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A New Generalization of Szász Operators

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Abstract— The purpose of this study is to define a new generalization of Szász operators. The paper proceeds to study the rate of convergence and approximation properties of the newly defined Szász operator on closed subintervals of the real axis. Subsequently, it investigates the Voronovskaja-type approximation and the local approximation of the new Szász operator using functions satisfying the Lipschitz condition. Additionally, this paper analyzes the rate of convergence for F and F' using the continuity modulus. Finally, it graphically illustrates the approximation of the new generalization of the Szász operator to a continuous function with a numerical example and provides a numerical table of error values in the approximation of a continuous function for different values of n and q.

Keywords — *Korovkin theorem, linear positive operators, Lipschitz conditions, rate of approximation, Voronovskaja-type approximation*

Mathematics Subject Classification (2020) 41A10, 41A36

1. Introduction

The Weierstrass approximation theorem constitutes a foundational result within the field of mathematical analysis. This theorem posits the notion that any continuous function on a closed interval can be uniformly approximated by a polynomial function. The development of the approximation theory commenced in 1885 with contributions from Weierstrass [1] and was further advanced by Bernstein's proof [2] of the Weierstrass theorem for $C[0,1]$ in 1912, where $C[0,1]$ denotes the set of all real-valued continuous functions defined on the closed interval $[0,1]$. The Korovkin theorem [3] provided significant impetus to the study of Bernstein polynomials and their generalizations, and this area of research continues to attract considerable interest in contemporary analysis.

Bernstein operators are of great significance, consequently, many researchers have dedicated their efforts to discovering numerous generalizations in a wide array of spaces and settings. For more details concerning these generalizations, see [4–13]. Moreover, Szász [14], Mirakjan [15], and Kantorovich [16,17] studied generalizations of the Bernstein operator in different spaces and investigated the approximation properties of the operators bearing their names. In 2013, İzgi [18] introduced the bivariate Bernstein operator and analyzed its approximation. In 2020, Çiçek [19] focused on the approximation by q-analysis in weighted spaces using Szász operators. Consequently, numerous researchers have explored Bernstein, Szász-Mirakjan, Baskakov, and Phillips, along with their Kantorovich and Durrmeyer variants. The

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following operators provide the foundation for the construction of the operator introduced in this paper: For any function F belonging to the set $C[0,\infty)$, the set of all real-valued continuous functions defined on $[0,\infty)$, the Szász-Mirakjan operator are defined as follows:

$$S_n(F;q) = e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}F\left(\frac{k}{n}\right), \quad n\in\mathbb{N} \text{ and } 0\le q<\infty \tag{1.1}$$

where the notation $\mathbb{N}$ denotes the set of all positive integers. In 2012, İzgi [20] defined a generalization of the Bernstein polynomials as follows:

$$F_{n,u,v}(F;q) = \left(\frac{n+v}{n+u}\right)^n \sum_{k=0}^{n}\binom{n}{k} q^k \left(\frac{n+v}{n+u}-q\right)^{n-k} F\left(\frac{k}{n}\frac{n+v}{n+u}\right) \tag{1.2}$$

where $n\in\mathbb{N}$, $0\le u\le v$, and $0\le q\le \frac{n+u}{n+v}$. Motivated by the aforementioned operators, the present study proposes a novel generalization of Szász operators and investigates their approximation properties.

The remainder of the paper is organized as follows: Section 2 presents some basic definitions and properties required for the subsequent sections. Section 3 defines a novel generalization of Szász operators, denoted by $N_n(F;q)$, and conducts the calculation of test functions and central moments of the operator $N_n(F;q)$. Section 4 demonstrates that the operator $N_n(F;q)$ satisfies the Korovkin conditions. Additionally, it analyzes the rate of approximation using the modulus of continuity and the approximation properties of functions in the Lipschitz class. Moreover, it signifies Voronovskaja-type asymptotic convergence through rigorous mathematical proof. Section 5 provides numerical illustrations, with the approach further elucidated through the use of graphs. The final section presents the conclusions and recommendations derived from the analysis.

2. Preliminaries

This section presents a definition and two properties to be used in the next sections. Throughout this study, let the notation $\mathbb{R}$ denote the set of all real numbers, and let $[0,A]$, where $A\in\mathbb{R}$, be a closed interval.

Theorem 2.1 (Korovkin Theorem)**.** [21] Suppose that (T_n) is a sequence of linear positive operators satisfying the following conditions on $[0,A]$:

$$T_n(1;q)\rightrightarrows 1, \quad T_n(t;q)\rightrightarrows q, \quad \text{and} \quad T_n\left(t^2;q\right)\rightrightarrows q^2$$

Then, for every function $F\in C[0,A]$ that is bounded on $\mathbb{R}$, the following convergence holds as $n\to\infty$:

$$T_n(F;q)\rightrightarrows F(q), \quad 0\le q\le A$$

or equivalently,

$$\|T_n(F)-F\|_{C[0,A]}\to 0, \quad n\to\infty$$

Here, the notation $\rightrightarrows$ denotes uniform convergence.

Definition 2.2. [22] Let $F\in C[0,A]$. The modulus of continuity for the function F is defined as follows:

$$W(F;\delta) = \sup_{q,t\in[0,A]}|F(q)-F(t)|, \quad \text{where } |t-q|\le\delta$$

Proposition 2.3. [22] Let $F\in C[0,A]$ and $\alpha>0$. Then, the following properties hold for all $\delta>0$ and for all $t,q\in[0,A]$:

i. $W(F;\delta)\ge 0$ and $W(F;\delta)\to 0$ as $\delta\to 0$

ii. $W(\digamma;\alpha\delta) \leq (1+\alpha)W(\digamma;\delta)$

iii. $|\digamma(t)-\digamma(q)| \leq W(\digamma;|t-q|)$

iv. $|\digamma(t)-\digamma(q)| \leq \left(1+\frac{|t-q|}{\delta}\right)W(\digamma;\delta)$

3. A Generalization of the Szász-Mirakjan Operators

This section introduces a novel generalization of the Szász-Mirakjan operators, formulated analogously to the operator provided in (1.2). The sequence of linear positive operators is defined as follows:

$$N_n(\digamma;q) = e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\digamma\left(\frac{k}{n}\frac{n+u}{n+v}\right) \tag{3.1}$$

such that $u,v,n \in \mathbb{N}$, $0 \leq u \leq v$, $\digamma \in C[u,v]$, and $0 \leq q < \infty$. The operator defined in (3.1) is linear and positive. When $u=v$ in (3.1), the operators in (1.1) are obtained. That is, the operator sequence in (3.1) is a generalization of the sequences in (1.1). Afterward, this section calculates the test functions and central moments of the operator $N_n(\digamma;q)$ in (3.1).

Lemma 3.1. Let $F \in C[0,A]$ and $q \in [0,\infty)$. Then, the operator $N_n(\digamma;q)$ satisfies the following properties:

i. $N_n(1;q)=1$

ii. $N_n(t;q) = q - \frac{v-u}{n+v}q$

iii. $N_n\left(t^2;q\right) = q^2 - \frac{2(v-u)n+v^2-u^2}{(n+v)^2}q^2 + \frac{1}{n}\left(\frac{n+u}{n+v}\right)^2 q$

iv. $N_n\left(t^3;q\right) = q^3 - \frac{3(n+v)(n+a)(v-u)+(v-u)^3}{(n+v)^3}q^3 + \frac{3}{n}\left(\frac{n+u}{n+v}\right)^3 q^2 + \frac{1}{n^2}\left(\frac{n+u}{n+v}\right)^3 q$

v. $$N_n\left(t^4;q\right) = q^4 + \left[\frac{-4(v-u)(n+v)^3}{(n+v)^4} + \frac{6(v-u)^2(n+v)^2}{(n+v)^4} + \frac{-4(n+v)(v-u)^3}{(n+v)^4} + \frac{(v-u)^4}{(n+v)^4}\right]q^4$$
$$+\frac{6}{n}\left(\frac{n+u}{n+v}\right)^4 q^3 + \frac{7}{n^2}\left(\frac{n+u}{n+v}\right)^4 q^2 + \frac{1}{n^3}\left(\frac{n+u}{n+v}\right)^4 q$$

PROOF. The proof of *i* and *ii* can be observed from the relevant definition.

iii. From the linearity of the operator $N_n(\digamma;q)$,

$$\begin{aligned} N_n(t^2;q) &= \left(\frac{n+u}{n+v}\right)^2 e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\left(\frac{k}{n}\right)^2 \\ &= \left(\frac{n+u}{n+v}\right)^2\left[e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\frac{k(k-1)}{n^2} + e^{-nx}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\frac{k}{n^2}\right] \\ &= \left(\frac{n+u}{n+v}\right)^2\left[q^2e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!} + \frac{q}{n}e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\right] \\ &= \left(\frac{n+u}{n+v}\right)^2\left[q^2+\frac{q}{n}\right] \end{aligned}$$

Thus,

$$N_n\left(t^2;q\right) = q^2 - \frac{2(v-u)n+v^2-u^2}{(n+v)^2}q^2 + \frac{1}{n}\left(\frac{n+u}{n+v}\right)^2 q$$

iv. The proof is similar to the proof of *i*.

v. From the linearity of the operator,

$$\begin{aligned} N_n(t^4;q) &= \left(\frac{n+u}{n+v}\right)^4 e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\left(\frac{k}{n}\right)^4 \\ &= \left(\frac{n+u}{n+v}\right)^4\left[e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\frac{k^3(k-1)}{n^4}+e^{-nq}\sum_{k=0}^{\infty}\frac{(nq)^k}{k!}\frac{k^3}{n^4}\right] \\ &= \left(\frac{n+u}{n+v}\right)^4\left[q^4+\frac{5}{n}q^3+\frac{4}{n^2}q^2\right]+\left(\frac{n+u}{n+v}\right)^4\left[\frac{1}{n}q^3+\frac{3}{n^2}q^2+\frac{1}{n^3}q\right] \\ &= \left(\frac{n+u}{n+v}\right)^4\left[q^4+\frac{6}{n}q^3+\frac{7}{n^2}q^2+\frac{1}{n^3}q\right] \end{aligned}$$

Hence,

$$N_n\left(t^4;q\right)=q^4+\left[\frac{-4(v-u)(n+v)^3}{(n+v)^4}+\frac{6(v-u)^2(n+v)^2}{(n+v)^4}+\frac{-4(n+v)(v-u)^3}{(n+v)^4}+\frac{(v-u)^4}{(n+v)^4}\right]q^4$$
$$+\frac{6}{n}\left(\frac{n+u}{n+v}\right)^4q^3+\frac{7}{n^2}\left(\frac{n+u}{n+v}\right)^4q^2+\frac{1}{n^3}\left(\frac{n+u}{n+v}\right)^4q$$

□

Lemma 3.2. Let $F\in C[0,A]$ and $q\in[0,\infty)$. Then, the following central moments for the operator $N_n(F;q)$ hold:

i. $N_n((t-q);q)=-\frac{v-u}{n+v}q$

ii. $N_n\left((t-q)^2;q\right)=\frac{(v-u)^2}{(n+v)^2}q^2+\frac{(n+a)^2}{n(n+v)^2}q$

iii. $N_n\left((t-q)^3;q\right)=\frac{-(v-u)^3}{(n+v)^3}q^3+\frac{3}{n}\left[\frac{(n+a)^2(u-v)}{(n+v)^3}\right]q^2+\frac{1}{n^2}\left(\frac{n+u}{n+v}\right)^3q$

iv. $N_n\left((t-q)^4;q\right)=\left[\frac{6(v-u)^2(n+v)^2}{(n+v)^4}+\frac{-4(n+v)(v-u)^2}{(n+v)^4}+\frac{(n+v)(v-u)^5}{(n+v)^4}+\frac{4(n+v)(v-u)^3}{(n+v)^4}\right.$
$\left.+\frac{12(v-u)(n+v)^2}{(n+v)^4}\right]q^4+\frac{6(n+u)^2(u-v)^2}{n(n+v)^4}q^3+\frac{(n+u)^3(3n+7u-4v)}{n^2(n+v)^4}q^2+\frac{1}{n^3}q$

Proof. The proof of *i* and *iii* can be observed from the relevant definition and Lemma 3.1.

ii. From Lemma 3.1 *ii* and *iii*,

$$\begin{aligned} N_n\left((t-q)^2;q\right) &= N_n\left(t^2;q\right)-2qN_n(t;q)+q^2 \\ &= \left(\frac{n+u}{n+v}\right)^2q^2+\left(\frac{n+u}{n+v}\right)^2\frac{q}{n}-2q^2\left(\frac{n+u}{n+v}\right)+q^2 \\ &= \frac{(b-a)^2}{(n+b)^2}q^2+\frac{(n+u)^2}{n(n+v)^2}q \end{aligned}$$

iv. From Lemma 3.1 *ii*-*v*,

$$\begin{aligned} N_n\left((t-q)^4;q\right) &= N_n\left(t^4;q\right)-4qN_n\left(t^3;q\right)+6q^2N_n\left(t^2;q\right)-4q^3N_n(t;q)+q^4 \\ &= \left(\tfrac{n+u}{n+v}\right)^4\left[q^4+\tfrac{6}{n}q^3+\tfrac{7}{n^2}q^2+\tfrac{1}{n^3}q\right] \\ &\quad -4q\left(\tfrac{n+u}{n+v}\right)^3\left[q^3+\tfrac{3}{n}q^2+\tfrac{1}{n^2}q\right]+6q^2\left(\tfrac{n+u}{n+v}\right)^2\left[q^2+\tfrac{1}{n}q\right] \\ &\quad -4q^3\left(\tfrac{n+u}{n+v}\right)[q]+q^4 \end{aligned}$$

Therefore,

$$N_n\left((t-q)^4;q\right) = \left[\frac{6(v-u)^2(n+v)^2}{(n+v)^4} + \frac{-4(n+v)(v-u)^2}{(n+v)^4} + \frac{(n+v)(v-u)^5}{(n+v)^4} + \frac{4(n+v)(v-u)^3}{(n+v)^4} + \frac{12(v-u)(n+v)^2}{(n+v)^4}\right]q^4 + \frac{6(n+u)^2(u-v)^2}{n(n+v)^4}q^3 + \frac{(n+u)^3(3n+7u-4v)}{n^2(n+v)^4}q^2 + \frac{1}{n^3}q$$

□

4. Direct and Local Approximations and Voronovskaja-Type Theorem

This section presents a theorem demonstrating that the operators N_n satisfy direct approximation, specifically, the Korovkin conditions. Then, it proves the local approximation properties, including approximation using the modulus of continuity and the rate of approximation for functions in the Lipschitz class. Moreover, this section investigates the rate of convergence of the operator $N_n(F;q)$ for functions F using the Voronovskaja-type asymptotic convergence formula [23].

Theorem 4.1. Let $F \in C[0,A]$ and F be bounded on $\mathbb{R}$. Then,

$$\lim_{n\to\infty} \|N_nF - F\|_{C[0,A]} = 0$$

Proof. Let $F \in C[0,A]$ and F be bounded on $\mathbb{R}$. It can be observed that $\|N_n 1 - 1\|_{C[0,A]} = 0$. From Lemma 3.1 *ii*,

$$\|N_nt - q\|_{C[0,A]} = \max_{0\le q\le A}\left|q - \frac{v-u}{n+v}q - q\right| = \frac{v-u}{n+1}A$$

Then,

$$\lim_{n\to\infty}\|N_nt - q\|_{C[0,A]} = \lim_{n\to\infty}\frac{v-u}{n+1}A = 0$$

Considering that $A^2 > A$, for $A > 1$, from Lemma 3.1 *iii*,

$$\left\|N_nt^2 - q^2\right\|_{C[0,A]} = \max_{0\le q\le A}\left|q^2 - \frac{2(v-u)n + v^2 - a^2}{(n+1)^2}q^2 + \frac{1}{n}\left(\frac{n+u}{n+v}\right)^2 q - q^2\right|$$

and thus

$$\lim_{n\to\infty}\left|-\frac{2(v-u)n + v^2 - a^2}{(n+1)^2}A^2 + \frac{1}{n}\left(\frac{n+u}{n+v}\right)^2 A\right| \le A^2\lim_{n\to\infty}\left(\frac{2(v-u)n + v^2 - a^2}{(n+1)^2} + \frac{1}{n}\left(\frac{n+u}{n+v}\right)^2\right) = 0$$

Then,

$$\lim_{n\to\infty}\left\|N_nt^2 - q^2\right\|_{C[0,A]} = 0$$

Thus, $N_n(1;q) \rightrightarrows 1$, $N_n(t;q) \rightrightarrows q$ and $N_n(t^2;q) \rightrightarrows q^2$. Therefore, from Theorem 2.1,

$$\lim_{n\to\infty}\|N_nF - F\|_{C[0,A]} = 0$$

□

Theorem 4.2. Let $F \in C[0,A]$. Then, the rate of convergence of $N_n(F;q)$ is obtained as follows:

$$\|N_n(F;q) - F(q)\| \le 2W\left(F;\sqrt{\frac{n^2 + 2b(b+1)n + v^2}{n(n+v)^2}}A\right)$$

Proof. Using the modulus of continuity, the Cauchy-Schwartz inequality, and Lemma 3.2, for $\delta = \sqrt{\frac{n^2+2b(b+1)n+v^2}{n(n+v)^2}}A$,

$$
\begin{aligned}
|N_n(F;q)-F(q)| &= |N_n(F(t)-F(q);q)| \\
&\leq N_n(W(F;|t-q|);q) \\
&\leq \left(1+\frac{1}{\delta}N_n(|t-q|;q)\right)W(F;\delta) \\
&\leq \left(1+\frac{1}{\delta}\sqrt{N_n\left((t-q)^2;q\right)}\right)W(F;\delta) \\
&\leq \left(1+\frac{1}{\delta}\sqrt{\frac{n^2+2b(b+1)n+v^2}{n(n+v)^2}A}\right)W(F;\delta) \\
&= 2W\left(F;\sqrt{\frac{n^2+2b(b+1)n+v^2}{n(n+v)^2}A}\right)
\end{aligned}
$$

□

Theorem 4.3. Let $F \in C[0,A]$ be differentiable and its derivative be continuous and bounded on the interval $[0,A]$. Then, for all $n \in \mathbb{N}$,

$$|N_n(F;q)-F(q)| \leq \frac{M}{n}+2\delta W(F';\delta)$$

where $\delta=\sqrt{\frac{n^2+2v(v+1)n+v^2}{n(n+v)^2}A}$.

Proof. Let $F \in C[0,A]$ be differentiable and its derivative be continuous and bounded on the interval $[0,A]$. Then, the Mean Value Theorem guarantees the existence of h between t and q, where $t,q \in [0,A]$ and $t \neq q$, such that

$$F'(h)=\frac{F(t)-F(q)}{t-q}$$

Adding $-F'(q)+F'(q)$ to the left side of the above equation yields the following equality:

$$F(t)-F(q)=(t-q)F'(q)+(t-q)(F'(h)+F'(q))$$

If the operator $N_n(F,q)$ is applied to the last equality, then the following result is obtained:

$$
\begin{aligned}
N_n(F;q)-F(q) &= F'(q)N_n((t-q);q)+N_n((t-q)(F'(h)+F'(q));q) \\
&= F'(q)\frac{-q(v-u)}{n+v}+N_n((t-q)(F'(h)-F'(q));q)
\end{aligned}
$$

In addition, since h lies between t and q, $|h-q| \leq |t-q|$. According to the modulus of continuity, the following result is derived:

$$
\begin{aligned}
|N_n(F;q)-F(q)| &\leq \frac{|-q(v-u)F'(q)|}{n+v}+N_n(|t-q|\,|F'(h)-F'(q)|\,;q) \\
&\leq \frac{|-q(v-u)F'(q)|}{n+v}+N_n\left(|t-q|\left(1+\frac{|t-q|}{\delta}\right);q\right)W(F';\delta_n) \\
&\leq \frac{|-q(v-u)F'(q)|}{n+v}+N_n\left(|t-q|+\frac{(t-q)^2}{\delta_n};q\right)W(F';\delta_n)
\end{aligned}
$$

Using the Cauchy-Schwartz inequality,

$$|N_n(F;q)-F(q)| \leq \frac{|-q(v-u)f'(q)|}{n+v}+\left[\left(N_n\left((t-q)^2;q\right)\right)^{\frac{1}{2}}+\frac{1}{\delta_n}N_n\left((t-q)^2;q\right)\right]W(F';\delta_n)$$

By taking the maximum value of $N_n\left((t-q)^2;q\right)$, then there exists a number $M>0$ such that $|F'(q)|<M$, since F' is bounded on the whole real axis. Therefore, for $\delta=\sqrt{\frac{n^2+2v(v+1)n+v^2}{n(n+v)^2}}A$,

$$|N_n(F;q)-F(q)|\leq\frac{M}{n}+2\delta W(F';\delta)$$

□

The following theorem investigates the rate of convergence of the operator $N_n(F;q)$ for functions of the Lipschitz class $\mathrm{Lip}_M(\alpha)$, where $M>0$ and $0\leq\alpha\leq 1$ [22]. Here, a function F belongs to $\mathrm{Lip}_M(\alpha)$ if

$$|F(t)-F(q)|\leq M|t-q|^{\alpha},\quad \text{for all } t,q\in\mathbb{R}$$

Theorem 4.4. If the function F satisfies the Lipschitz condition, then

$$|N_n(F;q)-F(q)|\leq M\left(\frac{n^2+2b(b+1)n+v^2}{n(n+v)^2}A^2\right)^{\frac{\alpha}{2}}$$

Proof. Assume that the function F satisfies the Lipschitz condition. Since $N_n(1;q)=1$ and the operator $N_n(F;q)$ is linear, then

$$|N_n(F;q)-F(q)|=|N_n(F;q)-F(q)N_n(1;q)|\leq(N_n(|F(t)-F(q)|;q)$$

Thus, from the Lipschitz condition, $|N_n(F;q)-F(q)|\leq MN_n\left(|t-q|^{\alpha};q\right)$. From the Hölder inequality, $N_n\left(|t-q|^{\alpha};q\right)\leq N_n\left((t-q)^2;q\right)^{\frac{\alpha}{2}}$. Therefore, from Lemma 3.2,

$$|N_n(F;q)-F(q)|\leq M\left(\frac{n^2+2b(b+1)n+v^2}{n(n+v)^2}A^2\right)^{\frac{\alpha}{2}}$$

□

The following theorem presents the rate of convergence of the operator $N_n(F;q)$ for functions F using the Voronovskaja-type asymptotic convergence formula [23].

Theorem 4.5. If $F\in C[0,A]$ and the functions F,F', and F'' are bounded on $[0,A]$, then

$$\lim_{n\to\infty}(n+v)\left(N_n(F;q)-F(q)\right)=(u-v)F'(q)+\frac{1}{2}qF''(q)$$

Proof. The Taylor expansion of the function F at a point q is as follows:

$$F(t)=F(q)+\frac{1}{1!}F'(q)(t-q)+\frac{1}{2!}F''(q)(t-q)^2+\frac{1}{3!}F'''(q)(t-q)^3+\frac{1}{4!}F^{(4)}(q)(t-q)^4+\cdots$$

Then,

$$F(t)-F(q)=\frac{1}{1!}F'(q)(t-q)+\frac{1}{2!}F''(q)(t-q)^2+(t-q)^2\mu(t-q)$$

where

$$\mu(t-q)=\left(\frac{1}{3!}F'''(q)(t-q)+\frac{1}{4!}F^{(4)}(q)(t-q)^2+\dots\right)$$

Since $\lim_{t\to q}\mu(t-q)=0$, then $\mu(t-q)$ is bounded. Thus,

$$N_n(F;q)-F(q)=N_n((t-q);q)F'(q)+\frac{1}{2}N_n\left((t-q)^2;q\right)F''(q)+N_n\left((t-q)^2\mu(t-q);q\right)$$

If both sides of the last equation are multiplied by $n+v$, then from Lemma 3.2

$$\begin{aligned}(n+v)\left(N_n(F;q)-F(q)\right)&=(n+v)\left(-\tfrac{v-u}{n+v}q\right)F'(q)+\tfrac{1}{2}(n+v)\left(\tfrac{(v-u)^2}{(n+v)^2}q^2+\tfrac{(n+a)^2}{n(n+v)^2}q\right)F''(q)\\&\quad+(n+v)N_n\left((t-q)^2\mu(t-q);q\right)\end{aligned}\tag{4.1}$$

From the Cauchy-Schwartz inequality,

$$(n+v)N_n\left((t-q)^2\mu(t-q);q\right) \leq \sqrt{(n+v)^2N_n\left((t-q)^4;q\right)} \times \sqrt{N_n(\mu(t-q))^2;q)}$$

Since $\mu(t-q) \in C[0,A]$, then $\mu(t-q)^2 \in C[0,A]$. From Theorem 4.3, $N_n(\mu(t-q))^2;q)^2 \to 0$ as $n \to \infty$. If the limit of both sides of the (4.1) is taken, then

$$\lim_{n\to\infty}(n+v)\left(N_n(F;q) - F(q)\right) = (u-v)F'(q) + \frac{1}{2}qF''(q)$$

□

5. Numerical Examples

This section compares the operator $N_n(F;q)$ and the classical Szász-Mirakjan operator $S_n(F;q)$ in the approximation of the specified function. It displays the approximation of the operators $N_n(F;q)$ and $S_n(F;q)$ to the function $F(q) = e^{\frac{-q}{5}} \sin 3\pi q$ in Figure 1, where $n = 30$, $u = 0.9$, and $v = 1$, and in Figure 2, where $n = 300$, $u = 1$, and $v = 10$. Moreover, this section depicts the approximation of the operators $N_n(F;q)$ and $S_n(F;q)$ to the function $g(q) = \frac{e^{\frac{-q}{3}} \sin \pi q}{1+q}$ in Figure 3, where $n = 30$, $u = 0,9$, and $v = 1$, and in Figure 4, where $n = 300$, $u = 1$, and $v = 10$. In the figures, the operator $N_n(F;q)$) is shown in red and the operator $S_n(F;q)$ in blue.

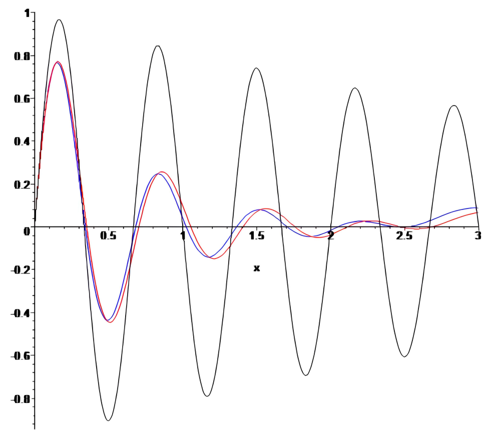

Figure 1. Convergence of the approximation to $F(q) = e^{\frac{-q}{5}} \sin 3\pi q$ with $n = 30$, $u = 0.9$, and $v = 1$

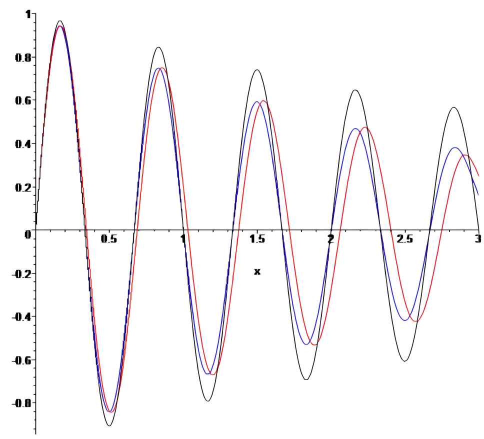

Figure 2. Convergence of the approximation to $F(q) = e^{\frac{-q}{5}} \sin 3\pi q$ with $n = 300$, $u = 1$, and $v = 10$

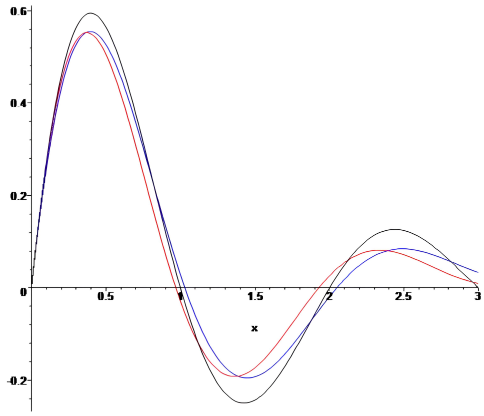

Figure 3. Convergence of the approximation to $g(q) = \frac{e^{\frac{-q}{3}} \sin \pi q}{1+q}$ with $n = 30$, $u = 0.9$, and $v = 1$

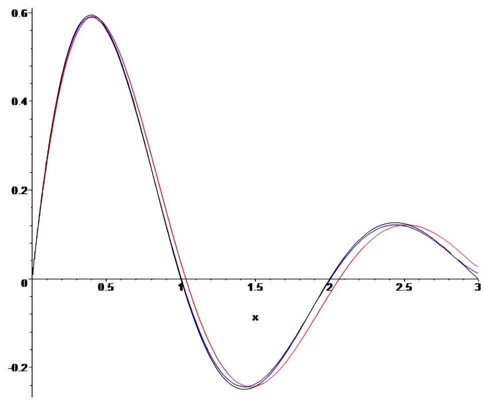

Figure 4. Convergence of the approximation to $g(q) = \frac{e^{\frac{-q}{3}} \sin \pi q}{1+q}$ with $n = 300$, $u = 1$, and $v = 10$

As demonstrated by the figures depicting the approximation of the functions $F(q) = e^{\frac{-q}{5}} \sin 3\pi q$ and $g(q) = \frac{e^{\frac{-q}{3}} \sin \pi q}{1+q}$, the operator $N_n(F; q)$ provides a superior approximation compared to the operator $S_n(F; q)$. Furthermore, the following table presents numerical values of the margins of error in the approximation. It can be observed from Table 1 that as the value of n increases, the distance between $N_n(F; q)$ and $F(q)$ converges to zero.

Table 1. Numerical values of $|N_n(F; q) - F(q)|$ for some q and n values

n/q	**0.2**	**0.5**	**0.8**	**1.2**	**1.5**	**1.8**
5	0.6834	0.9359	0.8137	0.7479	0.7396	0.6681
50	0.1418	0.3224	0.4221	0.4783	0.545	0.5384
100	0.075	0.1794	0.2477	0.3011	0.3596	0.3721
200	0.0386	0.0948	0.1347	0.1707	0.2095	0.2229
500	0.0157	0.0392	0.0567	0.0739	0.0923	0.0997
990	0.008	0.02	0.0291	0.0383	0.0481	0.0525

Afterward, this section presents a numerical table that illustrates the convergence behavior of the operators $N_n(F; q)$ and $S_n(F; q)$ to the function $F(q)$. The points used in Tables 1 and 2 correspond to those at which the greatest distances between the operators and the function are observed in the plots. In Table 2, $K(q)$ represents the rates of convergence of the operators $N_n(F; q)$ and $S_n(F; q)$ to the function F and is defined by $K(q) = \left| \frac{N_n(F;q) - F(q)}{S_n(F;q) - F(q)} \right|$. As observed in Table 2, the rate is approximately 1, that is, it is a constant number. This demonstrates that the convergence behaviors of both operators to the function are equivalent.

Table 2. Numerical values of $K(q)$ for different q and n values

n/q	**0.2**	**0.5**	**1.2**	**1.8**	**2.5**
5	0.9886	1.0016	1	1.0017	1.0112
50	0.99	0.9974	0.9951	1.0025	0.9998
100	0.9911	0.9982	0.9936	1.0037	0.9997
200	0.9918	0.9988	0.993	1.0048	0.9995
500	1.0006	1	0.9999	1.0009	0.9998
990	1.0003	1	0.9999	0.9999	0.9999

6. Conclusion

This study proposes a novel extension to Szász operators and conducts a comprehensive investigation into their convergence rate and approximation properties on closed subintervals of the real axis. It successfully carries out the analysis of the Voronovskaja-type approximation and local approximation properties of the new Szász operator, using functions satisfying the Lipschitz condition. Furthermore, this paper analyzes the rate of convergence for the functions F and F' in detail by the modulus of continuity. Then, it uses a numerical example to graphically illustrate the approximation of the new generalized Szász operator to a continuous function, accompanied by a numerical table presenting error values for different values of n and q. The findings demonstrate that the theoretical and practical efficacy of the proposed operator provides a solid foundation for future research in approximation theory and numerical analysis. Subsequent studies may be conducted in various spaces, including complex spaces, L_p spaces, and weighted spaces. Moreover, the operator $N_n(F;q)$ can be combined with other operators to derive a new generalizations.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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