$, Cent. Eur. J. Math., 8(6) (2010) 993-1008.
[8] G. Ganchev and V. Milousheva, Special Class of Meridian Surfaces in the Four-Dimensional Euclidean Space, arXiv: 1402.5848v1 [math.DG], 24 Feb. 2014.
[9] G. Ganchev and V. Milousheva, Geometric Interpretation of the Invariants of a Surface in $\mathbb{R}^4$ via Tangent Indicatrix and the Normal Curvature Ellipse, arXiv:0905.4453v1 [math.DG], 27 May 2009.
[10] W. Kühnel and M. Steller, On Closed Weingarten Surfaces, Monatsh. Math., 146 (2005), 113-126.
[11] Y. H. Kim and D. W. Yoon, Classification of ruled surfaces in Minkowski 3-spaces, J. Geom. Phys., 49 (2004), 89-100.
[12] W. Kühnel, Ruled W-surfaces, Arch. Math., 62 (1994), 475-480.
[13] R. Lopez, On linear Weingarten surfaces, International J. Math., 19 (2008), 439-448.
[14] R. Lopez, Special Weingarten surfaces foliated by circles, Monatsh. Math., 154 (2008), 289-302.
[15] M. I. Munteanu and A. I. Nistor, Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space, arXiv:0809.4745v1 [math.DG], 27 Sep 2008.
[16] J. Weingarten, Ueber eine Klasse auf einander abwickelbarer Flaachen, J. Reine Angew. Math. 59 (1861), 382–393.
[17] J. Weingarten, Ueber die Flachen, derer Normalen eine gegebene Flache beruhren, J. Reine Angew. Math. 62 (1863), 61-63.
[18] D. W. Yoon, Some properties of the helicoid as ruled surfaces, JP Jour. Geom. Topology, 2 (2002), 141-147.
[19] D. W. Yoon, Polynomial translation surfaces of Weingarten types in Euclidean 3-space, Cent. Eur. J. Math., 8(3) (2010), 430-436.

Kocaeli University, Science and Art Faculty, Department of Mathematics, Kocaeli-TURKEY
E-mail address: ogunay@kocaeli.edu.tr

Uludağ University, Science and Art Faculty, Department of Mathematics, Bursa-TURKEY
E-mail address: bbulca@uludag.edu.tr

Balikesir University, Science and Art Faculty, Department of Mathematics, Balikesir-TURKEY
E-mail address: benguk@balikesir.edu.tr

Uludağ University, Science and Art Faculty, Department of Mathematics, Bursa-TURKEY
E-mail address: arslan@uludag.edu.tr