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Tarun Saxena

Akhilesh Yadav

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Research article

Geometry of Almost Ricci Solitons on D-homothetically Deformed K-Paracontact Metric Manifolds

Tarun Saxena*and Akhilesh Yadav

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Abstract

The aim of present paper is to study geometrical aspects of the almost Ricci soliton on D-homothetically deformed K-paracontact metric manifold and find the condition when the deformed metric remains almost Ricci soliton. Further, we analyze the nature of almost Ricci soliton on D-homothetically deformed K-paracontact metric manifold when associated potential vector field is divergence-free, the conformal vector field, or point wise collinear with Reeb vector field. We also provide examples of D-homothetically deformed K-paracontact metric manifold whose metric represents Ricci soliton. Finally, we discuss the behavior of almost gradient Ricci solitons on D-homothetically deformed K-paracontact metric manifold and obtain condition when the deformed metric remains the almost gradient Ricci soliton.

Keywords: *K-paracontact metric manifold, D-homothetic deformation, almost Ricci soliton, almost gradient Ricci soliton.*

AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C21; 53C25; 53D10.*

1. Introduction

In the last few years, the research interest in Ricci solitons has been much increased. The concept of Ricci solitons was introduced by R. S. Hamilton [12], as a natural generalization of an Einstein metric. The study of Ricci solitons in the framework of paracontact metric manifold was first used by G. Calvaruso and D. Perrone in [9]. They introduced the H-paracontact metric manifold and studied how the harmonicity of the Reeb vector field ξ of the paracontact metric manifold is related to the paracontact Ricci solitons. In [1], authors discussed almost Ricci solitons on paracontact manifolds and proved that an almost gradient Ricci soliton with para-Sasakian metric is Einstein with a constant scalar curvature. In [3] and [4], Blaga studied Ricci solitons on paracontact metric manifold and proved the existence of η-Ricci solitons on Lorentzian para-Sasakian manifold and para-Kenmotsu manifold under several curvature conditions. In [14], authors studied η-Ricci soliton on three dimensional $N(k)$-paracontact metric manifolds and also, they proved three dimensional contractible $N(k)$-paracontact metric manifold is of constant curvature when associated metric is gradient Ricci soliton.

In [6], authors studied η-Ricci solitons in the framework of (ϵ)- almost paracontact metric manifolds, when the associated potential vector field is torse-forming vector field and when vector field is characteristic vector field of the manifold and also, obtained few results of parallel symmetric $(0,2)$ tensor field for η-Ricci solitons on (ϵ)- almost paracontact metric manifolds. In [2], authors studied Ricci and scalar curvatures with almost Ricci solitons in a generalized D-conformally deformed $(LCS)_n$-manifold by taking potential vector field as the Reeb vector field, solenoidal vector field and a gradient vector field. In [10], authors gave some results when a K-contact manifold and a D-homothetically deformed K-contact manifold both are almost Ricci solitons with the same potential vector field. In [16], Nagaraja and Kumar analyzed the nature of Ricci solitons on D-conformally deformed Kenmotsu manifold when potential vector field is orthogonal to Reeb vector field.

Motivated by these studies, we consider D-homothetically deformed K-paracontact metric manifolds whose metric is an almost Ricci soliton. The present paper is organized as follows: In sections 1 and 2, we introduce

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*** Corresponding author***

some basic information about Ricci solitons, paracontact metric manifolds and D-homothetic deformation on paracontact metric manifolds, which are required for this paper. In section 3, first we provide gradient of a smooth function, Hessian of a smooth function, divergence of a smooth vector field and Laplacian of a smooth function on a D-homothetically deformed K-paracontact metric manifold. Then we investigate almost Ricci solitons on D-homothetically deformed K-paracontact metric manifold and obtain condition when the deformed metric remains the almost Ricci soliton with some detailed examples. Further, we analyse the nature of an almost Ricci soliton when associated potential vector field is divergence-free, conformal vector field, or point wise collinear with the Reeb vector field. Next, in section 4, we discuss the behaviour of almost gradient Ricci solitons on D-homothetically deformed K-paracontact metric manifold and obtain condition when the deformed metric remains the almost gradient Ricci soliton.

2. Preliminaries

First of all, we give some basic introduction of Ricci solitons. A Riemannian metric g on a smooth manifold M is said to be a Ricci soliton, if there exist some constant λ, and a smooth vector field V on M satisfying,

$$(L_V g)(X_1, X_2) + 2S(X_1, X_2) + 2\lambda g(X_1, X_2) = 0, \tag{2.1}$$

for all smooth vector fields X_1, X_2 on M, where $L_V g$ denotes the Lie derivative of g along the vector field V on M and S is the Ricci tensor of g. Here, the vector field V, and the constant λ are known as potential vector field and the soliton constant, respectively. The Ricci soliton is called shrinking, steady, and expanding if $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, respectively. Obviously, Einstein metrics are the trivial Ricci solitons with potential vector field V as homothetic vector field or zero. Ricci solitons are the fixed points of the Hamilton's Ricci flow [11]

$$\frac{\partial g(t)}{\partial t} = -2S(g(t)), \tag{2.2}$$

viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. Also, the Ricci solitons model the formation of singularities in the Ricci flow and correspond to self-similar solutions [15].

If the potential vector field V in (2.1) is considered as the gradient of some smooth function f on M, then the Ricci soliton is called a gradient Ricci soliton, and (2.1) can be written as

$$Hess(f)(X_1, X_2) + S(X_1, X_2) + \lambda g(X_1, X_2) = 0, \tag{2.3}$$

for all smooth vector fields X_1, X_2 on M, where $Hess(f)$ is the Hessian of potential function f, S is the Ricci tensor of g and λ is a constant on M. In [19], authors generalized the notion of Ricci soliton to almost Ricci soliton by setting the soliton constant λ to be a smooth function on M.

Now, we recall some basic facts and formulas of paracontact metric manifold. We refer [7], [13] and [21] for more details.

Definition 2.1. A smooth manifold M of dimension $(2n+1)$ is called an almost paracontact manifold with structure (ϕ, ξ, η), where ϕ is a $(1,1)$- type tensor field, ξ is a global vector field and η is a 1-form on M such that

$$\phi^2 X = X - \eta(X)\xi, \tag{2.4}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.5}$$

and let the horizontal distribution $D = Ker(\eta)$ generated by η, called paracontact distribution such that the tensor field ϕ induces an almost paracomplex structure on each fibre on D, i.e., the eigen distributions D^+ and D^- of ϕ corresponding to the eigenvalues 1 and -1, respectively.

Also, if an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that

$$g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.6}$$

for all smooth vector fields X_1, X_2 on M, then M admits an almost paracontact metric structure and metric g is called compatible metric [21].

Definition 2.2. An almost paracontact metric manifold $M(\phi,\xi,\eta,g)$ of dimension $(2n+1)$ is said to be a paracontact metric manifold if it satisfies

$$d\eta(X_1,X_2)=g(X_1,\phi X_2), \tag{2.7}$$

for all vector fields X_1, X_2 on M, and the 1-form η is called paracontact form and the vector field ξ is called Reeb vector field.

On a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, we consider the self adjoint operator $h=\frac{1}{2}L_\xi\phi$, where L_ξ is the Lie derivative along ξ. The operator h satisfies, $Trh=0$, $Tr(h\phi)=0$, $h\xi=0$, $h\phi=-\phi h$.

Lemma 2.1. *[7] On a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, we have*

$$\nabla_X\xi=-\phi X+\phi hX, \tag{2.8}$$

$$\nabla_\xi h=-\phi+\phi h^2-\phi l, \tag{2.9}$$

$$g(Q\xi,\xi)=Trl=Tr(h^2)-2n, \tag{2.10}$$

for all vector fields X on M, where ∇ is the operator of covariant differentiation of g and Q is the Ricci operator associated with the $(0,2)$ Ricci tensor given by

$$S(X_1,X_2)=g(QX_1,X_2), \tag{2.11}$$

for all vector fields X_1, X_2 on M.

Definition 2.3. If the vector field ξ is Killing, equivalently $h=0$ then the paracontact metric manifold $M(\phi,\xi,\eta,g)$ is called K-paracontact metric manifold.

Lemma 2.2. *[7] On a K-paracontact manifold $M^{2n+1}(\phi,\xi,\eta,g)$, we have*

$$\nabla_X\xi=-\phi X, \tag{2.12}$$

$$R(X,\xi)\xi=-X+\eta(X)\xi, \tag{2.13}$$

$$Q\xi=-2n\xi, \tag{2.14}$$

$$(\nabla_X Q)\xi=Q\phi X-2n\phi X, \tag{2.15}$$

$$(\nabla_\xi Q)X=Q\phi X-\phi QX, \tag{2.16}$$

for all smooth vector fields X on M.

Definition 2.4. [18] A manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies

$$S(X_1,X_2)=ag(X_1,X_2)+b\eta(X_1)\eta(X_2), \tag{2.17}$$

for all smooth vector fields X_1, X_2 on M, where a and b are smooth functions on M. Also, a and b are constants for a K-paracontact metric manifold of dimension greater than 3. If $b=0$ in (2.17) then the manifold M becomes Einstein manifold.

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost paracontact metric manifold. For a positive constant α, we define a deformation on the almost paracontact metric manifold as [20]

$$\overline{\phi}=\phi, \tag{2.18}$$

$$\overline{\xi}=\frac{1}{\alpha}\xi, \tag{2.19}$$

$$\overline{\eta}=\alpha\eta, \tag{2.20}$$

$$\overline{g}=\alpha g+\alpha(\alpha-1)\eta\otimes\eta, \tag{2.21}$$

is called D-homothetic deformation of an almost paracontact metric manifold $M(\phi,\xi,\eta,g)$ and with the deformed structure $(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$, the space $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is also an almost paracontact metric manifold of dimension $(2n+1)$.

In this paper, we consider K-paracontact metric manifold with D-homothetically deformed structure. The relationship between the Levi-Civita connections ∇ and $\overline{\nabla}$ of g and $\overline{g}$, respectively, Riemannian curvature tensor R and $\overline{R}$, Ricci tensor S and $\overline{S}$, and scalar curvature r and $\overline{r}$ on a K-paracontact metric manifold M is given by [21]

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \frac{\alpha-1}{\alpha}g(\phi hX_1, X_2)\xi - (\alpha-1)\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}, \tag{2.22}$$

$$\begin{aligned}\overline{R}(X_1,X_2)X_3 =& R(X_1,X_2)X_3 - (\alpha-1)[2g(X_1,\phi X_2)\phi X_3 + g(X_1,\phi X_3)\phi X_2 \\ &+ g(X_3,\phi X_2)\phi X_1 + \eta(X_2)(\nabla_{X_1}\phi)X_3 + \eta(X_3)(\nabla_{X_1}\phi)X_2 \\ &- \eta(X_1)(\nabla_{X_2}\phi)X_3 - \eta(X_3)(\nabla_{X_2}\phi)X_1] + (\alpha-1)^2[\eta(X_1)\eta(X_3)X_2 \\ &- \eta(X_2)\eta(X_3)X_1],\end{aligned} \tag{2.23}$$

$$\overline{S}(X_2,X_3) = S(X_2,X_3) + 2(\alpha-1)g(X_2,X_3) - 2(\alpha-1)\{n(\alpha+1)+1\}\eta(X_2)\eta(X_3), \tag{2.24}$$

$$\overline{r} = \frac{1}{\alpha}r + 2n\left(1-\frac{1}{\alpha}\right), \tag{2.25}$$

for all smooth vector fields X_1, X_2 and X_3 on M.

3. Almost Ricci Solitons on D-homothetically deformed paracontact metric manifolds

Let $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ be D-homothetically deformed paracontact metric manifold of paracontact metric manifold $M(\phi,\xi,\eta,g)$. For any smooth function f on M, let $\overline{grad}(f)$ be the gradient of smooth function f on D-homothetically deformed K-paracontact metric manifold, then from definition of gradient, we have

$$\overline{g}(\overline{grad}(f), X) = Xf, \tag{3.1}$$

for all smooth vector fields X on M. Using (2.21) and definition of gradient on M in (3.1), we get

$$\alpha\overline{grad}(f) + \alpha(\alpha-1)\eta(\overline{grad}(f)) = grad(f). \tag{3.2}$$

Now, taking inner product of (3.2) with ξ, we obtain

$$\eta(\overline{grad}(f)) = \frac{1}{\alpha^2}\eta(grad(f)). \tag{3.3}$$

Using (3.3) and $g(grad(f), X) = Xf$ in (3.2), we get

$$\overline{grad}(f) = \frac{1}{\alpha}grad(f) - \frac{(\alpha-1)}{\alpha^2}(\xi f)\xi. \tag{3.4}$$

Next, let $\overline{Hess}(f)$ be the Hessian of a smooth function f on D-homothetically deformed K-paracontact metric manifold. Then from the definition of Hessian, we have

$$\overline{Hess}(f)(X_1,X_2) = \overline{g}(\overline{\nabla}_{X_1}\overline{grad}(f), X_2), \tag{3.5}$$

for all smooth vector fields X_1, X_2 on M. Using (2.21), (2.22) in (3.5), we get

$$\overline{Hess}(f)(X_1,X_2) = g(\nabla_{X_1}grad(f), X_2) + (\alpha-1)\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f, \tag{3.6}$$

for all smooth vector fields X_1, X_2 on M. Now, using definition of Hessian of a smooth function f on manifold M in (3.6), we obtain

$$\overline{Hess}(f)(X_1,X_2) = Hess(f)(X_1,X_2) + (\alpha-1)\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f, \tag{3.7}$$

for all smooth vector fields X_1, X_2 and smooth function f on M.

Now, let $\overline{div}X$ be the divergence of a smooth vector field X on D-homothetically deformed K-paracontact metric manifold. Then from definition of divergence, we have

$$\overline{div}X = \Sigma\overline{g}(\overline{\nabla}_{e_i}X, e_i). \tag{3.8}$$

Using (2.21) and (2.22) in (3.8), we obtain

$$\overline{div}X = \alpha\Sigma g(\nabla_{e_i}X, e_i) + \alpha(\alpha-1)\Sigma\nabla_{e_i}(\eta(X))\eta(e_i), \tag{3.9}$$

for all smooth vector fields X on M. Now, using definition of divergence on manifold M in (3.9), we get

$$\overline{div}X = \alpha divX, \tag{3.10}$$

for all smooth vector fields X on M.

Next, let $\overline{\Delta}(f)$ be the Laplacian of a smooth function f on D-homothetically deformed K-paracontact metric manifold. Then from definition of Laplacian, we have

$$\overline{\Delta}(f) = \overline{div}(\overline{grad}(f)). \tag{3.11}$$

Using (3.4) and (3.10) in (3.11), we obtain

$$\overline{\Delta}(f) = \frac{1}{\alpha}div(grad(f)). \tag{3.12}$$

Now, using definition of Laplacian on manifold M in (3.12), we get

$$\overline{\Delta}(f) = \frac{1}{\alpha}\Delta(f). \tag{3.13}$$

So, we can state the following Lemma, which will be used in this paper.

Lemma 3.1. *Let $M(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be D-homothetically deformed K-paracontact metric manifold. Then gradient of a smooth function, Hessian of a smooth function, divergence of a smooth vector field and Laplacian of a smooth function on the deformed manifold are given by,*

(i) $\overline{grad}(f) = \frac{1}{\alpha}grad(f) - \frac{(\alpha-1)}{\alpha^2}(\xi f)\xi,$

(ii) $\overline{Hess}(f)(X_1, X_2) = Hess(f)(X_1, X_2) + (\alpha-1)\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f,$

(iii) $\overline{div}X = \alpha divX,$

(iv) $\overline{\Delta}(f) = \frac{1}{\alpha}\Delta(f),$

respectively, for all smooth vector fields X, X_1, X_2 and smooth function f on M.

Now, we consider K-paracontact metric manifold $M(\phi, \xi, \eta, g)$ whose metric g represents almost Ricci soliton with potential vector field V, i.e.

$$(L_V g)(X_1, X_2) + 2S(X_1, X_2) + 2\lambda g(X_1, X_2) = 0, \tag{3.14}$$

for all smooth vector fields X_1, X_2 on M, where $L_V g$ denotes the Lie derivative of g along the vector field V on M, S is the Ricci tensor of g and λ is a smooth function on M. Now, taking Lie derivative of (2.21) along V, we get

$$(L_V\overline{g})(X_1, X_2) = \alpha(L_V g)(X_1, X_2) + \alpha(\alpha-1)\{(L_V\eta)(X_1)\eta(X_2) + \eta(X_1)(L_V\eta)(X_2)\}. \tag{3.15}$$

Next, taking Lie derivative of $\eta(X_1) = g(X_1, \xi)$ along V, we obtain

$$(L_V\eta)(X_1) = (L_V g)(X_1, \xi) + g(X_1, L_V\xi). \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\begin{aligned}(L_V\overline{g})(X_1, X_2) =& \alpha(L_V g)(X_1, X_2) + \alpha(\alpha-1)\{(L_V g)(X_1, \xi)\eta(X_2) \\ &+ g(X_1, L_V\xi)\eta(X_2) + \eta(X_1)(L_V g)(X_2, \xi) + \eta(X_1)g(X_2, L_V\xi)\},\end{aligned} \tag{3.17}$$

for all smooth vector fields X_1, X_2 on M.

Now, using (2.21), (2.24) and (3.17) in (3.14), we obtain

$$\begin{aligned}&\frac{1}{\alpha}(L_V\overline{g})(X_1, X_2) - (\alpha-1)\{(L_V g)(X_1, \xi)\eta(X_2) + g(X_1, L_V\xi)\eta(X_2) \\ &+ \eta(X_1)(L_V g)(X_2, \xi) + \eta(X_1)g(X_2, L_V\xi)\} + 2\overline{S}(X_1, X_2) - 4(\alpha-1)g(X_1, X_2) \\ &+ 4(\alpha-1)\{n(\alpha+1)+1\}\eta(X_1)\eta(X_2) + 2\frac{\lambda}{\alpha}\overline{g}(X_1, X_2) - 2\lambda(\alpha-1)\eta(X_1)\eta(X_2) = 0.\end{aligned}$$

Taking $\overline{V} = \frac{V}{\alpha}$ and $\overline{\lambda} = \frac{\lambda}{\alpha}$, the above equation can be written as

$$\begin{aligned}(L_{\overline{V}}\overline{g})(X_1, X_2) &+ 2\overline{S}(X_1, X_2) + 2\overline{\lambda}\overline{g}(X_1, X_2) - (\alpha - 1)\{(L_V g)(X_1, \xi)\eta(X_2)\\ &+ g(X_1, L_V\xi)\eta(X_2) + \eta(X_1)(L_V g)(X_2, \xi) + \eta(X_1)g(X_2, L_V\xi) + 4g(X_1, X_2)\}\\ &+ 4(\alpha - 1)\left\{n(\alpha + 1) + 1 + \frac{\lambda}{2}\right\}\eta(X_1)\eta(X_2) = 0,\end{aligned} \tag{3.18}$$

for all smooth vector fields X_1, X_2 on M. Thus, we can state

Theorem 3.1. *Let $M(\phi, \xi, \eta, g)$ be a K-paracontact metric manifold of dimension $(2n + 1)$ such that g represents an almost Ricci soliton with potential vector field V. Then under D-homothetic deformation, the deformed metric $\overline{g}$ remains almost Ricci soliton with potential vector field $\overline{V} = \frac{V}{\alpha}$ and soliton constant $\overline{\lambda} = \frac{\lambda}{\alpha}$ if and only if*

$$\begin{aligned}&(L_V g)(X_1, \xi)\eta(X_2) + g(X_1, L_V\xi)\eta(X_2) + \eta(X_1)(L_V g)(X_2, \xi) + \eta(X_1)g(X_2, L_V\xi)\\ &+ 4g(X_1, X_2) - 4\left\{n(\alpha + 1) + 1 + \frac{\lambda}{2}\right\}\eta(X_1)\eta(X_2) = 0,\end{aligned}$$

for all smooth vector fields X_1, X_2 on M.

In particular, taking the potential vector field V as Reeb vector field ξ, then $(L_{\overline{\xi}}\overline{g})(X_1, X_2) = (L_\xi g)(X_1, X_2)$. Thus, we have the following corollary.

Corollary 3.1. *On a K-paracontact metric manifold $M(\phi, \xi, \eta, g)$, let g represents an almost Ricci soliton such that the potential vector field V is the Reeb vector field. Then under D-homothetic deformation, the deformed metric $\overline{g}$ remains almost Ricci soliton with potential vector field as $\overline{\xi}$ and soliton constant $\overline{\lambda} = \frac{\lambda}{\alpha}$ if and only if*

$$g(\phi X_1, \phi X_2) = \left\{\frac{\lambda}{2} - n(\alpha + 1)\right\}\eta(X_1)\eta(X_2),$$

for all smooth vector fields X_1, X_2 on M.

Remark 3.1.

1. In [8], authors proved that if a para-Sasaki-like manifold M admits a para-Ricci-like soliton with potential vector field ξ and constants (α, β, γ), then the D-homothetic deformed manifold M also admits a para-Ricci-like soliton with potential vector field $\overline{\xi}$ and constants $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, \frac{\gamma + (1-\lambda)(\alpha+\beta)}{\lambda^2}\right)$.
2. In [5], Blaga proved that if (ξ, λ) defines an almost Ricci soliton in a $(2n + 1)$-dimensional D-homothetically deformed Kenmotsu manifold, then it is an expanding Ricci soliton with soliton constant $\overline{\lambda} = -\frac{2n}{\alpha^2}$.

Now, from (3.18) and Theorem3.1, we can write

$$(L_{\overline{V}}\overline{g})(X_1, X_2) + 2\overline{S}(X_1, X_2) + 2\overline{\lambda}\overline{g}(X_1, X_2) = 0, \tag{3.19}$$

for all smooth vector fields X_1, X_2 on M, where $\overline{V} = \frac{V}{\alpha}$ and $\overline{\lambda} = \frac{\lambda}{\alpha}$, $(L_{\overline{V}}\overline{g})$ denotes the Lie derivative of $\overline{g}$ along vector field $\overline{V}$ on M, $\overline{S}$ is the Ricci tensor of $\overline{g}$ and $\overline{\lambda}$ is the smooth function on M. Thus, on tracing (3.19), we obtain

$$\overline{div}(\overline{V}) + \overline{r} + \overline{\lambda}(2n + 1) = 0.$$

Now, we study almost Ricci soliton on D-homothetically deformed K-paracontact metric manifold, with potential vector field as divergence-free. Let us recall that a divergence-free vector field is one whose divergence vanishes everywhere which means the flow is incompressible. So, a vector field X is called a divergence-free vector field if $div(X) = 0$.

If potential vector field $\overline{V}$ is divergence-free then above equation can be written as

$$\overline{r} = -\overline{\lambda}(2n + 1). \tag{3.20}$$

Using (2.25) with (3.20), we get

$$r = -\overline{\lambda}(2n + 1) - 2n(\alpha - 1). \tag{3.21}$$

Using (2.21) and (2.24) in (3.19), we obtain

$$(L_{\overline{V}}\overline{g})(X_1, X_2) + 2S(X_1, X_2) + 4(\alpha - 1)g(X_1, X_2) - 4(\alpha - 1)\{n(\alpha + 1) + 1\} \\ \eta(X_1)\eta(X_2) + 2\overline{\lambda}\{\alpha g(X_1, X_2) + (\alpha - 1)\eta(X_1)\eta(X_2)\} = 0, \tag{3.22}$$

for all smooth vector fields X_1, X_2 on M. Now, taking trace of (3.22) and using $\overline{div}(\overline{V}) = 0$, we get

$$r = 2n\alpha^2 - 4n\alpha + 2n - \overline{\lambda}\{\alpha(2n + 1) + (\alpha - 1)\}. \tag{3.23}$$

Comparing (3.21) and (3.23), we obtain

$$\overline{\lambda} = 2n\alpha. \tag{3.24}$$

Using (3.24) in (3.20), we get

$$\overline{r} = -2n\alpha(2n + 1). \tag{3.25}$$

Using (3.24) in $\overline{\lambda} = \frac{\lambda}{\alpha}$, we get

$$\lambda = 2n\alpha^2. \tag{3.26}$$

Thus, we can state:

Theorem 3.2. *If* $(\overline{g}, \overline{V}, \overline{\lambda})$ *represents an almost Ricci soliton whose potential vector field is divergence-free on* $(2n + 1)$ *dimensional D-homothetically deformed K-paracontact metric manifold then the Ricci soliton is always expanding. Moreover, the almost Ricci soliton* (g, V, λ) *is always expanding.*

Remark 3.2. In [5], Blaga showed that if (V, λ) defines an almost Ricci soliton of solenoidal type in a $(2n + 1)$ dimensional D-homothetically deformed Kenmotsu manifold, then soliton constant $\overline{\lambda} = \xi(\eta(V)) - \frac{2n}{\alpha^2}$.

Next, we consider potential vector field V of almost Ricci soliton on D-homothetically deformed K-paracontact metric manifold as conformal vector field. Then

$$(L_V g)(X_1, X_2) = 2\mu g(X_1, X_2), \tag{3.27}$$

for all smooth vector fields X_1, X_2 on M and μ is a smooth function on M. Using (3.27) and $\overline{V} = \frac{V}{\alpha}$ in (3.17), we get

$$(L_{\overline{V}}\overline{g})(X_1, X_2) = 2\mu g(X_1, X_2) + 4\mu(\alpha - 1)\eta(X_1)\eta(X_2) + (\alpha - 1) \\ \{g(X_1, L_V\xi)\eta(X_2) + \eta(X_1)g(L_V\xi, X_2)\}, \tag{3.28}$$

for all smooth vector fields X_1, X_2 on M. Now, using (2.19), (2.20), (2.21) and (3.28) in (3.20), we obtain

$$\overline{S}(X_1, X_2) = -(\overline{\lambda} + \mu)\overline{g}(X_1, X_2) - \frac{2\mu}{\alpha^2}(\alpha - 1)\overline{\eta}(X_1)\overline{\eta}(X_2) + \frac{(\alpha - 1)}{\alpha} \\ \overline{\eta}(X_1)\overline{\eta}(X_2)\overline{\eta}(L_{\overline{V}}\overline{\xi}) - \frac{(\alpha - 1)}{2}\{\overline{g}(X_1, L_{\overline{V}}\overline{\xi})\overline{\eta}(X_2) + \overline{\eta}(X_1)\overline{g}(L_{\overline{V}}\overline{\xi}, X_2)\}, \tag{3.29}$$

for all smooth vector fields X_1, X_2 on M. Taking Lie derivative of $\overline{g}(\overline{\xi}, \overline{\xi}) = 1$ along $\overline{V}$ and using it in (3.29), we obtain

$$\overline{S}(X_1, X_2) = -(\overline{\lambda} + \mu)\overline{g}(X_1, X_2) - \frac{(\alpha - 1)}{\alpha^2}\{2\mu + \alpha(\alpha - 1)(2n - \overline{\lambda})\}\overline{\eta}(X_1)\overline{\eta}(X_2) \\ - \frac{(\alpha - 1)}{2}\{\overline{g}(X_1, L_{\overline{V}}\overline{\xi})\overline{\eta}(X_2) + \overline{\eta}(X_1)\overline{g}(L_{\overline{V}}\overline{\xi}, X_2)\}, \tag{3.30}$$

for all smooth vector fields X_1, X_2 on M. Hence, we get the following theorem.

Theorem 3.3. *Let $M(\phi, \xi, \eta, g)$ be a K-paracontact metric manifold of dimension $(2n+1)$ such that g represents an almost Ricci soliton with potential vector field V as conformal vector field. Then under D-homothetic deformation, the deformed K-paracontact metric manifold becomes η-Einstein K-paracontact metric manifold with functions $\overline{a} = -(\overline{\lambda} + \mu)$ and $\overline{b} = \frac{(1-\alpha)}{\alpha^2}\{2\mu + \alpha(\alpha-1)(2n-\overline{\lambda})\}$ if and only if*

$$\overline{g}(X_1, L_{\overline{V}}\overline{\xi})\overline{\eta}(X_2) + \overline{\eta}(X_1)\overline{g}(L_{\overline{V}}\overline{\xi}, X_2) = 0,$$

for all smooth vector fields X_1, X_2 on M.

Now, from (3.30) and Theorem3.3, we can write

$$\overline{S}(X_1, X_2) = -(\overline{\lambda} + \mu)\overline{g}(X_1, X_2) - \frac{(\alpha-1)}{\alpha^2}\{2\mu + \alpha(\alpha-1)(2n-\overline{\lambda})\}\overline{\eta}(X_1)\overline{\eta}(X_2), \tag{3.31}$$

for all smooth vector fields X_1, X_2 on M, where $\overline{\lambda}$ and μ are smooth functions on M. On tracing (3.31), we get

$$\overline{r} = -(\overline{\lambda} + \mu)(2n+1) - \frac{(\alpha-1)}{\alpha^2}\{2\mu + \alpha(\alpha-1)(2n-\overline{\lambda})\}. \tag{3.32}$$

Using (2.20), (2.21) and (2.24) in (3.31) and then contracting over X_1 and X_2, we get

$$\begin{aligned} r =& \alpha(\alpha-1)\{4n - (2\overline{\lambda} + \mu) - \alpha^2(2n - \overline{\lambda})\} - 2(\alpha-1)(n+\mu) \\ &- \alpha(\overline{\lambda} + \mu)(2n+1). \end{aligned} \tag{3.33}$$

Thus in view of (2.25), (3.32) and (3.33), we obtain

Theorem 3.4. *If $(\overline{g}, \overline{V}, \overline{\lambda})$ represents an almost Ricci soliton whose potential vector field is conformal vector field on $(2n+1)$ dimensional D-homothetically deformed K-paracontact metric manifold, then*

$$\overline{\lambda} = 2n + \frac{\mu}{\alpha}\left(\frac{\alpha^2 + 2\alpha - 2}{\alpha^2 - 3\alpha + 1}\right).$$

Now, we consider the potential vector field $\overline{V}$ of almost Ricci soliton on D-homothetically deformed K-paracontact metric manifold is point wise collinear with Reeb vector field $\overline{\xi}$. Then

$$\overline{V} = \sigma\overline{\xi}, \tag{3.34}$$

where σ is a non-zero smooth function on M. Taking covariant derivative of (3.34) along an arbitrary vector field X_1 on D-homothetically deformed K-paracontact metric manifold M and using $\overline{\nabla}_X\overline{\xi} = -\overline{\phi}X$, we get

$$\overline{\nabla}_{X_1}\overline{V} = (X_1\sigma)\overline{\xi} - \sigma(\overline{\phi}X_1). \tag{3.35}$$

Using (3.35) in (3.19), we obtain

$$(X_1\sigma)\overline{\eta}(X_2) + (X_2\sigma)\overline{\eta}(X_1) + 2\overline{S}(X_1, X_2) + 2\overline{\lambda}\overline{g}(X_1, X_2) = 0, \tag{3.36}$$

for all smooth vector fields X_1, X_2 on M. Now, replacing $\overline{\xi}$ in place of X_1 and X_2, and using $\overline{S}(\overline{\xi}, \overline{\xi}) = -2n$, we get

$$\overline{\xi}\sigma = 2n - \overline{\lambda}. \tag{3.37}$$

Taking $X_2 = \overline{\xi}$ in (3.36) and using (3.37), we get

$$X_1\sigma = (\overline{\xi}\sigma)\overline{\eta}(X_1), \tag{3.38}$$

for all smooth vector field X_1 on M. Now, by using the definition of gradient vector field it gives $\overline{D}\sigma = (\overline{\xi}\sigma)\overline{\xi}$. Taking covariant derivative of this along an arbitrary vector field X_1 on M, we obtain

$$\overline{\nabla}_{X_1}(\overline{D}\sigma) = X_1(\overline{\xi}\sigma)\overline{\xi} - (\overline{\xi}\sigma)(\overline{\phi}X_1). \tag{3.39}$$

Thus, making use of $\overline{g}(\overline{\nabla}_{X_1}\overline{D}\sigma, X_2) = \overline{g}(\overline{\nabla}_{X_2}\overline{D}\sigma, X_1)$ with (3.39), we get

$$X_1(\overline{\xi}\sigma)\overline{\eta}(X_2) - X_2(\overline{\xi}\sigma)\overline{\eta}(X_1) + 2(\overline{\xi}\sigma)d\overline{\eta}(X_1, X_2) = 0, \tag{3.40}$$

for all smooth vector fields X_1, X_2 on M. Choosing vector fields X_1 and X_2 orthogonal to $\overline{\xi}$ and using $d\overline{\eta} \neq 0$ in (3.40), we get $(\overline{\xi}\sigma) = 0$. Using this in (3.38) which implies σ is constant and hence (3.36) gives

$$\overline{S}(X_1, X_2) = -\overline{\lambda}\overline{g}(X_1, X_2), \tag{3.41}$$

for all smooth vector fields X_1, X_2 on M. Now tracing (3.41), we obtain

$$\overline{r} = -\overline{\lambda}. \tag{3.42}$$

Using (2.25) with (3.42), we get

$$r = -2n(\alpha - 1) - \overline{\lambda}\alpha. \tag{3.43}$$

Using (2.21) and (2.24) in (3.19), we obtain

$$\begin{aligned} &(L_{\overline{V}}\overline{g})(X_1, X_2) + 2S(X_1, X_2) + 4(\alpha - 1)g(X_1, X_2) - 4(\alpha - 1)\{n(\alpha + 1) + 1\} \\ &\eta(X_1)\eta(X_2) + 2\overline{\lambda}\{\alpha g(X_1, X_2) + (\alpha - 1)\eta(X_1)\eta(X_2)\} = 0, \end{aligned} \tag{3.44}$$

for all smooth vector fields X_1, X_2 on M. Now, tracing (3.44) and using $\overline{div}(\overline{V}) = 0$, we get

$$r = 2n\alpha^2 - 4n\alpha + 2n - \overline{\lambda}\{\alpha(2n + 1) + (\alpha - 1)\}. \tag{3.45}$$

Comparing (3.43) and (3.45), we obtain

$$\overline{\lambda} = \frac{2n\alpha(\alpha - 1)}{\alpha(2n + 1) - 1}. \tag{3.46}$$

Using (3.46) in $\overline{\lambda} = \frac{\lambda}{\alpha}$, we get

$$\lambda = \frac{2n\alpha^2(\alpha - 1)}{\alpha(2n + 1) - 1}.$$

Theorem 3.5. *If $(\overline{g}, \overline{V}, \overline{\lambda})$ represents an almost Ricci soliton whose potential vector field $\overline{V}$ is point-wise collinear with Reeb vector field $\overline{\xi}$ on $(2n + 1)$ dimensional D-homothetically deformed K-paracontact metric manifold then the potential vector field $\overline{V}$ is constant multiple of $\overline{\xi}$ and the deformed manifold is Einstein. Also, the soliton constant*

$$\overline{\lambda} = \frac{2n\alpha(\alpha - 1)}{\alpha(2n + 1) - 1}.$$

Example 3.1. We consider the three dimensional manifold $M = \mathbb{R}^3$ with standard coordinates (x, y, z) in $\mathbb{R}^3$. The vector fields $e_1 = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial z}$, $e_2 = \frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of M. Define the structure (ϕ, ξ, η) on M by $\phi(e_1) = e_1, \phi(e_2) = -e_2, \phi(e_3) = 0, \xi = e_3$ and the 1- form $\eta = 2ydx + dz$, which satisfies $\eta(e_1) = \eta(e_2) = 0$ and $\eta(e_3) = \eta(\xi) = 1$. So it can be verified $\phi^2 e_i = e_i - \eta(e_i)\xi$, for all i = 1, 2, 3, $\phi(\xi) = 0$ and $(\eta \circ \phi) = 0$.

Thus, the manifold M with structure (ϕ, ξ, η) is an almost paracontact manifold. Now, we define the pseudo-Riemannian metric g as $g(e_1, e_2) = 1, g(e_3, e_3) = g(\xi, \xi) = 1$ and for others $g(e_i, e_j) = 0$. We can compute

$$\begin{aligned} &[e_1, e_1] = 0, \quad [e_1, e_2] = 2e_3, \quad [e_1, e_3] = 0, \\ &[e_2, e_2] = 0, \quad [e_2, e_3] = 0, \quad [e_3, e_3] = 0. \end{aligned}$$

Then

$$\begin{aligned} d\eta(e_1, e_2) &= \frac{1}{2}\{e_1(\eta(e_2)) - e_2(\eta(e_1)) - \eta([e_1, e_2])\} = -1 = g(e_1, \phi e_2), \\ d\eta(e_1, e_3) &= \frac{1}{2}\{e_1(\eta(e_3)) - e_3(\eta(e_1)) - \eta([e_1, e_3])\} = 0 = g(e_1, \phi e_3), \\ d\eta(e_2, e_3) &= \frac{1}{2}\{e_2(\eta(e_3)) - e_3(\eta(e_2)) - \eta([e_2, e_3])\} = 0 = g(e_2, \phi e_3). \end{aligned}$$

Therefore, the manifold M with structure (ϕ, ξ, η, g) is a paracontact metric manifold. As we define the self adjoint operator, $h = \frac{1}{2}L_\xi\phi$, where L_ξ is the Lie derivative along ξ, we can easily verify $he_i = 0\ \forall i$, i.e. $h \equiv 0$. Hence, M is K-paracontact metric manifold. Now, using Koszul's formula the Levi-Civita connection ∇ is,

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = e_3, \quad \nabla_{e_1}e_3 = -e_1, \\ &\nabla_{e_2}e_1 = -e_3, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = e_2, \\ &\nabla_{e_3}e_1 = -e_1, \quad \nabla_{e_3}e_2 = e_2, \quad \nabla_{e_3}e_3 = 0. \end{aligned}$$

Using the definition of Riemannian curvature,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

we obtain

$$\begin{array}{rrr} R(e_1,e_1)e_1 = 0, & R(e_1,e_1)e_2 = 0, & R(e_1,e_1)e_3 = 0, \\ R(e_1,e_2)e_2 = -3e_2, & R(e_2,e_2)e_2 = 0, & R(e_3,e_2)e_2 = 0, \\ R(e_1,e_3)e_3 = -e_1, & R(e_2,e_3)e_3 = -e_2, & R(e_3,e_3)e_3 = 0, \\ R(e_3,e_1)e_2 = -e_3. & & \end{array}$$

Using these Riemannian curvature components, the Ricci tensor $S(X,Y) = \Sigma g(R(e_i,X)Y,e_i)$,

$$\begin{array}{lll} S(e_1,e_1) = -3, & S(e_2,e_2) = -3, & S(e_3,e_3) = -1, \\ S(e_1,e_2) = -1, & S(e_1,e_3) = 0, & S(e_2,e_3) = 0. \end{array}$$

Now, consider the potential vector field, $V = 3y\frac{\partial}{\partial x} + 3x\frac{\partial}{\partial y} - 3y^2\frac{\partial}{\partial z}$, then the metric g is an expanding Ricci soliton with soliton constant $\lambda = 1$.

Also, for any $\alpha > 0$, we can define D-homothetically deformed K-paracontact metric manifold M with structure

$$\begin{array}{lll} \overline{\phi}(e_1) = e_1, & \overline{\phi}(e_2) = -e_2, & \overline{\phi}(e_3) = 0, \\ \overline{\xi} = \frac{1}{\alpha}e_3, & \overline{\eta} = \alpha(2ydx + dz), & \end{array}$$

and D-homothetically deformed pseudo-Riemannian metric is $\overline{g}(e_1,e_2) = \alpha, \overline{g}(e_3,e_3) = \alpha^2$ and for others $\overline{g}(e_i,e_j) = 0$.

Then $(\overline{g}, \overline{V}, \overline{\lambda})$ is also an expanding Ricci soliton with potential vector field $\overline{V} = \frac{V}{\alpha}$ and $\overline{\lambda} = \frac{\lambda}{\alpha}$, which verifies Theorem3.1.

Example 3.2. Let us consider the three dimensional manifold $M = \mathbb{R}^3$ with standard coordinates (x,y,z) in $\mathbb{R}^3$. The vector fields $e_1 = e^z\frac{\partial}{\partial x}$, $e_2 = e^z\frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of M. Define the structure (ϕ,ξ,η) on M by

$$\begin{array}{lll} \phi(e_1) = e_1, & \phi(e_2) = -e_2, & \phi(e_3) = 0, \\ \xi = e_3, & \eta = dz, & \end{array}$$

which satisfies $\eta(e_1) = \eta(e_2) = 0$ and $\eta(e_3) = \eta(\xi) = 1$. So it can be verified $\phi^2 e_i = e_i - \eta(e_i)\xi$, for all i = 1, 2, 3, $\phi(\xi) = 0$ and $(\eta \circ \phi) = 0$.

Thus, the manifold M with structure (ϕ,ξ,η) is an almost paracontact manifold. Now, we define the pseudo-Riemannian metric g as $g(e_1,e_2) = -\frac{1}{2}, g(e_3,e_3) = g(\xi,\xi) = 1$ and for others $g(e_i,e_j) = 0$. Therefore, the manifold M with structure (ϕ,ξ,η,g) is an almost paracontact metric manifold. We can compute

$$\begin{array}{lll} [e_1,e_1] = 0, & [e_1,e_2] = 0, & [e_1,e_3] = -e_1, \\ [e_2,e_2] = 0, & [e_2,e_3] = -e_2, & [e_3,e_3] = 0. \end{array}$$

As we define the self adjoint operator, $h = \frac{1}{2}L_\xi\phi$, where L_ξ is the Lie derivative along ξ, we can easily verify $he_i = 0\ \forall i$, i.e. $h \equiv 0$. Hence, M is K-paracontact metric manifold. Now, using Koszul's formula the Levi-Civita connection ∇ is,

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = -\frac{e_3}{2}, & \nabla_{e_1}e_3 = -e_1, \\ \nabla_{e_2}e_1 = -\frac{e_3}{2}, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = e_2, \\ \nabla_{e_3}e_1 = -\frac{e_1}{2}, & \nabla_{e_3}e_2 = -\frac{e_2}{2}, & \nabla_{e_3}e_3 = 0. \end{array}$$

Using the definition of Riemannian curvature,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

we obtain

$$\begin{array}{rrr} R(e_1,e_1)e_1 = 0, & R(e_1,e_1)e_2 = 0, & R(e_1,e_1)e_3 = 0, \\ R(e_1,e_2)e_2 = \frac{e_2}{2}, & R(e_2,e_2)e_2 = 0, & R(e_3,e_2)e_2 = 0, \\ R(e_1,e_3)e_3 = -\frac{3}{2}e_1, & R(e_2,e_3)e_3 = \frac{3}{2}e_1, & R(e_3,e_3)e_3 = 0, \\ R(e_3,e_1)e_2 = \frac{e_3}{4}. & & \end{array}$$

Using these Riemannian curvature components, the Ricci tensor $S(X,Y) = \Sigma g(R(e_i,X)Y,e_i)$,

$$\begin{array}{lll} S(e_1,e_1) = \frac{1}{4}, & S(e_2,e_2) = -\frac{1}{4}, & S(e_3,e_3) = 0, \\ S(e_1,e_2) = 0, & S(e_1,e_3) = 0, & S(e_2,e_3) = 0. \end{array}$$

Now, consider the potential vector field, $V = -\frac{y}{2}\frac{\partial}{\partial x} + \frac{x}{2}\frac{\partial}{\partial y}$, then the metric g is a steady Ricci soliton.

Also, for any $\alpha > 0$, we can define D-homothetically deformed K-paracontact metric manifold M with structure

$$\overline{\phi}(e_1) = e_1, \quad \overline{\phi}(e_2) = -e_2, \quad \overline{\phi}(e_3) = 0,$$
$$\overline{\xi} = \frac{1}{\alpha}e_3, \qquad \overline{\eta} = \alpha dz,$$

and D-homothetically deformed pseudo-Riemannian metric is $\overline{g}(e_1, e_2) = -\frac{\alpha}{2}, \overline{g}(e_3, e_3) = \alpha^2$ and for others $\overline{g}(e_i, e_j) = 0$.

Then $(\overline{g}, \overline{V}, \overline{\lambda})$ is also a steady Ricci soliton with potential vector field $\overline{V} = \frac{V}{\alpha}$, which verifies Theorem3.1.

4. Almost Gradient Ricci Solitons on D-homothetically deformed paracontact metric manifolds

Here, we assume K-paracontact metric g as almost gradient Ricci soliton with potential function f, i.e.

$$Hess(f)(X_1, X_2) + S(X_1, X_2) + \lambda g(X_1, X_2) = 0, \tag{4.1}$$

for all smooth vector fields X_1, X_2 on M, where $Hess(f)$ is the Hessian of potential function f, S is the Ricci tensor of g and λ is a smooth function M. Using (2.21), (2.24) and (3.7) in (4.1), we get

$$\begin{aligned}&\overline{Hess}(f)(X_1, X_2) + \overline{S}(X_1, X_2) + \frac{\lambda}{\alpha}\overline{g}(X_1, X_2)\\ &- (\alpha - 1)\left[\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f + 2g(X_1, X_2) - 2\left\{n(\alpha + 1) + 1 - \frac{\lambda}{2}\right\}\eta(X_1)\eta(X_2)\right] = 0,\end{aligned}$$

for all smooth vector fields X_1, X_2 on M. Now taking $\overline{\lambda} = \frac{\lambda}{\alpha}$ in above equation and using (2.6), we get

$$\begin{aligned}&\overline{Hess}(f)(X_1, X_2) + \overline{S}(X_1, X_2) + \overline{\lambda}\overline{g}(X_1, X_2) - (\alpha - 1)\\ &\left[\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f - 2g(\phi X_1, \phi X_2) - 2\left\{n(\alpha + 1) - \frac{\lambda}{2}\right\}\eta(X_1)\eta(X_2)\right] = 0,\end{aligned} \tag{4.2}$$

for all smooth vector fields X_1, X_2 on M. Hence, we can state:

Theorem 4.1. *Let $M(\phi, \xi, \eta, g)$ be a K-paracontact metric manifold of dimension $(2n + 1)$ such that g represents an almost gradient Ricci soliton with potential function f. Then under D-homothetic deformation, the deformed metric $\overline{g}$ remains almost gradient Ricci soliton with potential function f and soliton constant $\overline{\lambda} = \frac{\lambda}{\alpha}$ if and only if*

$$g(\phi X_1, \phi X_2) = \left\{\frac{\lambda}{2} - n(\alpha + 1)\right\}\eta(X_1)\eta(X_2) - \frac{1}{2}\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f,$$

for all smooth vector fields X_1, X_2 on M.

Now, from Theorem4.1, we can write

$$\overline{Hess}(f)(X_1, X_2) + \overline{S}(X_1, X_2) + \overline{\lambda}\overline{g}(X_1, X_2) = 0, \tag{4.3}$$

for all smooth vector fields X_1, X_2 on M, where $\overline{\lambda} = \frac{\lambda}{\alpha}$, $\overline{Hess}(f)$ is the Hessian of potential function f on M, $\overline{S}$ is the Ricci tensor of $\overline{g}$ and $\overline{\lambda}$ is the smooth function on M. On taking trace of (4.3) and using (2.25) and (3.13), we get

$$r = -\overline{\lambda}\alpha(2n + 1) - 2n(\alpha - 1) - \Delta(f). \tag{4.4}$$

Using (2.21), (2.24) and (3.17) in (4.3), we obtain

$$\begin{aligned}&Hess(f)(X_1, X_2) + (\alpha - 1)\{\eta(X_1)\phi X_2 + \eta(X_2)\phi X_1\}f + S(X_1, X_2) + 2(\alpha - 1)g(X_1, X_2)\\ &- 2(\alpha - 1)\{n(\alpha + 1) + 1\}\eta(X_1)\eta(X_2) + \overline{\lambda}\{\alpha g(X_1, X_2) + \alpha(\alpha - 1)\eta(X_1)\eta(X_2)\} = 0,\end{aligned} \tag{4.5}$$

for all smooth vector fields X_1, X_2 on M. Now tracing (4.5), we get

$$r = -2n(\alpha - 1)^2 - \overline{\lambda}\alpha(2n + \alpha) - \Delta(f). \tag{4.6}$$

Comparing (4.4) and (4.6), we get

$$\overline{\lambda} = \frac{2n(2 - \alpha)}{\alpha}. \tag{4.7}$$

Now, using (4.7) in $\overline{\lambda} = \frac{\lambda}{\alpha}$, we obtain

$$\lambda = 2n(2 - \alpha).$$

Hence, we can state:

Theorem 4.2. *If $(\overline{g}, f, \overline{\lambda})$ represents an almost gradient Ricci soliton with potential function f on $(2n+1)$ dimensional D-homothetically deformed K-paracontact metric manifold then the gradient Ricci soliton is expanding, steady and shrinking according as $\alpha < 2$, $\alpha = 2$ and $\alpha > 2$, respectively. Moreover, the almost gradient Ricci soliton (g, f, λ) is also expanding, steady and shrinking according as $\alpha < 2$, $\alpha = 2$ and $\alpha > 2$.*

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Author's contributions

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Affiliations

TARUN SAXENA
ADDRESS: Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India.
E-MAIL: tarunsaxena254@gmail.com
ORCID ID: https://orcid.org/0009-0009-6613-3627

AKHILESH YADAV
ADDRESS: Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India.
E-MAIL: akhilesha68@gmail.com
ORCID ID: https://orcid.org/0000-0003-3990-857X