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Subdivisions of the Spectra for Difference Operator Δ^m over the Sequence Space ℓ_p

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ABSTRACT

This study focuses on the higher-order difference operator Δ^m, which is defined via an (m+1)-band matrix and serves as a generalization of classical operators such as Δ, Δ^2, $B(r,s)$ and $B(r,s,t)$. Within the framework of the sequence space ℓ_p for $1 < p < \infty$ we investigate the operator's boundedness, determine its spectrum and fine spectrum, and examine certain spectral decompositions. This work contributes to the broader understanding of difference operators on Banach spaces and their spectral behavior, providing a general framework that encompasses a range of previously studied operators as special cases.

Keywords: Spectrum, Fine spectrum, Generalized difference operator, Band matrix

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Introduction

Band matrices are structurally sparse matrices that frequently arise in mathematical modeling and numerical analysis, often appearing as discrete representations of difference operators. Investigating the spectral properties of such matrices yields deep and far-reaching implications in both theoretical and applied disciplines. In this context, the significance of studying the spectrum of m-band forward difference matrices can be categorized as follows:

1. Why should the spectrum of a band matrix be studied?

The spectrum of a band matrix encodes essential characteristics that determine the behavior of the associated linear system. In systems derived from finite difference methods, the spectral structure directly influences critical aspects such as solution stability, convergence behavior, and numerical accuracy. Moreover, spectral properties significantly affect the efficiency and speed of numerical algorithms. The banded structure also enables optimized memory usage in solution procedures, making such matrices highly advantageous in computational applications.

2. What are the applications of the spectrum of the forward difference matrix?

Forward difference matrices typically represent discrete approximations of first-order differential operators and thus play a foundational role in the modeling of time- or space-dependent systems. Their spectral characteristics provide crucial insights into the long-term behavior and stability of these systems. Applications of forward difference matrices span a wide range of areas including heat distribution (heat equations), wave propagation, biological systems, financial time series, and signal processing. In particular, the ability of the spectrum to reveal resonance behavior in multi-mass systems enhances the importance of forward difference matrices in dynamic system analysis.

3. What does the computation of the spectrum of the m-band forward difference operator contribute to science?

The m-band forward difference operator generalizes classical first-order difference operators by allowing interactions over a wider range of connections. Computing its spectrum contributes not only to the theoretical analysis of such discrete systems but also underpins the development of stability and accuracy analyses in numerical methods. Spectral information enables the design of effective solutions for advanced problems such as multi-point boundary value problems, discretizations of higher-order differential equations, and discrete approximations of quantum systems. Consequently, this line of research supports both the derivation of novel results in pure mathematics and the advancement of more efficient modeling techniques in applied sciences.

Preliminaries and Literature Review

Numerous significant contributions can be found in the literature regarding the computation spectral decomposition of difference and generalized difference operators defined over various sequence spaces. For instance, Altay and Başar examined the fine spectrum of the classical difference operator on c_0, c and ℓ_p for $1 < p < \infty$ in their works [1]-[2]. The spectral properties of $B(r,s)$ on c_0 and c were explored in [3]. Akhmedov and

Başar investigated the fine spectra of the classical difference operator on ℓ_p and bv_p for $1 < p < \infty$ in [4]-[5], while Durna et al. analyzed these properties on the space cs in [6]. Srivastava and Kumar studied the fine spectra of the generalized difference operator Δ_v on c_0 and ℓ_1 in [7]-[8], where the sequences $a = (a_k)$ and $b = (b_k)$ are convergent under certain conditions, $v = (v_k)$ is fixed or strictly decreasing, and $r \epsilon \mathbb{N}$.

In more recent developments, Akhmedov and El-Shabrawy [9], as well as Dutta and Baliarsingh [10]-[11], investigated the fine spectra of the operators $\Delta_{a,b}$, Δ^2 and Δ_v^r on the spaces c, c_0 and ℓ_1, respectively, under similar assumptions on the sequences involved. The fine spectral characteristics of lower triangular matrices were studied by Durna in [12]-[13], and upper triangular matrices on ℓ_p were analyzed in [14]. Durna et al. investigated spectral decomposition of the band matrix $\Delta_3^{a,b}$ on the sequence space h in [15]. Furthermore, the spectra of $U(a, 0, \mathrm{b})$ was considered by Durna in [16], and Rani et al. introduced the spectral analysis of a new class of band matrices in [17].

As these examples demonstrate, the study of spectra for bounded operators on Banach spaces is both rich and widely applicable. In recent years, significant attention has been devoted to the spectral analysis of such operators. Up to now, most spectral results on the space ℓ_p for $1 < p < \infty$ have been confined to operators of order at most two. In the present work, we extend the analysis by determining the spectrum of the generalized difference operator Δ^m, which corresponds to an $(m+1) -$ band matrix acting on ℓ_p for $1 < p < \infty$. Consequently, this study generalizes several previous results: those related to the classical difference operator Δ for $m = 1$ as in [4], the operator $B(1, -1)$ examined in [18], and the operator $B(1, -2, 1)$ for $m = 2$ studied in [19].

Fine Spectrum

In this section, the fine spectra of the generalized difference operator Δ^m $(m \epsilon \mathbb{N})$ on the ℓ_p $(1 < p < \infty)$. sequence spaces will be calculated.

Let $A: D(A) \subset X \to X$ be a bounded linear operator where X is a complex normed space. Let $A_\zeta := \zeta I - A$ for $\zeta \epsilon \mathbb{C}$. A_ζ^{-1} is known as the resolvent operator of A.

ζ∈C is a regular value of A such that

a) A_ζ^{-1} exists,

b) A_ζ^{-1} is continuous,

c) A_ζ^{-1} is defined on a set which is dense in X,

the set of all regular values is denoted by $\rho(A, X)$. The set $\sigma(A, X) = \mathbb{C} \backslash \rho(A, X)$ is called the spectrum of A.

The set $\sigma(A, X)$ is union of three disjoint sets as follows:

a) The point spectrum

$$\sigma_p(A, X) = \{\zeta \epsilon \mathbb{C}: A_\zeta^{-1} \textit{ does not exist}\}.$$

b) The continuous spectrum

$$\sigma_c(A, X) = \{\zeta \epsilon \mathbb{C}: \text{Domain set } of\ A_\zeta^{-1} \text{ is a dense subset of } X \text{ and } A_\zeta^{-1} \text{ is discontinuous}\}$$

c) The residual spectrum

$$\sigma_r(A, X) = \{\zeta \epsilon \mathbb{C}: A_\zeta^{-1} \text{exists and its domain of definition is not dense in X}\}$$

From the above definitions, the following table is obtained;

Table1: Disjoint divisions of the spectra of a bounded linear operator

		1	2	3
		A_ζ^{-1} exists and continuous	A_ζ^{-1} exists and discontinuous	A_ζ^{-1} does not exists
I	$R(A_\zeta) = X$	$\zeta \in \rho(A, X)$	$-$	$\zeta \in \sigma_p(A, X)$
I	$\overline{R(A_\zeta)} = X$	$\zeta \in \rho(A, X)$	$\zeta \in \sigma_c(A, X)$	$\zeta \in \sigma_p(A, X)$
III	$\overline{R(A_\zeta)} \neq X$	$\zeta \in \sigma_r(A, X)$	$\zeta \in \sigma_r(A, X)$	$\zeta \in \sigma_p(A, X)$

Main Result

The definition of the operator Δ^m $(m \in \mathbb{N})$, which is a generalization of difference matrices, on the ℓ_p $(1 < p < \infty)$ Banach space is as follows:

$$\begin{aligned}
&(\Delta x)_k = x_k - x_{k-1} \\
&(\Delta^2 x)_k = \Delta(\Delta x)_k = \Delta(x_k - x_{k-1}) = x_k - 2x_{k-1} + x_{k-2} \\
&(\Delta^3 x)_k = x_k - 3x_{k-1} + 3x_{k-2} - x_{k-3} \\
&\vdots \\
&(\Delta^m x)_k = x_k - \binom{m}{1} x_{k-1} + \binom{m}{2} x_{k-2} + \cdots + (-1)^m x_{k-m} \\
&= \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i} \\
&= x_k - m x_{k-1} + \frac{m(m-1)}{2!} x_{k-2} + \cdots + (-1)^m x_{k-m},
\end{aligned}$$

where $k \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0,1,2,\cdots\}$ and $x_k = 0$ for $k < 0$ ([20]).

It is easy to prove that the Δ^m-with $m+1$ bands- operator can be represented by an (a_{nk}) matrix . Herein is

$$a_{nk} = \begin{cases} (-1)^{n-k} \binom{m}{k} & , \quad \max\{0, n-m\} \le k \le n \\ 0 & , \quad 0 \le k < \max\{0, n-m\} \text{ or } k > n \end{cases}$$

for every $n, k \in \mathbb{N}_0$. An equivalent can be written as

$$\Delta^m = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -m & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{m(m-1)}{2} & -m & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & 0 & \cdots \\ 0 & (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (1)$$

In this study, the spectrum of the Δ^m operator on the ℓ_p $(1 < p < \infty)$ sequence space will be calculated, where ℓ_p $(1 < p < \infty)$ is the sequence space of the form

$$\ell_p = \left\{x = (x_n) \in w : \sum_{k=0}^{\infty} |x_k|^p \text{ converget}\right\}.$$

Let $T: \ell_p \to \ell_p$ be a bounded linear operator with matrix representation A. Then, the adjoint operator $T^*: \ell_p^* \to \ell_p^* \cong \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, has a matrix representation given by the transpose of A (see [21], p. 215]).

The operator Δ^m, in particular, is associated with a $(m+1)$-band matrix, which extends classical difference operators such as Δ, Δ^2, $B(r,s)$, and $B(r,s,t)$, depending on specific conditions. We begin by outlining foundational properties regarding the linearity and boundedness of the operator Δ^m.

Theorem 1 *Let T be the operator corresponding to the matrix $A = (a_{nk})$ ([22])*

(i) The necessary and sufficient condition for $T \in B(\ell_\infty)$ is

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty. \qquad (2)$$

(ii) The necessary and sufficient condition for $T \in B(\ell_1)$ is

$$\sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty. \qquad (3)$$

Lemma 1 *Let $1 < p < \infty$ and $A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$ then $A \in (\ell_p, \ell_p)$ ([22]).*

Lemma 2 *T has a dense range if and only if the T^* adjoint operator is one-to-one ([23] , Theorem II 3.7).*

Lemma 3 *$R(T^*) = X^*$ if and only if T has a bounded inverse ([23] , Theorem II 3.11).*

Theorem 2 *$\Delta^m \in B(\ell_p)$ ([24], Theorem 2).*

Theorem 3 *$\sigma_p(\Delta^m, \ell_p) = \emptyset$. ([24], Theorem 3).*

Theorem 4 *$\sigma_p((\Delta^m)^*, \ell_q) = \{\alpha \in \mathbb{C} : |1-\alpha| < 2^m - 1\} \cup \{0\}$ ([24], Theorem 4).*

To compute the spectrum of the operator Δ^m on the sequence space ℓ_p, we need the inverse of the operator $(\Delta^m - \alpha I)^{-1}$. It has been proved that, under the condition $|1-\alpha| > 2^m - 1$ in [20, Theorem 5], the operator $\Delta^m - \alpha I$ has an inverse, which is $(\Delta^m - \alpha I)^{-1} = (b_{nk})$ for

$$(b_{nk}) = \begin{pmatrix} \frac{1}{1-\alpha} & 0 & 0 & 0 & 0 & \cdots \\ \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & 0 & 0 & \cdots \\ \frac{m^2}{(1-\alpha)^3} - \frac{m(m-1)}{2!(1-\alpha)^2} & \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & 0 & \cdots \\ \frac{m^3}{(1-\alpha)^4} - \frac{m^2(m-1)}{(1-\alpha)^3} + \frac{m(m-1)(m-2)}{3!(1-\alpha)^2} & \frac{m^2}{(1-\alpha)^3} - \frac{m(m-1)}{2!(1-\alpha)^2} & \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (4)$$

Thus, the following theorem can be proved.

Theorem 5 *$\sigma(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C} : |1-\alpha| \le 2^m - 1\}$([24], Teorem 5).*

Theorem 6 $\sigma_r(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C} : |1-\alpha| < 2^m - 1\} \cup \{0\}$.

Proof. Since $\sigma_r(\Delta^m, cs) = \sigma_p((\Delta^m)^*, \ell_q) \backslash \sigma_p(\Delta^m, \ell_p)$, the desired result follows from Theorem 3 and Theorem 4.

Theorem 7 $\sigma_c(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C} : |1-\alpha| = 2^m - 1\} \backslash \{0\}$.

Proof. The requested result is acquired directly from Theorem 3, Theorem 6, and Theorem 7, given that the spectrum of a bounded linear operator is the union of the point spectrum, residue spectrum, and continuous spectrum, and these sets are disjoint.

Theorem 8 $1 \in III_1\sigma(\Delta^m, \ell_p)$, $(0 < p < 1)$.

Proof. It suffices to show that matrix

$$\Delta^m - I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -m & 0 & 0 & 0 & \cdots \\ \frac{m(m-1)}{2!} & -m & 0 & 0 & \cdots \\ -\frac{m(m-1)(m-2)}{3!} & \frac{m(m-1)}{2!} & -m & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is bounded in order to have bounded inverse. Indeed, one can easily see that

$$\|(\Delta^m - I)x\| = \sum_{v=1}^{m} \left|(-1)^{m-v}\binom{m}{v}\right| |x_v| \leq \left(\left\|\frac{m}{2}\right\|\right)^{m} \|x\|_1$$

for every $x = (x_n) \in \ell_1$. This means that $\Delta^m - I$ is bounded.

Now let $\alpha \neq 1$. From Theorem 6, if $\alpha \in \{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \cup \{0\}$, then $\Delta^m - \alpha I \in III_1 \cup III_2$.

Let $y = (y_n) \in \ell_p$. $x = (x_n) \in \ell_p$ must be found such that $((\Delta^m)^* - \alpha I)x = y$. The transpose of (4) gives the inverse matrix $(\Delta^* - \alpha I)^{-1}$. From Theorem 4 in [20], if $|1-\alpha| > 2^m - 1$ then $((\Delta^m)^* - \alpha I)^{-1} \in B(c)$. Since

$$(c; c) \Rightarrow (c; \ell_\infty) \Rightarrow (\ell_\infty; \ell_\infty)$$

$((\Delta^m)^* - \alpha I)^{-1} \in B(\ell_\infty)$ holds when $|1-\alpha| > 2^m - 1$. Hence, from Lemma 1 under the condition $|1-\alpha| > 2^m - 1$ is $((\Delta^m)^* - \alpha I)^{-1} \in B(\ell_p)$. Thus, the operator $((\Delta^m)^* - \alpha I)^{-1}$ is surjective. Therefore, from Lemma 3, the operator $\Delta^m - \alpha I$ has bounded inverse when $|1-\alpha| \leq 2^m - 1$. So, under the condition$|1-\alpha| \leq 2^m - 1$, $\alpha \notin III_1\sigma(\Delta, \ell_p)$. This completes the proof.

Corollary 1 $III_2\sigma(\Delta^m, \ell_p) = [\{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \cup \{0\}]\backslash\{1\}$.

Proof. Being that $\sigma_r(\Delta^m, \ell_p) = III_1\sigma(\Delta^m, \ell_p) \cup III_2\sigma(\Delta^m, \ell_p)$, the requested result follows from Theorems 6 and 8.

Corollary 2 $I_3\sigma(\Delta^m, \ell_p) = II_3\sigma(\Delta^m, \ell_p)$
$= III_3\sigma(\Delta^m, \ell_p) = \emptyset$

Proof. Since $\sigma_p(\Delta^m, \ell_p) = I_3\sigma(\Delta^m, \ell_p) \cup II_3\sigma(\Delta^m, \ell_p) \cup III_3\sigma(\Delta^m, \ell_p)$ and the sets $I_3\sigma(\Delta^m, \ell_p)$, $II_3\sigma(\Delta^m, \ell_p)$, $III_3\sigma(\Delta^m, \ell_p)$ are disjoint, the desired result follows from Theorem 3.

Theorem 9 *The follows are valid;*

(a) $\sigma_{ap}(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C}: |1-\alpha| \leq 2^m - 1\}\{1\}$,

(b) $\sigma_\delta(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C}: |1-\alpha| \leq 2^m - 1\}$,

(c) $\sigma_{co}(\Delta^m, \ell_p) = \{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \cup \{0\}$.

Proof.

(a)Since $\sigma_{ap}(\Delta^m, \ell_p) = \sigma(\Delta^m, \ell_p)\backslash III_1\sigma(\Delta^m, \ell_p)$ is satisfied from Table 1, the proof is acquired from Theorems 5 and 8.

(b) Being that

$$\sigma_\delta(\Delta^m, \ell_p) = \sigma(\Delta^m, \ell_p)\backslash I_3\sigma(\Delta^m, \ell_p)$$

is satisfied from Table 1, the proof is obtained from Theorem 5 and Corollary 2.

(c) Being that

$$\sigma_{co}(\Delta^m, \ell_p) = III_1\sigma(\Delta^m, \ell_p) \cup III_2\sigma(\Delta^m, \ell_p) \cup III_3\sigma(\Delta^m, \ell_p)$$

is satisfied from Table 1, the proof is obtanied from Theorem 6 ve Corollary 2.

From Proposition 1.3 in [25] the following results are obtained.

Corollary 3 *Let* $p^{-1} + q^{-1} = 1$ *and* $p > 1$. *In this case;*

(a) $\sigma_{ap}((\Delta^m)^*, \ell_q) = \{\alpha \in \mathbb{C}: |1-\alpha| \leq 2^m - 1\}$,

(b) $\sigma_\delta((\Delta^m)^*, \ell_q) = \{\alpha \in \mathbb{C}: |1-\alpha| \leq 2^m - 1\}\{1\}$,

(c) $\sigma_p((\Delta^m)^*, \ell_q) = \{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \cup \{0\}$ is valid.

Conclusion and Implications

The results obtained in this study provide a comprehensive spectral analysis of m-band forward difference matrices and generalize many of the findings previously established for more specific matrix classes such as the classical forward difference matrix Δ, the second-order difference matrix Δ^2, and the generalized band matrices $B(r,s)$ and $B(r,s,t)$. By extending the analysis to m-band structures, this work encompasses a broader class of discrete operators, allowing for the inclusion of more complex interactions within the underlying systems.

In particular, the spectral characteristics derived here reduce to the known results for Δ and Δ^2 when the bandwidth parameter m is chosen appropriately (e.g., $m = 1$ or $m = 2$), and likewise, special cases of the matrices $B(r,s)$ and $B(r,s,t)$ are covered by specific selections of the matrix entries in the generalized framework presented. This confirms the consistency of the current approach with earlier work while demonstrating its capacity to unify and extend those results under a single theoretical structure.

Therefore, the findings of this paper not only validate known spectral properties in simpler matrix models but also provide a natural pathway to analyze more sophisticated systems governed by higher-order or multi-coupling discrete operators. This unifying framework is expected to facilitate further research in the spectral theory of difference operators and to enhance the applicability of such analyses in both pure and applied mathematical contexts.

Conflicts of interest

There are no conflicts of interest in this work.

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References

[1] Altay, B., Başar, F., On the fine spectrum of the difference operator on c_0 and c, *Inf. Sci.,* 168 (2004) 217–224.

[2] Altay, B., Başar, F., The fine spectrum and the matrix domain of the difference operator Δ on the sequence space ℓ_p, $(1 < p < \infty)$, *Commun. Math. Anal.,* 2(2) (2007) 1–11.

[3] Altay, B., Başar, F., On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c, *Int. J. Math. Math. Sci*., 18 (2005) 3005–3013.

[4] Akhmedov, A.M., Başar, F., On the fine spectra of the difference operator Δ over the sequence space ℓ_p, *Demonstr. Math.,* 39(3) (2006) 586–595.

[5] Akhmedov, A.M., Başar, F., On the fine spectra of the difference operator Δ over the sequence space bv_p, $(1 \le p < \infty)$, *Acta. Math. Sin. (Engl. Ser.),* 23(10) (2007) 1757–1768.

[6] Durna, N., Yıldırım, M., Kılıç, R., Partition of the spectra for the generalized difference operator $B(r,s)$ on the sequence space cs, *Cumhuriyet Sci. J.*, 39(1) (2018) 7–15.

[7] Srivastava, P.D., Kumar, S., On the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0, *Commun. Math. Anal.,* 6(1) (2009) 8–21.

[8] Srivastava, P.D., Kumar, S., Fine spectrum of the generalized difference operator Δ_v on sequence space ℓ_1, *Thai. J. Math.,* 8(2) (2010) 221–233.

[9] Akhmedov, A.M., El-Shabrawy, S.R., On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c, *Comput. Math. Appl*., 61 (2011) 2994–3002.

[10] Dutta, S., Baliarsingh, P., On the fine spectra of the generalized rth difference operator Δ_v^r on the sequence space ℓ_1, *Appl. Math. Comput.*, 219 (2012) 1776–1784.

[11] Dutta, S., Baliarsingh, P., On the spectrum of $2 - nd$ order generalized difference operator Δ^2 over the sequence space c_0, *Bol. Soc. Paran. Mat.,* 31(2) (2013) 235–244.

[12] Durna, N., Yıldırım, M., Ünal, Ç**.,** On the fine spectrum of generalized lower triangular double band matrices Δ_{uv} Over The Sequence Space c_0, *Cumhuriyet Sci. J.,* 37(3) (2016) 281–291.

[13] Durna, N.**,** Subdivision of spectra for some lower triangular doule-band matrices as operators on c_0, *Ukrain. Mat. Zh.,* 70(7) (2018) 913–922.

[14] Durna, N., Kılıç, R., On the spectrum and fine spectrum of the upper triangular double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ over the sequence space ℓ_p, *Honam Mathematical J.*, 45(4) (2023) 598-609.

[15] Durna, N., Mursaleen, M. and Kılıç, R., Spectra and fine spectra of the generalized upper difference operator with triple repetition Δ_{ab}^3 on the Hahn sequence space, *Mathematica Slovaca*, 72(3) (2022) 719-736.

[16] Durna, N., Spectra and fine spectra of the upper triangular band matrix $U(a;0;b)$ on the Hahn sequence space, *Mathematical Communications*, 25(1) (2020) 49-66.

[17] Rani, J., Patra, A., Srivastava, P. D., Spectrum and Fine Spectrum of Band Matrices Generated by Oscillatory Sequences, *Operators and Matrices*, 18(4) (2024) 805-831.

[18] Bilgiç, H., Furkan, H., On the fine spectrum of the generalized difference operator $B(r;s)$ over the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$, *Nonlinear Anal.*, 68(3) (2008) 499-506.

[19] Furkan, H., Bilgiç, H., Altay, B., On the fine spectrum of the operator $B(r,s,t)$ over the sequence space ℓ_p and bv_p $(1 < p < \infty)$, *Comput. Math. Appl.,* 60(7) (2010) 897–908.

[20] Baliarsingh, P., Mursalen, M., Rakocevic, V., A survey on the spectra of the difference operators over the Banach space c, *RACSAM*, 115(57) (2021) 1–17.

[21] Taylor, R. B., Introduction to functional analysis Wiley, New York, (1980).

[22] Wilansky, A., Summability Through Functional Analysis, North-Holland Mathematics Studies, Amsterdam, (1984).

[23] Goldberg, S., Unbounded Linear Operators, McGraw Hill, New York (1966).

[24] Tuğutlu M., Durna N., On the Spectrum of the difference operator Δ^m over the sequence space ℓ_p, 1 st ICSAR, Konya, Turkey, 2022,362-366.

[25] Appell, J., Pascale, E.D., Vignoli, A., Nonlinear Spectral Theory, Walter de Gruyter, Berlin New York, (2004).