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PS-modules over generalized Malcev-Neumann series

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Abstract

In [1], the first author introduced a new class of extension rings called the *generalized Malcev-Neumann series ring* $R((S;\sigma;\tau))$ with coefficients in a ring R and exponents in a strictly ordered monoid S which extends the usual construction of Malcev-Neumann series rings. The conditions under which the generalized Malcev-Neumann series module $M\left((S)\right)_{R((S;\sigma;\tau))}$ is a PS-module are investigated in the present paper.

Keywords: PS-module, generalized Malcev-Neumann series rings.

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1. Introduction

Throughout this paper R denotes an associative ring with identity and M_R a unitary right R-module. According to Nicholson and Watters [2], M_R is called a *PS-module* if every simple submodule is projective, equivalently if its socle, $\mathrm{Soc}\left(M_R\right)$, is projective. Examples of PS-modules include nonsingular modules, regular modules in the sense of Zelmanowitz [3] and modules with zero socle. The class of PS-modules is closed under direct sums and submodules. In [4], Weimin proved that PS-modules are preserved by Morita equivalences and excellent extensions.

For any subset X of R, denote

$$\mathrm{l}_M(X) = \{m \in M \mid mX = 0\}.$$

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1.1. Theorem ([4]). *The following statements are equivalent for a right R-module M_R:*
(1) M_R is a PS-module.
(2) If L is a maximal right ideal of R then either $\mathrm{l}_M(L) = 0$ or $L = eR$, where $e^2 = e \in R$.

A left PS-module ${}_RM$ is defined analogously. A ring R is said to be a *left PS-ring* if ${}_RR$ is a PS-module. Every semiprime ring is a PS-ring. Every PP-ring is a PS-ring (where a ring R is called PP-ring if every principal left ideal is projective). In particular every Baer ring is a PS-ring (where a ring R is called Baer if every left (or right) annihilator is generated by an idempotent). A ring for which every simple singular module is injective is a PS-ring. If $\mathrm{l}_R(J(R)) = 0$, then R is a PS-ring. In fact $J(R) \subset L$ for every maximal right ideal so $\mathrm{l}_R(L) = 0$.

The notion of PS-rings is not left-right symmetric (cf. [2]). A ring R is *duo* if each one-sided ideal of R is a two-sided ideal. As a generalization of left duo rings, a ring R is called *weakly left duo* if for every $r \in R$ there is a natural number $n(r)$ such that $Rr^{n(r)}$ is a two-sided ideal of R. A ring R is weakly duo if it is weakly right and left duo. In [4], Weimin proved that a duo ring R is a PS-ring if and only if it is a right PS-ring. In [5], Dingguo generalized this result to weakly duo rings, as follows: A weakly duo reduced ring R is a PS-ring if and only if R is a right PS-ring.

If R is a PS-ring so also are $R[x]$ and $R[[x]]$. The converse of this result is false in general by the following example:

1.2. Example ([2], Example 3.2). If $R = \mathbb{Z}_4$, then $R[x]$ and $R[[x]]$ are PS-rings but R is not PS-ring.

The main aim of this paper is to investigate conditions for the generalized Malcev-Neumann series module $M((S))_{R((S;\sigma;\tau))}$ to be a PS-module.

2. PS-modules of generalized Malcev-Neumann series rings

Let $(S, ., \leqslant)$ be a strictly ordered monoid (that is, $(S, \leqslant)$ is an ordered monoid satisfying the condition that, if $s < s'$, then $st < s't$ and $ts < ts'$ for $s, s', t \in S$). Recall that a subset X of $(S, \leqslant)$ is said to be *artinian* if every strictly decreasing sequence of elements of X is finite and that X is *narrow* if every subset of pairwise order-incomparable elements of X is finite. Suppose the two maps $\sigma : S \longrightarrow \mathrm{End}(R)$ and $\tau : S \times S \longrightarrow \mathrm{U}(R)$ (the group of invertible elements of R). Let $A = R((S;\sigma;\tau))$ denote the set of all formal sums $f = \sum_{x\in S} a_x\overline{x}$ such that $\mathrm{supp}(f) = \{x \in S \mid a_x \neq 0\}$ is an artinian and narrow subset of S, with componentwise addition and the multiplication rule is given by

$$\left(\sum_{x\in S} a_x\overline{x}\right)\left(\sum_{y\in S} b_y\overline{y}\right) = \sum_{z\in S}\left(\sum_{\{(x,y)\mid xy=z\}} a_x\sigma_x(b_y)\,\tau(x,y)\right)\overline{z},$$

for each $\sum_{x\in S} a_x\overline{x}$ and $\sum_{y\in S} b_y\overline{y} \in A$. In order to ensure the associativity, it is necessary to impose two additional conditions on σ and τ namely that for all $x, y, z \in S$,
(i) $\sigma_x(\tau(y,z))\tau(x,yz) = \tau(x,y)\tau(xy,z)$,
(ii) $\sigma_x\sigma_y = \eta(x,y)\sigma_{xy}$, where $\eta(x,y)$ denotes the automorphism of R defined by

$$\eta(x,y)(r) = \tau(x,y)r\tau(x,y)^{-1} \text{ for all } r \in R.$$

It is now routine to check that $A = R((S;\sigma;\tau))$ is a ring which is called the *ring of generalized Malcev-Neumann series*. We can assume that the identity element of A is $\overline{1}$, this means that

$$\sigma_1 = \mathrm{Id}_R \text{ and } \tau(x,1) = \tau(1,x) = 1 \text{ for any } x \in S.$$

In this case $r \mapsto r\overline{1}$ is an embedding of R as a subring into A.

For each $f \in A\backslash\{0\}$ we denote by $\pi(f)$ the set of minimal elements of $\mathrm{supp}(f)$. If $(S, \leq)$ is a strictly totally ordered monoid, then $\mathrm{supp}(f)$ is a nonempty well-ordered subset of S and $\pi(f)$ consists of only one element.

Clearly, the above construction generalizes the construction of Malcev-Neumann series rings, in case of $S = G$ (an ordered group), which introduced independently by Malcev and Neumann (see [6] and [7]).

If the order $\leq$ is the trivial order, then $A = R((S;\sigma;\tau))$ is the usual crossed product ring $R[S;\sigma;\tau]$. Also, if the monoid S has the trivial order and τ is trivial, then $A = R((S;\sigma;\tau))$ is the usual skew monoid ring $R[S;\sigma]$. Whoever if the monoid S has the trivial order and σ is trivial, then $A = R((S;\sigma;\tau))$ is the usual twisted monoid ring $R[S;\tau]$. Finally, if the monoid S has the trivial order and σ and τ are trivial, then $A = R((S;\sigma;\tau))$ is the usual monoid ring $R[S]$, (see, Sections 3.2 and 3.3 in [8]).

Moreover, if α is a ring endomorphism of R and set $S = \mathbb{Z}_{\geq 0}$ endowed with the trivial order. Define $\sigma : S \longrightarrow \mathrm{End}(R)$ via $\sigma(x) = \alpha^x$ for every $x \in \mathbb{Z}_{\geq 0}$ and $\tau(x, y) = 1$ for any $x, y \in \mathbb{Z}$. We have $A = R((S;\sigma;\tau))$ is the usual skew polynomial ring $R[x, \alpha]$. Whoever if $\leq$ is the usual order, then $A = R((S;\sigma;\tau))$ is the usual skew power series ring $R[[x, \sigma]]$. If α is a ring automorphism of R, $S = \mathbb{Z}$ and $\leq$ is the usual order, then $A = R((S;\sigma;\tau))$ is the usual ring of skew Laurent power series $R\left[\left[x, x^{-1}, \alpha\right]\right]$.

In the same time, if we set also $\sigma(s) = \sigma_s = \mathrm{Id}_R \in \mathrm{End}(R)$ for all $s \in S$, then it is easy to check that polynomial rings, Laurent polynomial rings, formal power series rings and Laurent power series rings are special cases of $A = R((S;\sigma;\tau))$.

If M_R is a unitary right R-module, then the *Malcev-Neumann series module* $B = M((S))$ is the set of all formal sums $\sum\limits_{x\in S} m_x\overline{x}$ with coefficients in M and artinian and narrow supports, with pointwise addition and scalar multiplication rule is defined by

$$\left(\sum_{x\in S} m_x\overline{x}\right)\left(\sum_{y\in S} a_y\overline{y}\right) = \sum_{z\in S}\left(\sum_{\{(x,y)|xy=z\}} m_x\sigma_x(a_y)\,\tau(x,y)\right)\overline{z},$$

where $\sum\limits_{x\in S} m_x\overline{x} \in B$ and $\sum\limits_{y\in S} a_y\overline{y} \in A$. One can easily check that (i) and (ii) ensure that $M((S))$ is a unitary right A-module. For each $\varphi \in B\backslash\{0\}$ we denote by $\pi(\varphi)$ the set of minimal elements of $\mathrm{supp}(\varphi)$. If $(S, \leq)$ is a strictly totally ordered monoid, then $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of S and $\pi(\varphi)$ consists of only one element.

Let V be a subset of M_R, then

$$V((S)) = \left\{\varphi = \sum_{x\in S} m_x\overline{x} \in B\,|0 \neq m_x \in V \text{ and } x \in \mathrm{supp}(\varphi)\right\}.$$

2.1. Definition ([9]). A right R-module M_R is called S-compatible if, for each $m \in M$, $a \in R$ and $x \in S$, $ma = 0$ if and only if $m\sigma_x(a) = 0$.

A ring R is called S-compatible if R_R is an S-compatible R-module.

Now, we are able to deliver our theorem.

2.2. Theorem. *Let $(S, ., \leq)$ be a strictly totally ordered monoid which satisfies the condition that $1 \leq s$ for every $s \in S$ and M_R an S-compatible module. If M_R is a PS-module, then $B_A = M((S))_{R((S;\sigma;\tau))}$ is a PS-module.*

Proof. Let L be a maximal right ideal of A. We will show that either $\mathrm{l}_B(L) = 0$ or $L = hA$, where $h^2 = h \in A$. Since $(S, ., \leq)$ is a strictly totally ordered monoid, $\mathrm{supp}(f)$ is a nonempty well-ordered subset of S, for every $0 \neq f = \sum\limits_{x\in S} a_x\overline{x} \in A$. We denote by

$\pi(f)$ the smallest element of support f.
For any $s \in S$, set

$$I_s = \{a_s \in \operatorname{supp}(f) \,|\, f \in L \text{ and } \pi(f) = s\} \subset R \text{ and } I = \bigcup_{s \in S} I_s.$$

Let J be the right ideal of R generated by I. If $J = R$, then there exist $s_1, \ldots, s_n \in S$, $f_1, \ldots, f_n \in L$ and $r_1, \ldots, r_n \in R$ such that

$$1 = a_{s_1} r_1 + \cdots + a_{s_n} r_n,$$

where $a_{s_i} \in I_{s_i}$ and $\pi(f_i) = s_i$, for every $1 \leq i \leq n$. We will show that $\mathrm{l}_B(L) = 0$. Suppose that $\varphi = \sum_{y \in S} m_y \overline{y} \in \mathrm{l}_B(L)$ and $\varphi \neq 0$. Then $\operatorname{supp}(\varphi)$ is a nonempty well-ordered subset of S. Let $t = \pi(\varphi)$. If

$$m_t \sigma_t(a_{s_i}) \tau(t, s_i) \neq 0 \text{ for some } 1 \leq i \leq n,$$

then the coefficient of φf_i at $t s_i = \pi(\varphi f_i)$ is non zero. This means that $\varphi f_i \neq 0$ for some $1 \leq i \leq n$, a contradiction. Thus

$$m_t \sigma_t(a_{s_i}) \tau(t, s_i) = 0 \text{ for all } 1 \leq i \leq n.$$

Since $\tau(t, s_i) \in \mathrm{U}(R)$ and M_R is an S-compatible module, we get

$$m_t a_{s_i} = 0 \text{ for all } 1 \leq i \leq n.$$

Consequently,

$$\begin{aligned} m_t &= m_t 1 = m_t(a_{s_1} r_1 + \cdots + a_{s_n} r_n) \\ &= (m_t a_{s_1}) r_1 + \cdots + (m_t a_{s_n}) r_n = 0, \end{aligned}$$

a contradiction. Thus $\mathrm{l}_B(L) = 0$. Suppose that $J \neq R$. We will show that J is a maximal right ideal of R. Let $r \in R - J$. If $r\overline{1} \in L$, then $r \in I_1 \subset I$ and so $r \in J$, a contradiction. Therefore $r\overline{1} \notin L$. Since L is a maximal right ideal of A,

$$A = L + (r\overline{1})A.$$

It follows that there exist $f = \sum_{x \in S} a_x \overline{x} \in L$ and $g = \sum_{y \in S} b_y \overline{y} \in A$ such that $\overline{1} = f + (r\overline{1})g$. Thus

$$1 = a_1 + r\sigma_1(b_1)\tau(1, 1) = a_1 + r b_1.$$

If $a_1 = 0$, then $1 = r b_1 \in rR$. So $R = J + rR$.
If $a_1 \neq 0$, then $1 \in \operatorname{supp}(f)$. Since $1 \leq s$ for every $s \in S$, $\pi(f) = 1$. Thus $a_1 \in I_1 \subset I \subset J$, which implies that $R = J + rR$.
Hence J is a maximal right ideal of R. Since M_R is a PS-module, it follows that either $\mathrm{l}_M(J) = 0$ or $J = eR$, where $e^2 = e \in R$. According to that we have the following two cases:
Case(1). Suppose that $\mathrm{l}_M(J) = 0$. We will show that $\mathrm{l}_B(L) = 0$. Let $\varphi = \sum_{y \in S} m_y \overline{y} \in \mathrm{l}_B(L)$ and $\varphi \neq 0$. Then $\operatorname{supp}(\varphi)$ is a nonempty well-ordered subset of S. Let $s = \pi(\varphi)$. For any $r \in J$, there exist $s_1, \ldots, s_n \in S$, $f_1, \ldots, f_n \in L$ and $r_1, \ldots, r_n \in R$ such that

$$r = a_{s_1} r_1 + \cdots + a_{s_n} r_n,$$

where $a_{s_i} \in I_{s_i}$ and $\pi(f_i) = s_i$, for every $1 \leq i \leq n$. Since $\varphi \in \mathrm{l}_B(L)$, $f_1, ..., f_n \in L$, we get $\varphi f_i = 0$ for every $1 \leq i \leq n$. If

$$m_s \sigma_s(a_{s_i}) \tau(s, s_i) \neq 0 \text{ for some } 1 \leq i \leq n,$$

then the coefficient of φf_i at $ss_i = \pi(\varphi f_i)$ is non zero. This means that $\varphi f_i \neq 0$ for some $1 \leq i \leq n$, a contradiction. Thus

$$m_s \sigma_s (a_{s_i}) \tau (s, s_i) = 0 \text{ for all } 1 \leq i \leq n.$$

Since $\tau(s, s_i) \in \mathrm{U}(R)$ and M_R is an S-compatible module, we get

$$m_s a_{s_i} = 0 \text{ for all } 1 \leq i \leq n.$$

Consequently,

$$m_s r = m_s (a_{s_1} r_1 + \cdots + a_{s_n} r_n) = (m_s a_{s_1}) r_1 + \cdots + (m_s a_{s_n}) r_n = 0.$$

Therefore $m_s \in \mathrm{l}_M(J) = 0$ and $\pi(\varphi) = s$. Thus $\varphi = 0$, a contradiction. Hence $\mathrm{l}_B(L) = 0$.
Case(2). Suppose that $J = eR$, where $e^2 = e \in R$. We will show that $L = (e\overline{1})A$, where

$$(e\overline{1})^2 = (e\overline{1})(e\overline{1}) = e\sigma_1(e)\tau(1,1)\,\overline{1}.\overline{1} = (e\overline{1}) \in A.$$

To show that $(e\overline{1})A \subseteq L$, we need to prove that $(e\overline{1}) \in L$. If $(e\overline{1}) \notin L$, then $A = L + (e\overline{1})A$. Thus there exist $f \in L$ and $g \in A$ such that $\overline{1} = f + (e\overline{1})g$. Thus

$$1 = a_1 + e\sigma_1(b_1)\tau(1,1) = a_1 + eb_1.$$

If $a_1 = 0$, then $1 = eb_1 \in eR = J$, a contradiction.
If $a_1 \neq 0$, then $1 \in \operatorname{supp}(f)$. Since $1 \leq s$ for every $s \in S$, $\pi(f) = 1$. Thus $a_1 \in I_1 \subset I \subset J$, which implies that $a_1 \in J$ and $J = eR$. Hence $1 = a_1 + eb_1 \in J + eR = J$, a contradiction. Therefore $(e\overline{1}) \in L$ which implies that $(e\overline{1})A \subseteq L$.
Conversely, suppose that $f \in L$ and $\pi(f) = s$, then $a_s \in I_s \subset I \subset J = eR$ and so $a_s = ea_s$. We claim that $a_u = ea_u$ for any $u \in \operatorname{supp}(f)$.
Suppose that $a_v = ea_v$ for each $v < u$. Consider the following element $f_u \in A$ defined by:

$$f_u = \sum_{u<v} a_u \overline{u} + \sum_{u \geq v} 0\overline{u} = \sum_{u<v} a_u \overline{u}.$$

Thus $\pi(f - f_u) = u$. By hypothesis it is easy to see that $f_u = \sum_{u<v} ea_u \overline{u} = (e\overline{1})f_u \in (e\overline{1})A \subset L$. Thus $f - f_u \in L$. By analogy with the proof above, it follows that $a_u = ea_u$, which implies that $f = (e\overline{1})f \in (e\overline{1})A$. Thus $L = (e\overline{1})A$ and the result follows since $(e\overline{1})$ is an idempotent of A. ∎

In particular, if we set $M_R = R_R$ we get the following:

2.3. Corollary. *Let $(S, ., \leq)$ be a strictly totally ordered monoid which satisfies the condition that $1 \leq s$ for every $s \in S$ and R an S-compatible ring. If R is a right PS-ring, then $A = R((S;\sigma;\tau))$ is a right PS-ring.*

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