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On neutrosophic submodules of a module

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Abstract

The target of this study is to observe some of the algebraic structures of a single valued neutrosophic set. So, we introduce the concept of a neutrosophic submodule of a given classical module and investigate some of the crucial properties and characterizations of the proposed concept.

Keywords: neutrosophic set, single valued neutrosophic set, ring, module.

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1. Introduction

Neutrosopy is a branch of philosophy introduced by Smarandache in 1980. It is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics. While neutrosophic set generalizes the fuzzy set, neutrosophic probability generalizes the classical and imprecise probability, neutrosophic statistics generalizes the classical and imprecise statistics, neutrosophic logic however generalizes fuzzy logic, intuitionistic logic, Boolean logic, multi-valued logic, paraconsistent logic and dialetheism. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. The use of neutrosophic theory becomes inevitable when a situation involving indeterminacy is to be modeled since fuzzy set theory is limited to modeling a situation involving uncertainty. From scientific and engineering point of view, the definition of a neutrosophic set was specified to the single valued neutrosophic set. The single valued neutrosophic set was introduced for the first time by F. Smarandache, Neutrosophy / Neutrosophic probability, set, and logic, American Res. Press, see pages 7-8, 1998 [10], which is also mentioned by Denis Howe, from England, in The Free Online Dictionary of Computing, 1999, and by Wang et al.[11]. The single valued neutrosophic set is a

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generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent set etc.

The introduction of neutrosophic theory has led to the establishment of the concept of neutrosophic algebraic structures. Vasantha Kandasamy and Florentin Smarandache [6] for the first time introduced the concept of algebraic structures which has caused a paradigm shift in the study of algebraic structures. Single valued neutrosophic set is also applied to algebraic and topological structures (see [1, 2, 3, 7, 8, 9]). Çetkin and Aygün [4] proposed the definitions of neutrosophic subgroups [3] and neutrosophic subrings [4] of a given classical group and classical ring, respectively. In this paper, as a continuation of the studies [3] and [4], we present the concept of neutrosophic submodules and also we investigate crucial properties and characterizations of the proposed concept.

2. Preliminaries

In this chapter, we give some preliminaries about single valued neutrosophic sets and set operations, which will be called neutrosophic sets, for simplicity.

2.1 Definition [10] A neutrosophic set A on the universe of X is defined as $A = \{< x, t_A(x), i_A(x), f_A(x) >, x \in X\}$ where $t_A, i_A, f_A : X \to]^-0, 1^+[$ and $^-0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non standard subsets of $]^-0, 1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence throughout this work, the following specified definition of a neutrosophic set known as single valued neutrosophic set is considered.

2.2 Definition [11] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A on X is characterized by truth-membership function t_A, indeterminacy-membership function i_A and falsity-membership function f_A. For each point x in $X, t_A(x), i_A(x), f_A(x) \in [0, 1]$.

A neutrosophic set A can be written as

$$A = \sum_{i=1}^{n} < t(x_i), i(x_i), f(x_i) > /x_i,\ x_i \in X.$$

2.3 Example[11] Assume that $X = \{x_1, x_2, x_3\}, x_1$ is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are in $[0, 1]$. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued neutrosophic set of X defined by

$A =< 0.3, 0.4, 0.5 > /x_1+ < 0.5, 0.2, 0.3 > /x_2+ < 0.7, 0.2, 0.2 > /x_3$.

Since the membership functions t_A, i_A, f_A are defined from X into the unit interval $[0, 1]$ as $t_A, i_A, f_A : X \to [0, 1]$, a (single valued) neutrosophic set A will be denoted by a mapping defined as $A : X \to [0, 1] \times [0, 1] \times [0, 1]$ and $A(x) = (t_A(x), i_A(x), f_A(x))$, for simplicity.

2.4 Definition [8, 11] Let A and B be two neutrosophic sets on X. Then

(1) A is contained in B, denoted as $A \subseteq B$, if and only if $A(x) \leq B(x)$. This means that $t_A(x) \leq t_B(x), i_A(x) \leq i_B(x)$ and $f_A(x) \geq f_B(x)$. Two sets A and B is called equal, i.e., $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(2) the union of A and B is denoted by $C = A \cup B$ and defined as $C(x) = A(x) \vee B(x)$ where $A(x) \vee B(x) = (t_A(x) \vee t_B(x), i_A(x) \vee i_B(x), f_A(x) \wedge f_B(x))$, for each $x \in X$. This means that $t_C(x) = \max\{t_A(x), t_B(x)\}, i_C(x) = \max\{i_A(x), i_B(x)\}$ and $f_C(x) = \min\{f_A(x), f_B(x)\}$.

(3) the intersection of A and B is denoted by $C = A \cap B$ and defined as $C(x) = A(x) \wedge B(x)$ where $A(x) \wedge B(x) = (t_A(x) \wedge t_B(x), i_A(x) \wedge i_B(x), f_A(x) \vee f_B(x))$, for

each $x \in X$. This means that $t_C(x) = \min\{t_A(x), t_B(x)\}, i_C(x) = \min\{i_A(x), i_B(x)\}$ and $f_C(x) = \max\{f_A(x), f_B(x)\}$.

(4) the complement of A is denoted by A^c and defined as $A^c(x) = (f_A(x), 1 - i_A(x), t_A(x))$, for each $x \in X$. Here $(A^c)^c = A$.

2.1. Proposition. [11] *Let A, B and C be the neutrosophic sets on the common universe X. Then the following properties are valid.*

(1) $A \cup B = B \cup A, A \cap B = B \cap A$.

(2) $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$.

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(4) $A \cap \widetilde{\emptyset} = \widetilde{\emptyset}, A \cup \widetilde{\emptyset} = A, A \cup \widetilde{X} = \widetilde{X}, A \cap \widetilde{X} = A$, *where* $t_{\widetilde{\emptyset}} = i_{\widetilde{\emptyset}} = 0, f_{\widetilde{\emptyset}} = 1$ *and* $t_{\widetilde{X}} = i_{\widetilde{X}} = 1, f_{\widetilde{X}} = 0$.

(5) $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$.

2.6 Definition Let A and B be two neutrosophic sets on X and Y, respectively. Then the cartesian product of A and B which is denoted by $A \times B$ is a neutrosophic set on $X \times Y$ and it is defined as $(A \times B)(x, y) = A(x) \times B(y)$ where $A(x) \times B(y) = (t_{A\times B}(x, y), i_{A\times B}(x, y), f_{A\times B}(x, y))$, i.e.,

$t_{A\times B}(x, y) = t_A(x) \wedge t_B(y)$, $i_{A\times B}(x, y) = i_A(x) \wedge i_B(y)$ and $f_{A\times B}(x, y) = f_A(x) \vee f_B(y)$.

2.7 Definition [3] Let A be a neutrosophic set on X and $\alpha \in [0, 1]$. Define the α-level sets of A as follows:

$(t_A)_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}, (i_A)_\alpha = \{x \in X \mid i_A(x) \geq \alpha\}$, and $(f_A)^\alpha = \{x \in X \mid f_A(x) \leq \alpha\}$.

2.8 Definition [3] Let $g : X_1 \to X_2$ be a function and A, B be the neutrosophic sets of X_1 and X_2, respectively. Then the image of a neutrosophic set A is a neutrosophic set of X_2 and it is defined as follows:

$g(A)(y) = (t_{g(A)}(y), i_{g(A)}(y), f_{g(A)}(y)) = (g(t_A)(y), g(i_A)(y), g(f_A)(y)), \forall y \in X_2$ where

$$g(t_A)(y) = \begin{cases} \bigvee t_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases}, \quad g(i_A)(y) = \begin{cases} \bigvee i_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases},$$

$$g(f_A)(y) = \begin{cases} \bigwedge f_A(x), & \text{if } x \in g^{-1}(y); \\ 1, & \text{otherwise.} \end{cases}$$

And the preimage of a neutrosophic set B is a neutrosophic set of X_1 and it is defined as follows:

$g^{-1}(B)(x) = (t_{g^{-1}(B)}(x), i_{g^{-1}(B)}(x), f_{g^{-1}(B)}(x)) = (t_B(g(x)), i_B(g(x)), f_B(g(x))) = B(g(x)), \forall x \in X_1$.

3. Neutrosophic submodules

In this section, we define the concept of a neutrosophic submodule of a given classical module over a ring and also investigate its elementary properties and characterizations. Throughout this paper, R denotes a commutative ring with unity 1.

3.1 Definition Let M be a module over a ring R. A neutrosophic set A on M is called a neutrosophic submodule of M if the following conditions are satisfied:

(M1)$A(0) = \widetilde{X}$, i.e.,

$t_A(0) = 1,\ i_A(0) = 1,\ f_A(0) = 0$.

(M2)$A(x + y) \geq A(x) \wedge A(y)$, for each $x, y \in M$ i.e.,

$t_A(x + y) \geq t_A(x) \wedge t_A(y), i_A(x + y) \geq i_A(x) \wedge i_A(y)$ and $f_A(x + y) \leq f_A(x) \vee f_A(y)$.

(M3)$A(rx) \geq A(x)$, for each $x \in M,\ r \in R$, i.e.,

$t_A(rx) \geq t_A(x), i_A(rx) \geq i_A(x)$ and $f_A(rx) \leq f_A(x)$.

The collection of all neutrosophic submodules of M is denoted by $NSM(M)$.

3.2 Example Let us take the classical ring $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Since each ring is a module on itself, we consider $M = \mathbb{Z}_4$ as a classical module. Define the single valued neutrosophic set A as follows:

$A = \{< 1, 1, 0 > /\overline{0} + < 0.6, 0.3, 0.6 > /\overline{1} + < 0.8, 0.1, 0.4 > /\overline{2} + < 0.6, 0.3, 0.6 > /\overline{3}\}$.

It is clear that the neutrosophic set A is a neutrosophic submodule of the module M.

3.3 Definition Let A, B be neutrosophic sets on M. Then their sum $A + B$ is a neutrosophic set on M, defined as follows:

$t_{A+B}(x) = \vee\{t_A(y) \wedge t_B(z) \mid x = y + z,\ y, z \in M\}$,
$i_{A+B}(x) = \vee\{i_A(y) \wedge i_B(z) \mid x = y + z,\ y, z \in M\}$,
$f_{A+B}(x) = \wedge\{f_A(y) \vee f_B(z) \mid x = y + z,\ y, z \in M\}$.

3.4 Definition Let A be a neutrosophic set on M, then $-A$ is a neutrosophic set on M, defined as follows:

$t_{-A}(x) = t_A(-x), i_{-A}(x) = i_A(-x)$ and $f_{-A}(x) = f_A(-x)$, for each $x \in M$.

3.5 Definition Let A be a neutrosophic set on an R-module M and $r \in R$. Define neutrosophic set rA on M as follows:

$t_{rA}(x) = \vee\{t_A(y) \mid y \in M,\ x = ry\}$, $i_{rA}(x) = \vee\{i_A(y) \mid y \in M,\ x = ry\}$ and $f_{rA}(x) = \wedge\{f_A(y) \mid y \in M,\ x = ry\}$,

3.1. Proposition. *If A is a neutrosophic submodule of an R-module M, then $(-1)A = -A$.*

Proof. Let $x \in M$ be arbitrary.

$$t_{(-1)A}(x) = \bigvee_{x=(-1)y} t_A(y) = \bigvee_{y=-x} t_A(x) = t_A(-x) = t_{-A}(x).$$

Since similarly $i_{(-1)A}(x) = i_{-A}(x)$ and $f_{(-1)A}(x) = f_{-A}(x)$, for each $x \in M$, the following is valid,

$(-1)A = (t_{(-1)A}, i_{(-1)A}, f_{(-1)A}) = (t_{-A}, i_{-A}, f_{-A}) = -A.$ □

3.2. Proposition. *If A and B are neutrosophic sets on M, with $A \subseteq B$, then $rA \subseteq rB$, for each $r \in R$.*

Proof. It is straightforward by the definition. □

3.3. Proposition. *If A is a neutrosophic set on M, then $r(sA) = (rs)A$, for each $r, s \in R$.*

Proof. Let $x \in M$ and $r, s \in R$ be arbitrary.

$$f_{r(sA)}(x) = \bigwedge_{x=ry} f_{sA}(y) = \bigwedge_{x=ry} \bigwedge_{y=sz} f_A(z) = \bigwedge_{x=r(sz)} f_A(z) = \bigwedge_{x=(rs)z} f_A(z) = f_{(rs)A}(x).$$

By the similar calculations the other equalities are obtained, so

$r(sA) = (t_{r(sA)}, i_{r(sA)}, f_{r(sA)}) = (t_{(rs)A}, i_{(rs)A}, f_{(rs)A}) = (rs)A.$ □

3.4. Proposition. *If A and B are neutrosophic sets on M, then $r(A + B) = rA + rB$, for each $r \in R$.*

Proof. Let A and B are neutrosophic sets on M, $x \in M$ and $r \in R$.

$$
\begin{aligned}
i_{r(A+B)}(x) &= \bigvee_{x=ry} i_{A+B}(y) \\
&= \bigvee_{x=ry} \bigvee_{y=z_1+z_2} (i_A(z_1) \wedge i_B(z_2)) \\
&= \bigvee_{x=rz_1+rz_2} (i_A(z_1) \wedge i_B(z_2)) \\
&= \bigvee_{x=x_1+x_2} ((\bigvee_{x_1=rz_1} i_A(z_1)) \wedge (\bigvee_{x_2=rz_2} i_B(z_2))) \\
&= \bigvee_{x=x_1+x_2} (i_{rA}(x_1) \wedge i_{rB}(x_2)) = i_{rA+rB}(x).
\end{aligned}
$$

The other equalities are obtained similarly.

Hence, $r(A+B) = (t_{r(A+B)}, i_{r(A+B)}, f_{r(A+B)}) = (t_{rA+rB}, i_{rA+rB}, f_{rA+rB}) = rA + rB$. □

3.5. Proposition. *If A is a neutrosophic set on M, then $t_{rA}(rx) \geq t_A(x), i_{rA}(rx) \geq i_A(x)$ and $f_{rA}(rx) \leq f_A(x)$.*

Proof. It is straightforward by the definition. □

3.6. Proposition. *If A and B are neutrosophic sets on M, then*
(1) $t_B(rx) \geq t_A(x)$, for each $x \in M$, if and only if $t_{rA} \leq t_B$.
(2) $i_B(rx) \geq i_A(x)$, for each $x \in M$, if and only if $i_{rA} \leq i_B$.
(3) $f_B(rx) \leq f_A(x)$, for each $x \in M$, if and only if $f_{rA} \geq f_B$.

Proof. (1) Suppose $t_B(rx) \geq t_A(x)$, for each $x \in M$, then $t_{rA}(x) = \bigvee_{x=ry, y \in M} t_A(y)$. So, $t_{rA} \leq t_B$.

Conversely, suppose $t_{rA} \leq t_B$ is satisfied. Then $t_{rA}(x) \leq t_B(x)$, for each $x \in M$. Hence, $t_B(rx) \geq t_{rA}(rx) \geq t_A(x)$, for each $x \in M$ (by Proposition 3.5).

(2) and (3) are proved in a similar way. □

3.7. Proposition. *If A and B are neutrosophic sets on M, then*
(1) $t_{rA4sB}(rx+sy) \geq t_A(x) \wedge t_B(y)$,
(2) $i_{rA4sB}(rx+sy) \geq i_A(x) \wedge i_B(y)$,
(3) $f_{rA4sB}(rx+sy) \leq f_A(x) \vee f_B(y)$, for each $x, y \in M, r, s \in R$.

Proof. It is proved by using Definition 3.3, Definition 3.5 and Proposition 3.5. □

3.8. Proposition. *If A, B, C are neutrosophic sets on M, then the followings are satisfied for each $r, s \in R$;*
(1) $t_C(rx+sy) \geq t_A(x) \wedge t_B(y)$, for all $x, y \in M$ if and only if $t_{rA+sB} \leq t_C$.
(2) $i_C(rx+sy) \geq i_A(x) \wedge i_B(y)$, for all $x, y \in M$ if and only if $i_{rA+sB} \leq i_C$.
(3) $f_C(rx+sy) \leq f_A(x) \vee f_B(y)$, for all $x, y \in M$ if and only if $f_{rA+sB} \geq f_C$.

Proof. It is proved by using Proposition 3.7. □

3.9. Theorem. *Let A be a neutrosophic set on M and $r, s \in R$. Then*
(1) $t_{rA} \leq t_A \Leftrightarrow t_A(rx) \geq t_A(x), i_{rA} \leq i_A \Leftrightarrow i_A(rx) \geq i_A(x)$ and $f_{rA} \geq f_A \Leftrightarrow f_A(rx) \leq f_A(x)$, for each $x \in M$.
(2) $t_{rA+sA} \leq t_A \Leftrightarrow t_A(rx+sy) \geq t_A(x) \wedge t_A(y)$,
$i_{rA+sA} \leq i_A \Leftrightarrow i_A(rx+sy) \geq i_A(x) \wedge i_A(y)$,
and $f_{rA+sA} \geq f_A \Leftrightarrow f_A(rx+sy) \leq f_A(x) \vee f_A(y)$.

Proof. The proof follows from Propositions 3.6 and 3.8. □

3.10. Theorem. *Let A be a neutrosophic set on M. Then A is a neutrosophic submodule of M if and only if A is a neutrosophic subgroup of the additive group M, in the sense of* [3], *and satisfies the conditions $t_{rA} \leq t_A$, $i_{rA} \leq i_A$ and $f_{rA} \geq f_A$, for each $r \in R$.*

Proof. Proof is clear from the definition of a neutrosophic subgroup in [3], and Theorem 3.9. □

3.11. Theorem. *Let A be a neutrosophic set on M. Then $A \in NSM(M)$ if and only if the following properties are satisfied:*

(i) $A(0) = \widetilde{X}$.

(ii) $A(rx + sy) \geq A(x) \wedge A(y)$, *for each* $x, y \in M$, $r, s \in R$.

Proof. Let A be a neutrosophic submodule of M and $x, y \in M$. From the condition (M1) of Definition 3.1, it is obvious that $A(0) = \widetilde{X}$. From (M2) and (M3), the followings are true,

$t_A(rx + sy) \geq t_A(rx) \wedge t_A(sy) \geq t_A(x) \wedge t_A(y)$,
$i_A(rx + sy) \geq i_A(rx) \wedge i_A(sy) \geq i_A(x) \wedge i_A(y)$ and
$f_A(rx + sy) \leq f_A(rx) \vee f_A(sy) \leq f_A(x) \vee f_A(y)$. for each $x, y \in M$, $r, s \in R$.

Hence,

$$\begin{aligned} A(rx + sy) &= (t_A(rx + sy), i_A(rx + sy), f_A(rx + sy)) \\ &\geq (t_A(x) \wedge t_A(y), i_A(x) \wedge i_A(y), f_A(x) \vee f_A(y)) \\ &= (t_A(x), i_A(x), f_A(x)) \wedge (t_A(y), i_A(y), f_A(y)) \\ &= A(x) \wedge A(y). \end{aligned}$$

Conversely, suppose A satisfies the conditions (i) and (ii). Then it is clear by hypothesis $A(0) = \widetilde{X}$.

$t_A(x + y) = t_A(1.x + 1.y) \geq t_A(x) \wedge t_A(y)$
$i_A(x + y) = i_A(1.x + 1.y) \geq i_A(x) \wedge i_A(y)$
$f_A(x + y) = f_A(1.x + 1.y) \leq f_A(x) \vee f_A(y)$.

So, $A(x + y) \geq A(x) \wedge A(y)$ and the condition (M2) of Definition 3.1 is satisfied.

Now let us show the validity of condition (M3).By the hypothesis,

$t_A(rx) = t_A(rx + r0) \geq t_A(x) \wedge t_A(0) = t_A(x)$
$i_A(rx) = i_A(rx + r0) \geq i_A(x) \wedge i_A(0) = i_A(x)$
$f_A(rx) = f_A(rx + r0) \leq f_A(x) \vee f_A(0) = f_A(x)$, for each $x, y \in M$, $r \in R$.

Therefore, (M3) of Definition 3.1 is satisfied. □

3.12. Theorem. *If A and B are neutrosophic submodules of a classical module M, then the intersection $A \cap B$ is also a neutrosophic submodule of M.*

Proof. Since $A, B \in NSM(M)$, we have $A(0) = \widetilde{X}$, $B(0) = \widetilde{X}$.

$t_{A\cap B}(0) = t_A(0) \wedge t_B(0) = 1$
$i_{A\cap B}(0) = i_A(0) \wedge i_B(0) = 1$
$f_{A\cap B}(0) = f_A(0) \vee f_B(0) = 0$.

Hence $(A \cap B)(0) = \widetilde{X}$ and we obtain the condition (M1) of Definition 3.1 is satisfied.

Let $x, y \in M$, $r, s \in R$. By Theorem 3.11, it is enough to show that

$(A \cap B)(rx + sy) \geq (A \cap B)(x) \wedge (A \cap B)(y)$, i.e.,

$t_{A\cap B}(rx+sy) \geq t_{A\cap B}(x) \wedge t_{A\cap B}(y), i_{A\cap B}(rx+sy) \geq i_{A\cap B}(x) \wedge i_{A\cap B}(y)$ and $f_{A\cap B}(rx+sy) \leq f_{A\cap B}(x) \vee f_{A\cap B}(y)$.

Now we consider the truth-membership degree of the intersection,

$$\begin{aligned} t_{A\cap B}(rx + sy) &= t_A(rx + sy) \wedge t_B(rx + sy) \\ &\geq (t_A(x) \wedge t_A(y)) \wedge (t_B(x) \wedge t_B(y)) \\ &= (t_A(x) \wedge t_B(x)) \wedge (t_A(y) \wedge t_B(y)) = t_{A\cap B}(x) \wedge t_{A\cap B}(y). \end{aligned}$$

The other inequalities are proved similarly. Hence, $A \cap B \in NSM(M)$. □

A nonempty subset N of M is a submodule of M if and only if $rx + sy \in N$ for all $x, y \in M,\ r, s \in R$.

3.13. Proposition. *Let M be a module over R. $A \in NSM(M)$ if and only if for all $\alpha \in [0,1]$, α-level sets of A, $(t_A)_\alpha, (i_A)_\alpha$ and $(f_A)^\alpha$ are classical submodules of M where $A(0) = \widetilde{X}$.*

Proof. Let $A \in NSM(M),\ \alpha \in [0,1],\ x, y \in (t_A)_\alpha$ and $r, s \in R$ be any elements. Then $t_A(x) \geq \alpha,\ t_A(y) \geq \alpha$ and $t_A(x) \wedge t_A(y) \geq \alpha$. By using Theorem 3.11, we have $t_A(rx + sy) \geq t_A(x) \wedge t_A(y) \geq \alpha$. Hence $rx + sy \in (t_A)_\alpha$. Therefore $(t_A)_\alpha$ is a classical submodule of M for each $\alpha \in [0,1]$.

Similarly, for $x, y \in (i_A)_\alpha,\ (f_A)^\alpha$ we obtain $rx+sy \in (i_A)_\alpha,\ (f_A)^\alpha$ for each $\alpha \in [0,1]$. Consequently, $(i_A)_\alpha,\ (f_A)^\alpha$ are classical submodules of M for each $\alpha \in [0,1]$.

Conversely, let $(t_A)_\alpha$ be a classical submodules of M for each $\alpha \in [0,1]$. Let $x, y \in M,\ \alpha = t_A(x) \wedge t_A(y)$. Then $t_A(x) \geq \alpha$ and $t_A(y) \geq \alpha$. Thus, $x, y \in (t_A)_\alpha$. Since $(t_A)_\alpha$ is a classical submodule of M, we have $rx + sy \in (t_A)_\alpha$ for all $r, s \in R$. Hence, $(t_A)(rx + sy) \geq \alpha = t_A(x) \wedge t_A(y)$.

Similarly we obtain $(i_A)(rx + sy) \geq i_A(x) \wedge i_A(y)$.

Now we consider $(f_A)^\alpha$. Let $x, y \in M$, $\alpha = f_A(x) \vee f_A(y)$. Then $f_A(x) \leq \alpha,\ f_A(y) \leq \alpha$. Thus $x, y \in (f_A)^\alpha$. Since $(f_A)^\alpha$ is a submodule of M, we have $rx + sy \in (f_A)^\alpha$ for all $r, s \in R$. Thus $(f_A)(rx + sy) \leq \alpha = f_A(x) \vee f_A(y)$.

It is also obvious that $A(0) = \widetilde{X}$. Hence the conditions of Theorem 3.11 are satisfied. □

3.14. Proposition. *Let A and B be two neutrosophic sets on X and Y, respectively. Then the following equalities are satisfied for the α-levels.*
$(t_{A\times B})_\alpha = (t_A)_\alpha \times (t_B)_\alpha,\ (i_{A\times B})_\alpha = (i_A)_\alpha \times (i_B)_\alpha$ *and* $(f_{A\times B})^\alpha = (f_A)^\alpha \times (f_B)^\alpha$.

Proof. Let $(x, y) \in (t_{A\times B})_\alpha$ be arbitrary. So,

$$\begin{aligned} t_{A\times B}(x,y) \geq \alpha &\Leftrightarrow t_A(x) \wedge t_B(y) \geq \alpha \\ &\Leftrightarrow t_A(x) \geq \alpha \ \text{ and } \ t_B(y) \geq \alpha \\ &\Leftrightarrow (x,y) \in (t_A)_\alpha \times (t_B)_\alpha. \end{aligned}$$

$(i_{A\times B})_\alpha = (i_A)_\alpha \times (i_B)_\alpha$ is proved in a similar way.

Let $(x, y) \in (f_{A\times B})^\alpha$ be arbitrary. Hence,

$$\begin{aligned} f_{A\times B}(x,y) \leq \alpha &\Leftrightarrow f_A(x) \vee f_B(y) \leq \alpha \\ &\Leftrightarrow (f_A(x) \leq \alpha, \quad f_B(y) \leq \alpha \\ &\Leftrightarrow (x,y) \in (f_A)^\alpha \times (f_B)^\alpha. \end{aligned}$$
□

3.15. Theorem. *Let $A, B \in NSM(M)$. Then the product $A \times B$ is also a neutrosophic submodule of M.*

Proof. We know that direct product of two submodules is a submodule. So, by Proposition 3.13 and Proposition 3.14, we obtain the result. □

3.16. Proposition. *Let A and B be two neutrosophic sets on X and Y, respectively and $g : X \to Y$ be a mapping. Then the followings hold:*
(i) $g((t_A)_\alpha) \subseteq (t_{g(A)})_\alpha,\ g((i_A)_\alpha) \subseteq (i_{g(A)})_\alpha,\ g((f_A)^\alpha) \supseteq (f_{g(A)})^\alpha$.
(ii) $g^{-1}((t_B)_\alpha) = (t_{g^{-1}(B)})_\alpha,\ g^{-1}((i_B)_\alpha) = (i_{g^{-1}(B)})_\alpha,\ g^{-1}((f_B)^\alpha) = (f_{g^{-1}(B)})^\alpha$.

Proof. (i) Let $y \in g((t_A)_\alpha)$. Then there exists $x \in (t_A)_\alpha$ such that $g(x) = y$. Hence $t_A(x) \geq \alpha$. So, $\bigvee\limits_{x\in g^{-1}(y)} t_A(x) \geq \alpha$, i.e., $t_{g(A)}(y) \geq \alpha$ and $y \in (t_{g(A)})_\alpha$. Hence $g((t_A)_\alpha) \subseteq (t_{g(A)})_\alpha$. Similarly, we obtain other inclusions.

(ii)

$$
\begin{aligned}
(t_{g^{-1}(B)})_\alpha &= \{x \in X : t_{g^{-1}(B)}(x) \geq \alpha\} \\
&= \{x \in X : t_B(g(x)) \geq \alpha\} \\
&= \{x \in X : g(x) \in (t_B)_\alpha\} \\
&= \{x \in X : x \in g^{-1}((t_B)_\alpha)\} = g^{-1}((t_B)_\alpha)
\end{aligned}
$$

The other equalities are obtained in a similar way. □

3.17. Theorem. *Let M, N be the classical modules and $g : M \to N$ be a homomorphism of modules. If B is a neutrosophic submodule of N, then the preimage $g^{-1}(B)$ is a neutrosophic submodule of M.*

Proof. By Proposition 3.16 (ii), we have
$g^{-1}((t_B)_\alpha) = (t_{g^{-1}(B)})_\alpha,\ g^{-1}((i_B)_\alpha) = (i_{g^{-1}(B)})_\alpha,\ g^{-1}((f_B)^\alpha) = (f_{g^{-1}(B)})^\alpha$.. Since preimage of a submodule is a submodule, by Proposition 3.13 we obtain the result. □

3.18. Corollary. *If $g : M \to N$ is a homomorphism of modules and $\{B_j : j \in I\}$ is a family of neutrosophic submodules of N, then $g^{-1}(\cap B_j)$ is a neutrosophic submodule of M.*

3.19. Theorem. *Let M, N be the classical modules and $g : M \to N$ be a homomorphism of modules. If A is a neutrosophic submodule of M, then the image $g(A)$ is a neutrosophic submodule of N.*

Proof. By Proposition 3.13, it is enough to show that $(t_{g(A)})_\alpha,\ (i_{g(A)})_\alpha,\ (f_{g(A)})^\alpha$ are submodules of N for all $\alpha \in [0, 1]$.

Let $y_1, y_2 \in (t_{g(A)})_\alpha$. Then $t_{g(A)}(y_1) \geq \alpha$ and $t_{g(A)}(y_2) \geq \alpha$. There exist $x_1, x_2 \in M$ such that $t_A(x_1) \geq t_{g(A)}(y_1) \geq \alpha$ and $t_A(x_2) \geq t_{g(A)}(y_2) \geq \alpha$. Then $t_A(x_1) \geq \alpha$, $t_A(x_2) \geq \alpha$ and $t_A(x_1) \wedge t_A(x_2) \geq \alpha$. Since A is a neutrosophic submodule of M, for any $r, s \in R$, we have $t_A(rx_1 + sx_2) \geq t_A(x_1) \wedge t_A(x_2) \geq \alpha$. Hence,

$rx_1 + sx_2 \in (t_A)_\alpha \Rightarrow g(rx_1 + sx_2) \in g((t_A)_\alpha) \subseteq (t_{g(A)})_\alpha$

$\Rightarrow rg(x_1) + sg(x_2) \in (t_{g(A)})_\alpha \Rightarrow ry_1 + sy_2 \in (t_{g(A)})_\alpha$.

Therefore, $(t_{g(A)})_\alpha$ is a submodule of N. Similarly, $(i_{g(A)})_\alpha,\ (f_{g(A)})^\alpha$ are classical submodules of N for each $\alpha \in [0, 1]$. By Proposition 3.13, $g(A)$ is a neutrosophic submodule of N. □

3.20. Corollary. *If $g : M \to N$ is a surjective module homomorphism and $\{A_i : i \in I\}$ is a family of neutrosophic submodule of M, then $g(\cap A_i)$ is a neutrosophic submodule of N.*

4. Conclusion

Modules over a ring are a generalization of abelian groups (which are modules over $\mathbb{Z}$) [5]. From the philosophical point of view, it has been shown that a neutrosophic set generalizes a classical set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set etc. A single valued neutrosophic set is an instance of neutrosophic set which can be used in real scientific and engineering problems. Therefore, the study of single valued neutrosophic sets and their properties have a considerable significance in the sense of applications as well as in understanding the fundamentals of uncertainty. So, as a continuation of the studies [3, 4], we decided to introduce the concept of a neutrosophic submodule and examine its elementary properties. Consequently, this study is concerned with carrying over to neutrosophic modules various concepts and results of neutrosophic subgroup theory concerned in [3].

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