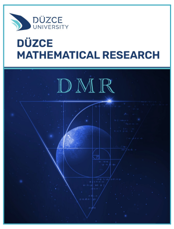

Düzce Mathematical Research
ISSN: 3108-5741 (electronic)
https://dergipark.org.tr/en/pub/dmr
Publisher: Düzce University

1:1 (2025), 11–21

Research Article

A note on conformable fractional Newton-type inequalities via functions of bounded variation

Mehmet Sarıışık [1], Fatih Hezenci [1,*], Hüseyin Budak [2]

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye*

[2] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Türkiye*

Abstract. In this paper, we establish an equality in order to obtain conformable fractional Newton-type inequalities. Moreover, we prove some Newton-type inequalities associated with conformable fractional operators for functions of bounded variation. Furthermore, some results are presented by using special choices of the obtained inequalities.

Keywords. Conformable fractional integrals, functions of bounded variations, Newton-type inequalities

1. Introduction

The Hermite-Hadamard and Simpson-type inequalities have attracted considerable attention in recent years. In particular, Simpson-type inequalities are derived from Simpson's classical rules of numerical integration. Simpson's classical numerical integration rules are as follows:

i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

Simpson's second rule corresponds to the three-point Newton-Cotes quadrature formula. Therefore, evaluations involving a quadratic kernel with three nodes are often referred to as Newton-type results. In the literature, such results are commonly known as Newton-type inequalities. These inequalities have been extensively studied by numerous researchers. For example, Gao and Shi [1] studied Newton-type inequalities for functions whose second derivatives are convex. In [2], some error estimates of Newton-type quadrature formula by bounded variation and Lipschitzian functions were presented. In [3], the authors derived Newton-type inequalities for convex functions within the framework of quantum calculus. Furthermore, Noor et al. investigated Newton-type inequalities related

*Corresponding author
Email addresses: mehmet03srsk@gmail.com (Mehmet Sarıışık), fatihezenci@gmail.com (Fatih Hezenci), hsyn.budak@gmail.com (Hüseyin Budak)
Received : 28.07.2025 Accepted: 23.08.2025 Published online: 28.11.2025

Cite as: M. Sarıışık, F. Hezenci, and H. Budak (2025), *A note on conformable fractional Newton-type inequalities via functions of bounded variations*, Düzce Math. Res., **1**(1), 11-21.

to harmonic convex functions in [4], and to $p-$harmonic convex functions in [5]. For additional information and unresolved aspects concerning Newton-type inequalities, the reader may consult [6–8] and the references cited therein.

Fractional calculus has gained significant attention in recent years due to its wide range of applications across various scientific fields. In light of its importance, various fractional integral operators have been introduced and studied. By employing Hermite–Hadamard-type and Simpson-type inequalities, bounds for newly developed formulas can be established. Notably, Hermite–Hadamard-type and trapezoidal-type inequalities were first investigated using Riemann-Liouville fractional integrals in [9].

Definition 1.1 ([10, 11]). If we consider $f \in L_1[a,b]$, then the Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a \tag{1.1}$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b, \tag{1.2}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and its described as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

Let us note that $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

The paper [12] presents several Newton-type inequalities for differentiable convex functions, derived through the use of Riemann-Liouville fractional integrals. In addition, the authors obtained some inequalities of Riemann-Liouville fractional Newton-type for functions of bounded variation. Moreover, in [13], the authors derived several Newton-type inequalities for functions whose absolute first derivatives, raised to a given power, satisfy the condition of arithmetic-harmonic convexity. Furthermore, the authors of the paper [14] present a method to examine some Newton-type inequalities for various function classes using Riemann-Liouville fractional integrals. These types of inequalities have been extensively studied by several researchers (see, e.g., [15–17], and references therein).

Numerous forms of fractional integrals, including Riemann-Liouville and conformable fractional integrals, have been examined in the context of integral inequalities. The growing applicability of these concepts has recently drawn significant attention from researchers in mathematics, physics, and engineering [18, 19]. Moreover, fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes and engineering problems [20, 21]. In [22], the authors developed a new form of fractional derivative, called the conformable derivative, based on the fundamental limit definition of the standard derivative. This operator exhibits several desirable analytical properties. The conformable derivative satisfies several key properties that are not fulfilled by the Riemann-Liouville and Caputo definitions. In [23], the author proved that the conformable approach in [22] cannot yield good results when compared to the Caputo definition for specific functions. This limitation of the conformable definition has been addressed through various extensions of the conformable approach, as proposed in [24, 25].

In [26], the authors introduced the fractional conformable integral operators and investigated several of their fundamental properties. They also explored the relationships between these operators and other existing fractional integral definitions. The fractional conformable integral operators are defined as follows:

Definition 1.2 ([26]). The fractional conformable integral operator ${}^{\beta}\mathscr{J}^{\alpha}_{a+}f(x)$ and ${}^{\beta}\mathscr{J}^{\alpha}_{b-}f(x)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are presented by

$${}^{\beta}\mathscr{J}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}dt, \quad t > a \tag{1.3}$$

and

$$^{\beta}\mathscr{J}^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt,\quad t<b, \tag{1.4}$$

respectively for $f\in L_1[a,b]$.

Let us consider $\alpha=1$ in equalities (1.3) and (1.4). Then, the fractional integral in (1.3) and (1.4) coincides with the Riemann-Liouville fractional integral in (1.1) and (1.2), respectively. There have been a great number of research papers written on these subjects, [27, 28] and the references therein.

2. A crucial equality

Lemma 2.1 ([29]). Suppose that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function (a,b) such that $f'\in L_1[a,b]$. Then, the following equality holds:

$$\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\alpha^{\beta}\Gamma(\alpha+1)}{2(b-a)^{\alpha\beta}}\left[^{\beta}\mathscr{J}^{\alpha}_{b-}f(a)+{}^{\beta}\mathscr{J}^{\alpha}_{a+}f(b)\right]$$

$$=\frac{\alpha^{\beta}(b-a)}{2}\left[I_1+I_2+I_3\right],$$

where

$$\begin{cases} I_1=\int\limits_0^{\frac{1}{3}}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{8\alpha^{\beta}}\right]\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\ I_2=\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\ I_3=\int\limits_{\frac{2}{3}}^{1}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{7}{8\alpha^{\beta}}\right]\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt. \end{cases}$$

3. Newton-type inequalities for functions of bounded variation

Theorem 3.1. *Consider that $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$. Then, one can obtain*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[^{\beta}\mathscr{J}^{\alpha}_{b-}f(a)+{}^{\beta}\mathscr{J}^{\alpha}_{a+}f(b)\right]\right|$$

$$\leq\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{8}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{7}{8}\right|\right\}\bigvee_a^b(f).$$

Here, $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

Proof. Describe the function $K^{\beta}_{\alpha}(x)$ and $L^{\beta}_{\alpha}(x)$ by

$$K^{\beta}_{\alpha}(x)=\begin{cases}\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\frac{(b-a)^{\alpha\beta}}{8}, & a\leq x\leq\frac{2a+b}{3},\\ \left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\frac{(b-a)^{\alpha\beta}}{2}, & \frac{2a+b}{3}<x\leq\frac{a+2b}{3},\\ \left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\frac{7(b-a)^{\alpha\beta}}{8}, & \frac{a+2b}{3}<x\leq b,\end{cases}$$

and

$$L_{\alpha}^{\beta}(x)=\begin{cases}\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{7(b-a)^{\alpha\beta}}{8}, & a\leq x\leq \frac{2a+b}{3},\\ \left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{(b-a)^{\alpha\beta}}{2}, & \frac{2a+b}{3}<x\leq \frac{a+2b}{3},\\ \left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{(b-a)^{\alpha\beta}}{8}, & \frac{a+2b}{3}<x\leq b,\end{cases}$$

respectively. It follows that

$$\begin{aligned}&\int_a^b\left[K_{\alpha}^{\beta}(x)-L_{\alpha}^{\beta}(x)\right]df(x) \qquad (3.1)\\ &=\int_a^{\frac{2a+b}{3}}\left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}+\frac{3(b-a)^{\alpha\beta}}{4}\right]df(x)\\ &+\int_{\frac{a+b}{2}}^{\frac{a+2b}{3}}\left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}\right]df(x)\\ &+\int_{\frac{a+2b}{3}}^{b}\left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{3(b-a)^{\alpha\beta}}{4}\right]df(x).\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}&\int_a^{\frac{2a+b}{3}}\left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}+\frac{3(b-a)^{\alpha\beta}}{4}\right]df(x) \qquad (3.2)\\ &=\left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}+\frac{3(b-a)^{\alpha\beta}}{4}\right]f(x)\Big|_a^{\frac{2a+b}{3}}\\ &-\alpha\beta\int_a^{\frac{2a+b}{3}}\left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}\right]f(x)dx\\ &-\alpha\beta\int_a^{\frac{2a+b}{3}}\left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}\right]f(x)dx\\ &=\left[\left[(b-a)^{\alpha}-\left(\frac{2(b-a)}{3}\right)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-\left(\frac{b-a}{3}\right)^{\alpha}\right]^{\beta}+\frac{3(b-a)^{\alpha\beta}}{4}\right]f\left(\frac{2a+b}{3}\right)\\ &+\frac{(b-a)^{\alpha\beta}}{4}f(a)\\ &-\alpha\beta\int_a^{\frac{2a+b}{3}}\left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(b-x)^{1-\alpha}}dx\\ &-\alpha\beta\int_a^{\frac{2a+b}{3}}\left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(x-a)^{1-\alpha}}dx.\end{aligned}$$

Similarly, we have

$$\int_{\frac{a+b}{2}}^{\frac{a+2b}{3}} \left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}\right] df(x) \tag{3.3}$$

$$= \left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}\right] f(x)\Big|_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}$$

$$-\alpha\beta\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}\right] f(x)dx$$

$$-\alpha\beta\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}\right] f(x)dx$$

$$= \left[\left[(b-a)^{\alpha}-\left(\frac{b-a}{3}\right)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-\left(\frac{2(b-a)}{3}\right)^{\alpha}\right]^{\beta}\right] f\left(\frac{a+2b}{3}\right)$$

$$-\left[\left[(b-a)^{\alpha}-\left(\frac{2(b-a)}{3}\right)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-\left(\frac{b-a}{3}\right)^{\alpha}\right]^{\beta}\right] f\left(\frac{2a+b}{3}\right)$$

$$-\alpha\beta\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}\right] \frac{f(x)}{(b-x)^{1-\alpha}}dx$$

$$-\alpha\beta\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}\right] \frac{f(x)}{(x-a)^{1-\alpha}}dx,$$

and

$$\int_{\frac{a+2b}{3}}^{b} \left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{3(b-a)^{\alpha\beta}}{4}\right] df(x) \tag{3.4}$$

$$= \left[\left[(b-a)^{\alpha}-(b-x)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{3(b-a)^{\alpha\beta}}{4}\right] f(x)\Big|_{\frac{a+2b}{3}}^{b}$$

$$-\alpha\beta\int_{\frac{a+2b}{3}}^{b} \left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}\right] f(x)dx$$

$$-\alpha\beta\int_{\frac{a+2b}{3}}^{b} \left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}\right] f(x)dx$$

$$= \frac{(b-a)^{\alpha\beta}}{4} f(b)$$

$$-\left[\left[(b-a)^{\alpha}-\left(\frac{b-a}{3}\right)^{\alpha}\right]^{\beta}-\left[(b-a)^{\alpha}-\left(\frac{2(b-a)}{3}\right)^{\alpha}\right]^{\beta}-\frac{3(b-a)^{\alpha\beta}}{4}\right] f\left(\frac{a+2b}{3}\right)$$

$$-\alpha\beta\int_{\frac{a+2b}{3}}^{b}\left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(b-x)^{1-\alpha}}dx$$

$$-\alpha\beta\int_{\frac{a+2b}{3}}^{b}\left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(x-a)^{1-\alpha}}dx.$$

By putting the equalities from (3.2) to (3.4) in (3.1), we have

$$\begin{aligned}\int_a^b\left[K_\alpha^\beta(x)-L_\alpha^\beta(x)\right]df(x)=&\frac{(b-a)^{\alpha\beta}}{4}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\\&-\alpha\beta\int_a^b\left[\left((b-a)^{\alpha}-(b-x)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(b-x)^{1-\alpha}}dx\\&-\alpha\beta\int_a^b\left[\left((b-a)^{\alpha}-(x-a)^{\alpha}\right)^{\beta-1}\right]\frac{f(x)}{(x-a)^{1-\alpha}}dx\\=&2(b-a)^{\alpha\beta}\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\\&-\alpha^{\beta}\Gamma(\beta+1)\left[{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)\right].\end{aligned}$$

That is,

$$\begin{aligned}&\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)\right]\\&=\frac{1}{2(b-a)^{\alpha\beta}}\int_a^b\left[K_\alpha^\beta(x)-L_\alpha^\beta(x)\right]df(x).\end{aligned}\tag{3.5}$$

By taking the absolute value of both sides of (3.5), we readily obtain

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)\right]\right|\\&\leq\frac{1}{2(b-a)^{\alpha\beta}}\left|\int_a^b\left[K_\alpha^\beta(x)-L_\alpha^\beta(x)\right]df(x)\right|\\&\leq\frac{1}{2(b-a)^{\alpha\beta}}\left[\left|\int_a^b K_\alpha^\beta(x)df(x)\right|+\left|\int_a^b L_\alpha^\beta(x)df(x)\right|\right].\end{aligned}\tag{3.6}$$

It is known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is bounded variation on $[a,b]$ then $\int_a^b g(x)df(x)$ exist and

$$\left|\int_a^b g(x)df(x)\right|\leq\sup_{x\in[a,b]}|g(x)|\bigvee_a^b(f).\tag{3.7}$$

With the help of the inequality (3.7), we have

$$\begin{aligned}
\left|\int_a^b K_a^\beta(x)df(x)\right| &\leq \left|\int_a^{\frac{2a+b}{3}}\left[\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{8}\right]df(x)\right| \\
&\quad+\left|\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}\left[\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{2}\right]df(x)\right| \\
&\quad+\left|\int_{\frac{a+2b}{3}}^{b}\left[\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{7(b-a)^{\alpha\beta}}{8}\right]df(x)\right| \\
&\leq \sup_{x\in\left[a,\frac{2a+b}{3}\right]}\left|\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{8}\right|\bigvee_a^{\frac{2a+b}{3}}(f) \\
&\quad+\sup_{x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]}\left|\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{2}\right|\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f) \\
&\quad+\sup_{x\in\left[\frac{a+2b}{3},b\right]}\left|\left[(b-a)^\alpha-(b-x)^\alpha\right]^\beta-\frac{7(b-a)^{\alpha\beta}}{8}\right|\bigvee_{\frac{a+2b}{3}}^{b}(f) \\
&= (b-a)^{\alpha\beta}\left[\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^\alpha\right]^\beta-\frac{1}{8}\right|\right\}\bigvee_a^{\frac{2a+b}{3}}(f)\right. \\
&\quad+\max\left\{\left|\left[1-\left(\frac{1}{3}\right)^\alpha\right]^\beta-\frac{1}{2}\right|,\left|\left[1-\left(\frac{2}{3}\right)^\alpha\right]^\beta-\frac{1}{2}\right|\right\}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f) \\
&\quad\left.+\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{1}{3}\right)^\alpha\right]^\beta-\frac{7}{8}\right|\right\}\bigvee_{\frac{a+2b}{3}}^{b}(f)\right] \\
&\leq (b-a)^{\alpha\beta}\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^\alpha\right]^\beta-\frac{1}{8}\right|,\left|\left[1-\left(\frac{1}{3}\right)^\alpha\right]^\beta-\frac{1}{2}\right|,\right. \\
&\quad\left.\left|\left[1-\left(\frac{2}{3}\right)^\alpha\right]^\beta-\frac{1}{2}\right|,\left|\left[1-\left(\frac{1}{3}\right)^\alpha\right]^\beta-\frac{7}{8}\right|\right\}\bigvee_a^b(f).
\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}
\left|\int_a^b L_\alpha^\beta(x)df(x)\right| &\leq \left|\int_a^{\frac{2a+b}{3}}\left[\left[(b-a)^\alpha-(x-a)^\alpha\right]^\beta-\frac{7(b-a)^{\alpha\beta}}{8}\right]df(x)\right| \qquad (3.8) \\
&\quad+\left|\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}\left[\left[(b-a)^\alpha-(x-a)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{2}\right]df(x)\right| \\
&\quad+\left|\int_{\frac{a+2b}{3}}^{b}\left[\left[(b-a)^\alpha-(x-a)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{8}\right]df(x)\right| \\
&\leq \sup_{x\in\left[a,\frac{2a+b}{3}\right]}\left|\left[(b-a)^\alpha-(x-a)^\alpha\right]^\beta-\frac{7(b-a)^{\alpha\beta}}{8}\right|\bigvee_a^{\frac{2a+b}{3}}(f) \\
&\quad+\sup_{x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]}\left|\left[(b-a)^\alpha-(x-a)^\alpha\right]^\beta-\frac{(b-a)^{\alpha\beta}}{2}\right|\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{x\in\left[\frac{a+2b}{3},b\right]}\left|\left[(b-a)^{\alpha}-(x-a)^{\alpha}\right]^{\beta}-\frac{(b-a)^{\alpha\beta}}{8}\right|\bigvee_{\frac{a+2b}{3}}^{b}(f)\\
=\;&(b-a)^{\alpha\beta}\left[\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{7}{8}\right|\right\}\bigvee_{a}^{\frac{2a+b}{3}}(f)\right.\\
&+\max\left\{\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|\right\}\bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}}(f)\\
&\left.+\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{8}\right|\right\}\bigvee_{\frac{a+2b}{3}}^{b}(f)\right]\\
\leq\;&(b-a)^{\alpha\beta}\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{8}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\right.\\
&\left.\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{7}{8}\right|\right\}\bigvee_{a}^{b}(f).
\end{aligned}
$$

If we substitute inequalities (3.8) and (3.8) into (3.6), then we obtain

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)+{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)\right]\right|\\
&\leq\max\left\{\frac{1}{8},\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{8}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{2}{3}\right)^{\alpha}\right]^{\beta}-\frac{1}{2}\right|,\left|\left[1-\left(\frac{1}{3}\right)^{\alpha}\right]^{\beta}-\frac{7}{8}\right|\right\}\bigvee_{a}^{b}(f),
\end{aligned}
$$

which gives the desired results of Theorem 3.1. □

Remark 3.2. *If we consider $\alpha=1$ in Theorem 3.1, then we have the following Newton-type inequality for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\
&\leq\max\left\{\frac{1}{8},\left|\left(\frac{1}{3}\right)^{\beta}-\frac{1}{8}\right|,\left|\left(\frac{2}{3}\right)^{\beta}-\frac{1}{2}\right|,\left|\left(\frac{1}{3}\right)^{\beta}-\frac{1}{2}\right|,\left|\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right|\right\}\bigvee_{a}^{b}(f),
\end{aligned}
$$

which is proved by Hezenci et al. in paper [16, Theorem 14].

Remark 3.3. *If we assign $\alpha=\beta=1$ in Theorem 3.1, then one can readily obtain*

$$
\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\leq\frac{5}{24}\bigvee_{a}^{b}(f).
$$

This is established by Alomari in paper [30].

4. Conclusion

Some new versions of Newton-type inequalities are considered for the case of differentiable convex functions with the help of the conformable fractional integrals. More precisely, several Newton-type inequalities for differentiable convex functions are evaluated by using the power-mean and Hölder inequality. Moreover, some new results are given by using special choices of obtained inequalities. In addition, some Newton-type inequalities are given for bounded and Lipschitzian functions. Finally, we prove several conformable fractional Newton-type inequalities for functions of bounded variation.

In future work, the concepts underlying our results related to Newton-type inequalities for conformable fractional integrals may pave new paths for mathematicians in this field. Furthermore, one could explore generalizing our results by employing different versions of convex function classes or other types of fractional integral operators. Finally, these types of inequalities could be derived for convex functions using conformable fractional integrals within the framework of quantum calculus.

Article Information

Acknowledgements: The authors gratefully acknowledge the editor and the anonymous reviewers for their valuable comments and constructive suggestions.

Author's Contributions: The authors have read and approved.

Artificial Intelligence Statement: Artificial intelligence was not employed in this study.

Conflict of Interest Disclosure: The authors declare that they have no conflict interests.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] S. Gao, W. Shi (2012), *On new inequalities of Newton's type for functions whose second derivatives absolute values are convex*, Int. J. Pure Appl. Math., **74**(1), 33–41.

[2] S. Erden, S. Iftikhar, P. Kumam, P. Thounthong (2020), *On error estimations of Simpson's second type quadrature formula*, Math. Methods Appl. Sci., **47**(13), 11232–11244. https://doi.org/10.1002/mma.7019

[3] M.A. Ali, H. Budak, Z. Zhang (2022), *A new extension of quantum Simpson's and quantum Newton's type inequalities for quantum differentiable convex functions*, Math. Methods Appl. Sci., **45**(4), 1845–1863. https://doi.org/10.1002/mma.7889

[4] M.A. Noor, K.I. Noor, S. Iftikhar (2016), *Some Newton's type inequalities for harmonic convex functions*, J. Adv. Math. Stud., **9**(1), 07–16.

[5] M.A. Noor, K.I. Noor, S. Iftikhar (2018), *Newton inequalities for p-harmonic convex functions*, Honam Math. J., **40**(2), 239–250. https://dx.doi.org/10.5831/HMJ.2018.40.2.239

[6] S. Iftikhar, P. Kumam, S. Erden (2020), *Newton's-type integral inequalities via local fractional integrals*, Fractals, **28**(03), 2050037. https://doi.org/10.1142/S0218348X20500371

[7] S. Iftikhar, S. Erden, M.A. Ali, J. Baili, H. Ahmad (2022), *Simpson's second-type inequalities for co-ordinated convex functions and applications for cubature formulas*, Fractal Fract., **6**(1), 33. https://doi.org/10.3390/fractalfract6010033

[8] M.A. Noor, K.I. Noor, M.U. Awan (2015), *Some Newton's type inequalities for geometrically relative convex functions*, Malaysian J. Math. Sci., **9**(3), 491–502.

[9] M.Z. Sarıkaya, E. Set, H. Yaldız, N. Başak (2013), *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, **57**, 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

[10] R. Gorenflo, F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, Springer Verlag, Wien, 1997.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[12] T. Sitthiwirattham, K. Nonlaopon, M.A. Ali, H. Budak (2022), *Riemann-Liouville fractional Newton's type inequalities for differentiable convex functions*, Fractal Fract. **6**(3), 175. https://doi.org/10.3390/fractalfract6030175

[13] S. Erden, S. Iftikhar, P. Kumam, M.U. Awan (2020), *Some Newton's like inequalities with applications*, RACSAM Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat., **114**(4), 1–13. https://doi.org/10.1007/s13398-020-00926-z

[14] F. Hezenci, H. Budak (2025), *Fractional Newton-type integral inequalities by means of various function classes*, Math. Methods Appl. Sci., **48**(1), 1198–1215. https://doi.org/10.1002/mma.10378

[15] S. Iftikhar, S. Erden, P. Kumam, M.U. Awan (2020), *Local fractional Newton's inequalities involving generalized harmonic convex functions*, Adv. Difference Equ., **2020**(1), 1–14. https://doi.org/10.1186/s13662-020-02637-6

[16] F. Hezenci, H. Budak, P. Kösem (2023), *A new version of Newton's inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain J. Math., **53**(1), 49–64. https://doi.org/10.1216/rmj.2023.53.49

[17] F. Hezenci, H. Budak (2023), *Some perturbed Newton type inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain J. Math., **53**(4), 1117–1127. https://doi.org/10.1216/rmj.2023.53.1117

[18] V.V. Uchaikin, *Fractional derivatives for physicists and engineers*, Springer: Berlin/Heidelberg, Germany, 2013.

[19] G.A. Anastassiou, *Generalized fractional calculus: New advancements and applications*, Springer: Switzerland, 2021.

[20] N. Attia, A. Akgül, D. Seba, A. Nour (2020), *An efficient numerical technique for a biological population model of fractional order*, Chaos, Solitons & Fractals, **141**, 110349. https://doi.org/10.1016/j.chaos.2020.110349

[21] A. Gabr, A.H. Abdel Kader, M.S. Abdel Latif (2021), *The effect of the parameters of the generalized fractional derivatives on the behavior of linear electrical circuits*, Int. J. Appl. Comput. Math., **7**, 247. https://doi.org/10.1007/s40819-021-01160-w

[22] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh (2014), *A new definition of fractional derivative*, J. Comput. Appl. Math., **264**, 65–70. https://doi.org/10.1016/j.cam.2014.01.002

[23] A.A. Abdelhakim (2019), *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fract. Calc. Appl. Anal., **22**, 242–254. https://doi.org/10.1515/fca-2019-0016

[24] D. Zhao, M. Luo (2017), *General conformable fractional derivative and its physical interpretation*, Calcolo, **54**, 903–917. https://doi.org/10.1007/s10092-017-0213-8

[25] A. Hyder, A.H. Soliman (2020), *A new generalized θ-conformable calculus and its applications in mathematical physics*, Phys. Scr., **96**, 015208. https://doi.org/10.1088/1402-4896/abc6d9

[26] F. Jarad, E. Uğurlu, T. Abdeljawad, D. Baleanu (2017), *On a new class of fractional operators*, Adv. Difference Equ., **2017**, 247. https://doi.org/10.1186/s13662-017-1306-z

[27] T. Abdeljawad (2015), *On conformable fractional calculus*, J. Comput. Appl. Math., **279**, 57–66. https://doi.org/10.1016/j.cam.2014.10.016

[28] H. Budak, C. Ünal, F. Hezenci (2024), *A study on error bounds for Newton-type inequalities in conformable fractional integrals*, Math. Slovaca, **74**(2), 313–330. https://doi.org/10.1515/ms-2024-0024

[29] F. Hezenci, P. Karagözoglu, H. Budak, *Some error bounds for Newton formula in conformable fractional operators*, submitted.

[30] M.W. Alomari (2012), *A companion of Dragomir's generalization of Ostrowski's inequality and applications in numerical integration*, Ukrainian Math. J., **64**, 435–450.