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Variational Principle for Studying the Long-Term Stability of Structural Materials Subjected to Neutron Irradiation Taking into Account Geometric Nonlinearity

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Abstract: In various branches of modern engineering, including nuclear power engineering and rocket and space technology, structural elements in various forms are widely used. During operation, they can be subjected to both force and non-mechanical loads (thermal, radiation). During operation, they are subjected to radiation loads. Thus, neutron fluxes, penetrating deep into the material, radically change its mechanical properties. Moreover, when irradiated for several years and at high temperatures, as is the case in nuclear reactors, creep deformations become significant, and irradiation and temperature affect the creep of materials differently. Therefore, in those areas of technology where neutron radiation and high temperatures are present, when designing structures, it is necessary to take into account the effect of radiation and temperature on the mechanical properties of the material. The task is further complicated when taking into account geometric nonlinearity.It is practically impossible to obtain an exact solution to such problems, therefore the development of approximate methods is of particular importance. In nonlinear problems, one of the effective approximate methods of solution is the variational method. To solve long-term stability problems by the variational method, it is necessary to develop these methods to be able to take into account geometric nonlinearity and changes in mechanical characteristics. This means that it is necessary to construct a functional that would take into account changes in the mechanical characteristics of the body, taking into account creep deformation and geometric nonlinearity.The article proposes a functional for studying the stability of structural materials (the stress-strain state (SSS) of a body) under neutron irradiation, taking into account geometric nonlinearity and creep deformations. It is proven that the Euler equations of the functional take the form of an equation describing the stress-strain state of a thin shell under irradiation, taking into account geometric nonlinearity and creep deformations.

Keywords: Creep deformation, Variational principle, Neutron, Deformation components, Stress-strain state

Introduction

The development of nuclear energy, rocket, and space technology requires a particularly careful approach to the issues of strength and stability of structures under the influence of radiation. The study of the effects of neutron radiation on the strength and stability of structures is of particular importance. Neutron streams, penetrating deep into the material, sharply alter its mechanical properties. Moreover, when radiation exposure continues for several years and the temperature is high, as is the case in nuclear reactors, creep deformations become significant, with radiation and temperature affecting the creep of materials differently.

Radiation-induced changes in the mechanical properties and swelling of structural materials can significantly affect the performance of active zone elements, and these factors should be considered when determining the kinetics of stressed and deformed states (Likhachev & Pupko, 1975). Therefore, many issues related to the calculation of the stress-strain state and the determination of performance considering the specifics of the structures require further development and improvement. In particular, it is necessary to develop mathematical methods for solving systems of equations that describe the kinetics of the stress-strain state of structures. A more detailed study of the patterns of deformation and swelling processes of dividing and structural materials is needed (Likhachev & Pupko, 1975). In connection with the widespread use of nuclear energy for peaceful purposes, many scientists began to address the issues of the impact of radiation on the mechanical properties of materials.

The first observations of measurements of elastic properties as a result of irradiation were made for graphite (Wassilew C. et al., 1987). It was found that irradiation of graphite in a reactor leads to the formation of a harder and more brittle material. The Young's modulus increases by approximately three times after irradiation with a dose. The first theoretical calculations were carried out for metals by Denis (1952).

Detailed calculations, taking into account the relaxation of the nearest neighboring atoms, led to the following conclusions: the presence of a small fraction of implanted atoms and vacancies results in a significant increase in the elastic modulus of copper—by 5–7% per 1% of implanted atoms. It was established that in the case of vacancies alone, the reduction in the modulus occurs mainly due to the influence of the entire mass of the material. Consequently, the increase in the elastic modulus is primarily associated with the presence of implanted atoms. A considerable increase in the modulus was experimentally observed by Thompson and Holmes (1956) in copper monocrystals after irradiation in a reactor. In the book by Glesston and Edlund (1954), the following formula is derived, which shows the law by which the intensity of radiation decreases with depth in the material:

$$J_z = J_0 e^{-N\sigma z} \text{ or } J(z) = J_0 \exp(-\mu z)$$

where σ -is the microscopic cross-section, $N\sigma$ -is the macroscopic cross-section, N is the number of nuclei in $1 sm^3$ of the irradiated material, $\mu = const$ - is the macroscopic cross-section, $[\mu] = 1/cм$, J_0 - is the irradiation intensity on the surface, J_z - is at a depth from the surface. The value of σ - depends on the energy of the bombarding neutron. The macroscopic cross-section is defined as:

$$\mu = N \cdot \sigma \text{ or } \mu = N_0 \cdot \frac{\rho}{A} \cdot \sigma ,$$

where $N_0 = 6{,}02 \cdot 10^{23}$ -is Avogadro's number, ρ -is the density of the irradiated material, A -is the atomic weight of the material.

Many works are devoted to studying the effect of irradiation on the radiation creep of structural materials and fuel elements (Gulgazli& Efendiev, 2017; Gorokhovet al., 2020; Breslavsky & Tatarinova, 2023; Onimus et al., 2020). In the book by Likhachev and Pupko (1975) the creep strain rate for an irradiated body is presented in the following form:

$$\dot{\varepsilon}_{ij}^{c} = \frac{3}{2} F_c \frac{S_{ij}}{\sigma_u}$$

where - S_{ij} is the stress deviator components, σ_u - is the stress intensity, $F_c = F_c(\sigma_u, T, D)$ is determined from experience, $\dot{\varepsilon}_{ij}^{c}$ -is the creep deformation rate, T- is the temperature, D- is the radiation dose.

Formulation of the Variational Principle

Basic Ratios

Let a body of volume V be irradiated by an intense flux of neutrons: on part S_σ of the surface, surface forces are specified, and on the remaining part S_u of the surface, displacements are specified. Then the equilibrium equation, taking into account geometric nonlinearity, the relationship between the rates of the components of deformations and stresses, and the boundary conditions in the Cartesian coordinate system have the following form (Volmir,1972; Amenzade, 1976; Aliyev et, al., 2024):

$$\left[\sigma_{ij}(u_{\alpha,i} + \delta_{\alpha i})\right]_{,j} = 0 , \tag{1}$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} + \dot{\varepsilon}_{ij}^{c} + \dot{\varepsilon}_{ij}^{v} , \tag{2}$$

$$\dot{\varepsilon}_{ij}^{e} = \frac{1}{E}\left[\dot{\sigma}_{ij} - \nu(3\dot{\sigma}\delta_{ij} - \dot{\sigma}_{ij})\right] - \frac{1}{E}\left[\left(3\delta_{ij}\sigma - \sigma_{ij}\right)\left(\frac{\partial \nu}{\partial T}\dot{T} + \frac{\partial \nu}{\partial D}\dot{D}\right)\right] -$$

$$-\frac{1}{E^2}\left[\sigma_{ij} - \nu\left(3\sigma\delta_{ij} - \sigma_{ij}\right)\right]\left(\frac{\partial E}{\partial T}\dot{T} + \frac{\partial E}{\partial D}\dot{D}\right),$$

$$\dot{\varepsilon}_{ij}^{p} = \left[F_\sigma \dot{\sigma}_u + F_T \dot{T} + F_D \dot{D}\right] \cdot S_{ij} ,$$

$$\dot{\varepsilon}_{ij}^{c} = \frac{3S_{ij}}{2\sigma_u} \cdot F_c ,$$

$$\dot{\varepsilon}_{ij}^{v} = \frac{\partial(\alpha T)}{\partial T} \cdot \dot{T}\delta_{ij} + \frac{1}{3}\dot{S}\delta_{ij} ,$$

$$\begin{cases} \sigma_{ij}\left(\delta_{\alpha i} + u_{\alpha,i}\right) \cdot n_j = \overline{N}_\alpha & на \ S_\sigma , \\ u_i = \overline{u}_i & на \ S_u , \end{cases} \tag{3}$$

where σ_{ij} -are the stress tensor components, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} + \dot{\varepsilon}_{ij}^{c} + \dot{\varepsilon}_{ij}^{v}$ are the strain tensor components, ε_{ij}^{e} - and ε_{ij}^{p} - are, respectively, the elastic and plastic strain components, ε_{ij}^{c} -are the creep strain components, ε_{ij}^{v} - are the volume strain components that arise due to temperature and irradiation, $E(D,T)$ - is Young's modulus, $\nu(D,T)$ - is Poisson's ratio, S_{ij} - are the stress deviator components, α - is the coefficient of thermal expansion, $\overline{N}_i$ -are the components of the surface force vector, $\overline{u}_i$ - are the displacements on the surface S_u, a δ_{ij} -are the Kronecker symbols, $\sigma_u = \sqrt{\frac{3}{2}S_{ij}S_{ij}}$ -are the intensities of shear stresses, $\sigma = \frac{1}{3}\sigma_{ii}$ - is the hydrostatic pressure, $F_\sigma(\sigma_u,T,D), F_T(\sigma_u,T,D), F_D(\sigma_u,T,D)$ do not depend on the type of stress state and are determined from experience under uniaxial tension as follows (Likhachev & Pupko, 1975):

$$F_\sigma = \frac{3}{2\sigma_u}\left(\frac{1}{E_k} - \frac{1}{E}\right),$$

where $E_k(\sigma_u,T,D) = \frac{d\sigma_0}{d\varepsilon_0}$ is the tangent modulus, σ_0 and ε_0 are the stress and strain under uniaxial tension, respectively,

$$F_T = \frac{3}{2\sigma_u}\left(\beta + \frac{1}{E^2} \cdot \frac{dE}{dT} \cdot \sigma_u\right),$$

where $\beta(\sigma_u,T,D) = \frac{d\varepsilon_0}{dT}$ is the coefficient of temperature compliance, $F_D = \frac{3}{2\sigma_u}\gamma$, $\gamma(\sigma_u,T,D) = \frac{d\varepsilon_0}{dD}$ is the coefficient of radiation compliance, $F_c(\sigma_u,T,D)$ - is determined from experience, $S(T,D)$ - is the swelling function, n_α - are the components of the normal to the surface of the body. The dot over the values means differentiation with respect to time, and the comma means covariant differentiations. The summation from 1 to 3 is performed over repeating indices.

Functional of the Variational Principle

The proposed functionality for this task is as follows:

$$J=\int_V\left\{\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}+\frac{1}{2}\sigma_{ij}\dot{u}_{\alpha,j}\dot{u}_{\alpha,i}-\dot{\sigma}_{ij}\left[\frac{1}{2E}\left(\dot{\sigma}_{ij}-\nu\left(3\dot{\sigma}\delta_{ij}-\dot{\sigma}_{ij}\right)\right)-\frac{1}{E}\left(3\sigma\delta_{ij}-\sigma_{ij}\right)\left(\frac{\partial\nu}{\partial T}\dot{T}+\frac{\partial\nu}{\partial D}\dot{D}\right)-\right.\right.$$

$$-\frac{1}{E^2}\left(\sigma_{ij}-\nu(3\sigma\delta_{ij}-\sigma_{ij})\right)\cdot\left(\frac{\partial E}{\partial T}\dot{T}+\frac{\partial E}{\partial D}\dot{D}\right)+\left(\frac{1}{2}F_\sigma\dot{\sigma}_u+F_T\dot{T}+E_D\dot{D}\right)S_{ij}+\frac{3}{2}F_c\cdot\frac{S_{ij}}{\sigma_u}+$$

$$\left.\left.+\frac{\partial(\alpha T)}{\partial T}\dot{T}\delta_{ij}+\frac{1}{3}\dot{S}\delta_{ij}\right]\right\}dV-\left(\int_{S_\sigma}\dot{\bar{N}}_i\dot{u}_i ds+\int_{S_u}\left(\dot{u}_i-\dot{\bar{u}}\right)\dot{N}_i ds\right) \qquad (4)$$

Proof

Let us prove that the Euler equations of the proposed functional (4) give the system (1), (2) and (3). Let us calculate the first variation of the functional (4). Let us assume that only the speeds of displacements and stresses vary.

$$\delta J=\int_V\left\{\dot{\sigma}_{ij}\delta\dot{\varepsilon}_{ij}+\dot{\varepsilon}_{ij}\delta\dot{\sigma}_{ij}+\sigma_{ij}\dot{u}_{\alpha,i}\delta\dot{u}_{\alpha,j}-\left[\frac{1}{2E}\left(\dot{\sigma}_{ij}-\nu\left(3\dot{\sigma}\delta_{ij}-\dot{\sigma}_{ij}\right)\right)-\right.\right.$$

$$-\frac{1}{E}\left(3\sigma\delta_{ij}-\sigma_{ij}\right)\left(\frac{\partial\nu}{\partial T}\dot{T}+\frac{\partial\nu}{\partial D}\dot{D}\right)-\frac{1}{E^2}\left(\sigma_{ij}-\nu(3\sigma\delta_{ij}-\sigma_{ij})\right)\times$$

$$\times\left(\frac{\partial E}{\partial T}\dot{T}+\frac{\partial E}{\partial D}\dot{D}\right)+\left(\frac{1}{2}F_\sigma\dot{\sigma}_u+F_T\dot{T}+F_D\dot{D}\right)S_{ij}+\frac{3}{2}F_c\cdot\frac{S_{ij}}{\sigma_u}+$$

$$\left.+\frac{\partial(\alpha T)}{\partial T}\dot{T}\delta_{ij}+\frac{1}{3}\dot{S}\delta_{ij}\right]\delta\dot{\sigma}_{ij}-$$

$$\left.-\dot{\sigma}_{ij}\left[\frac{1}{2E}\left(\delta\dot{\sigma}_{ij}-\nu\left(3\delta_{ij}\delta\dot{\sigma}-\delta\dot{\sigma}_{ij}\right)\right)+\frac{1}{2}F_\sigma S_{ij}\delta\dot{\sigma}_u\right]\right\}dV-$$

$$-\int_{S_\sigma}\dot{\bar{N}}_i\delta\dot{u}_i ds+\int_{S_u}\left(\dot{u}_i-\dot{\bar{u}}\right)\delta\dot{N}_i ds\,. \qquad (5)$$

Here we took into account $\delta\dot{\bar{N}}_i=0$ - on S_σ and $\delta\dot{\bar{u}}_i=0$ on S_u. We transform the integrand. It is known that

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}+u_{\alpha,i}u_{\alpha,j}\right)$$

then

$$\dot{\varepsilon}_{ij}=\frac{1}{2}\left(\dot{u}_{i,j}+\dot{u}_{j,i}+\dot{u}_{\alpha,i}u_{\alpha,j}+u_{\alpha,i}\dot{u}_{\alpha,j}\right).$$

$$\delta\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\delta\dot{u}_{i,j} + \delta\dot{u}_{j,i} + u_{\alpha,j}\delta\dot{u}_{\alpha,i} + u_{\alpha,i}\delta\dot{u}_{\alpha,j}\right) =$$

$$= \frac{1}{2}\left(\delta\dot{u}_{i,j} + u_{\alpha,i}\delta\dot{u}_{\alpha,j}\right) + \frac{1}{2}\left(\delta\dot{u}_{j,i} + u_{\alpha,j}\delta\dot{u}_{\alpha,i}\right) \tag{6}$$

In (6) taking into account $\delta\dot{u}_{i,j} = \delta_{\alpha i}\delta\dot{u}_{\alpha,j}$ and $\delta\dot{u}_{j,i} = \delta_{\alpha j}\delta\dot{u}_{\alpha,i}$, we then obtain:

$$\delta\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\delta_{\alpha i} + u_{\alpha,i}\right)\delta\dot{u}_{\alpha,j} + \frac{1}{2}\left(\delta_{\alpha j} + u_{\alpha,j}\right)\delta\dot{u}_{\alpha,i} \tag{7}$$

Taking into account (7), we transform the first term (5) as follows:

$$\int_V \dot{\sigma}_{ij}\delta\dot{\varepsilon}_{ij}dV = \int_V \dot{\sigma}_{ij}\left(\delta_{\alpha i} + u_{\alpha,i}\right)\delta\dot{u}_{\alpha,j}dV =$$

$$\int_V \left[\dot{\sigma}_{ij}\left(\delta_{\alpha j} + u_{\alpha,i}\right)\delta\dot{u}_{\alpha}\right]_{,j} dV - \int_V \left[\dot{\sigma}_{ij}\left(\delta_{\alpha i} + u_{\alpha,i}\right)\right]_{,j} \delta\dot{u}_{\alpha}dV \tag{8}$$

When obtaining (8), we took into account the symmetry of the tensor σ_{ij} . Applying the Gauss-Ostrogradsky theorem, from (8) we obtain:

$$\int_V \dot{\sigma}_{ij}\delta\dot{\varepsilon}_{ij}dV = \int_{S_\sigma + S_u} \dot{\sigma}_{ij}\left(\delta_{\alpha i} + u_{\alpha,i}\right)\delta\dot{u}_{\alpha}n_j dS - \int_V \left[\dot{\sigma}_{ij}\left(\delta_{\alpha i} + u_{\alpha,i}\right)\right]_{,j} \delta\dot{u}_{\alpha}dV , \tag{9}$$

where n_j - are the components of the normal to the surface *S*.

We transform the third term in (5) as follows:

$$\int_V \sigma_{ij}\dot{u}_{\alpha,i}\delta\dot{u}_{\alpha,j}dV = \int_V \left[\sigma_{ij}\dot{u}_{\alpha,i}\delta\dot{u}_{\alpha}\right]_{,j} dV - \int_V \left[\sigma_{ij}\dot{u}_{\alpha,i}\delta\dot{u}_{\alpha}\right]_{,j} \delta\dot{u}_{\alpha}dV =$$

$$= \int_{S_\sigma + S_u} \sigma_{ij}\dot{u}_{\alpha,i}\delta\dot{u}_{\alpha}n_j dS - \int_V \left[\sigma_{ij}\dot{u}_{\alpha,i}\right]_{,j} \delta\dot{u}_{\alpha}dV \tag{10}$$

Let us transform the following terms into (5):

$$\int_V \left[\frac{1}{2E}\left(\dot{\sigma}_{ij} - \nu\left(3\dot{\sigma}\delta_{ij} - \dot{\sigma}_{ij}\right)\right)\delta\dot{\sigma}_{ij} + \frac{1}{2E}\left(\delta\dot{\sigma}_{ij} - \nu\left(3\delta_{ij}\delta\dot{\sigma} - \delta\dot{\sigma}_{ij}\right)\right)\dot{\sigma}_{ij}\right]dV =$$

$$= \int_V \left[\frac{1}{E}\left(\dot{\sigma}_{ij} - \nu\left(3\dot{\sigma}\delta_{ij} - \dot{\sigma}_{ij}\right)\right)\delta\dot{\sigma}_{ij}\right]dV . \tag{11}$$

In obtaining (11) we used the following equalities:

$$\dot{\sigma}\delta\dot{\sigma}_{ij}\delta_{ij} = 3\dot{\sigma}\delta\dot{\sigma}, \quad \dot{\sigma}_{ij}\delta\dot{\sigma}\delta_{ij} = 3\dot{\sigma}\delta\dot{\sigma},$$

and also took into account that σ_{ij} is a symmetric tensor.

$$\int_V \left(\frac{1}{2} F_\sigma \dot{\sigma}_u S_{ij} \delta\dot{\sigma}_{ij} + F_\sigma \dot{\sigma}_{ij} S_{ij} \delta\dot{\sigma}_u \right) dV = \int_V F_\sigma \dot{\sigma}_u S_{ij} \delta\dot{\sigma}_{ij} dV \tag{12}$$

When receiving (12) we used the fact that

$$\frac{1}{2} F_\sigma \dot{\sigma}_u S_{ij} \delta\dot{\sigma}_{ij} = \frac{1}{2} F_\sigma \left(\sqrt{\frac{3}{2} S_{\alpha\beta} S_{\alpha\beta}} \right)^{\cdot} S_{ij} \cdot \delta\left(\dot{S}_{ij} + \dot{\sigma}\delta_{ij}\right) =$$

$$= \frac{3}{4} F_\sigma \cdot \frac{1}{\sigma_u} \dot{S}_{\alpha\beta} S_{\alpha\beta} S_{ij} \delta\dot{S}_{ij},$$

$$\frac{1}{2} F_\sigma \cdot \frac{3}{2} \cdot \frac{1}{\sigma_u} \cdot \dot{S}_{\alpha\beta} S_{\alpha\beta} S_{ij} \delta\dot{\sigma}\delta_{ij} = 0,$$

$$\frac{1}{2} F_\sigma \cdot \dot{\sigma}_{ij} S_{ij} \delta\dot{\sigma}_u = \frac{1}{2} F_\sigma \cdot \cdot S_{ij} \left(\dot{S} + \dot{\sigma}\delta_{ij}\right) \delta \left(\sqrt{\frac{3}{2} S_{\alpha\beta} S_{\alpha\beta}} \right)^{\bullet} =$$

$$= \frac{3}{4} F_\sigma \cdot \frac{1}{\sigma_u} \dot{S}_{\alpha\beta} S_{\alpha\beta} S_{ij} \delta\dot{S}_{ij}.$$

Thus, for the first variation, taking into account the transformations (8), (10) (11), (12) we obtain:

$$\delta J = \int_{S_\sigma} \left[\sigma_{ij} \left(u_{\alpha,i} + \delta_{\alpha i} \right) n_j - \overline{N}_\alpha \right]^{\bullet} \delta\dot{u}_\alpha dS + \int_V \left[\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^e_{ijj} - \dot{\varepsilon}^p_{ij} - \dot{\varepsilon}^c_{ij} - \dot{\varepsilon}^Y_{ij} \right) \delta\dot{\sigma}_{ij} \right] dV -$$

$$- \int_V \left[\sigma_{ij} \left(u_{\alpha,i} + \delta_{\varepsilon i} \right) \right]_{,j} \delta\dot{u}_\alpha dV - \int_{Su} \left(\dot{u}_i - \dot{\overline{u}}_i \right) \delta\dot{N}_i ds.$$

Equating δJ to zero and using the fundamental lemma of the calculus of variations, we obtain (Elsholts,1969):

$$\left[\sigma_{ij} \left(u_{\alpha,i} + \delta_{\varepsilon i} \right) \right]^{\bullet}_{,j} = 0, \tag{13}$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ijj} + \dot{\varepsilon}^p_{ij} + \dot{\varepsilon}^c_{ij} + \dot{\varepsilon}^v_{ij}, \tag{14}$$

$$\begin{array}{ll} \left[\sigma_{ij} \left(u_{\alpha,i} + \delta_{\alpha i} \right) n_j \right]^{\bullet} = \dot{\overline{N}}_\alpha & на \ S_\sigma \\ \dot{u}_i = \dot{\overline{u}}_i & на \ S_u \end{array} \tag{15}$$

Integrating (13), (14), (15) over time and taking into account that there was no irradiation at the initial moment of time, we obtain the complete system (1), (2), (3). Thus, we have shown that the stationary state of the proposed functional is achieved on functions describing the stress-strain state of a deformable solid under irradiation, taking into account geometric nonlinearity and creep deformation.

Long-Term Stability of the Rod

Statement of the Problem

A prismatic rod of constant thickness $2h$ is rigidly fixed at the ends and is irradiated with a neutron flux so that the irradiation intensity is constant along the lateral surface. It is assumed that the rod is thin, i.e. $\gamma = h/l << 1$ ($2l$ is the length of the rod), temperature $T = const$, Poisson's coefficient $\nu = const$. We assume that the irradiation is one-sided.

Let us take a right-hand Cartesian coordinate system *Oxyz* so that the *x*-axis is directed along the rod axis and passes through the center of gravity of the cross-section, and the *y*- and *z*-axes are directed along the principal axes of inertia of the cross-section (Figure 1).

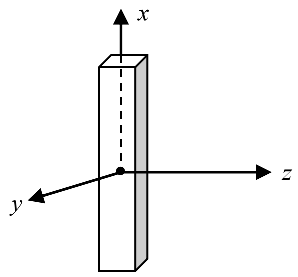

Figure 1.

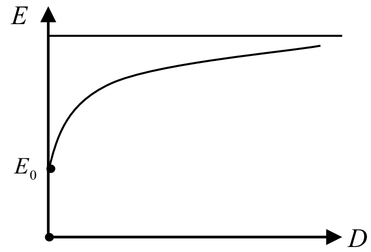

Figure 2.

We make the following assumptions: 1) The points of the middle surface do not move along the *x*-axis, and the geometric nonlinearity exists only in the normal direction *z*; 2) The Kirchhoff-Love hypothesis is satisfied, i.e., $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$, where the *z*-coordinate is directed along the normal; 3) The points of the beam do not move along the *y*-axis, and the other quantities do not depend on *y*. Then, the components of the strain tensor of the middle surface and the components of its bending are expressed through the components of the displacement vector on the middle surface as follows:

$$\varepsilon = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \quad \theta = -\frac{\partial^2 w}{\partial r^2}, \tag{16}$$

where $w(x,t)$ - displacement of the mid-layer point in the direction of the z-axis; $\varepsilon(x,t), \theta(x,t)$ - accordingly, deformation and change of curvature on the mid-surface of the rod, *t* - time.

In solving this problem, following theoretical and experimental work, we assume that under irradiation, Young's modulus increases monotonically depending on the irradiation dose *D*, but is limited from above, and therefore Young's modulus *E* (*D*) is selected in the form (Cybulskise, 1971; Murray,1972; Onimus et al., 2020) (Fig. 2).

$$E(D) = E_0 \cdot \frac{kD + J\tau_0}{D + J\tau_0} \text{ or } E(D) = E_0 \cdot \frac{kD + J_0 e^{-\mu h}\tau_0}{D + J_0 e^{-\mu h}\tau_0} \tag{17}$$

where E_0 - is the Young's modulus of the unirradiated material, $k = const > 1$-is a dimensionless quantity, $\tau_0 = const = 1$, τ_0-is a measure of time.

It is assumed that the dose of irradiation *D* depends linearly on *t* (Glesson & Edlund, 1954; Onimus et al., 2020), i.e.,

$$D = Jt = J_0 \exp(-\mu z)\, t.$$

Expression (17) for unilateral irradiation can be rewritten as:

$$E(z,t) = E_0 \cdot \frac{ke^{\mu z} \cdot t + \tau_0}{e^{\mu z} \cdot t + \tau_0}.$$

In nuclear reactors, neutrons are fast, i.e., they have an energy of $U > 1$ Mew. Based on the above and considering that the rod is thin, we can conclude that $\mu z << 1$. Therefore, when expanding in a series $\frac{1}{E(z,t)}$ by μz we take only the linear terms and introduce the dimensionless time $\tau = \frac{t}{\tau_0}$, we get:

$$\frac{1}{E(z,t)} = \frac{1}{E_0}\left[\frac{\tau+1}{k\tau+1} - \frac{\mu\tau(k-1)}{(k\tau+1)^2}\cdot z\right]$$

In the linear stressed state, i.e., the stress components

$$\sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma_{22} = \sigma_{33} = 0, \sigma_{11} = \sigma$$

and the components of the deviator and the intensity of shear stresses, respectively, have the form:

$$S_{12} = S_{13} = S_{23} = 0, S_{11} = \frac{2}{3}\sigma, S_{22} = S_{33} = -\frac{\sigma}{3}, \sigma_u = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \sigma$$

Thus, based on the above, the equation of state for this problem has the form:

$$\dot{\tilde{\varepsilon}} = \dot{\tilde{\varepsilon}}^e + \dot{\tilde{\varepsilon}}^c + \dot{\tilde{\varepsilon}}^v, \tag{18}$$

where $\tilde{\varepsilon}$ is the deformation component at an arbitrary point of the rod; $\tilde{\varepsilon}^e$-is the elastic deformation component, $\tilde{\varepsilon}^c$-is the creep deformation component, and $\tilde{\varepsilon}^v$-is the volumetric deformation component.

According to the work (Amenzade,197Gulgazli, 2012), the rates of the elastic deformation component, creep deformation, and volumetric deformation are:

$$\dot{\tilde{\varepsilon}}^e = \left[\frac{\sigma}{E}\right]^{\bullet}, \quad \dot{\tilde{\varepsilon}}^c = F_c, \ \dot{\tilde{\varepsilon}}^v = \frac{1}{3}\dot{S},$$

where S is the volumetric swelling caused only by irradiation and determined empirically, and the function $F_c = F_c(\sigma_i, D)$ is defined empirically.

Then, the equation of state (18) will be presented in the following form (Likhachev & Pupko, 1975; Gulgazli, & Efendiev, 2017):

$$\dot{\tilde{\varepsilon}} = \left[\frac{\sigma}{E}\right]^{\bullet} + F_c + \frac{1}{3}\dot{S} \tag{19}$$

Following theoretical and experimental works, the dependence of S and F_c on D can be approximated as follows (Cybulskise, 1971; Murray,1972; Onimus et al., 2020):

$$S = KD, \quad F_c = \Omega(T,D)\cdot\sigma_u$$

where B and K are empirical constants, $\overline{U}$ - is the average neutron energy in reactors with fast neutrons, and the function $\Omega(D)$ has the following form: $\Omega = B\overline{U}J_0 e^{-\mu(h-z)}$.

In equation (19), the first term expresses the rate of elastic deformation, the second - the rate of creep deformation, and the third - the rate of volumetric swelling in the corresponding direction.

Then, the variational equation for this problem has the form (4):

$$\int_V \left[\dot{\tilde{\sigma}}\delta\dot{\tilde{\varepsilon}} + \dot{\tilde{\varepsilon}}\delta\dot{\tilde{\sigma}} + \tilde{\sigma}\frac{\partial\dot{w}}{\partial x}\delta\frac{\partial\dot{w}}{\partial x} - \left(\frac{\dot{\tilde{\sigma}}}{2E} - \frac{\tilde{\sigma}}{E^2}\frac{\partial E}{\partial D}\cdot\dot{D} + \frac{\dot{S}}{3}\right)\delta\dot{\tilde{\sigma}} - \left(F_c + \frac{\dot{\tilde{\sigma}}}{2E}\right)\delta\dot{\tilde{\sigma}}\right]dv = 0. \tag{20}$$

Based on the known relationships of the components of the finite deformation tensor in a layer of the rod, removed at a distance z from the central layer, we have the form (Volmir,1972):

$$\widetilde{\varepsilon} = \widetilde{\varepsilon} + z\theta \tag{21}$$

The component of the stress tensor at an arbitrary point of the rod has the form (Volmir,1972):

$$\widetilde{\sigma} = \frac{1}{2h} N(x,t) + \frac{3z}{2h^3} M(x,t) \tag{22}$$

where

$$N(x,t) = \int_{-h}^{h} \widetilde{\sigma}(x.z,t)dz; \qquad M(x,t) = \int_{-h}^{h} z\widetilde{\sigma}(x,z.t)dz\,. \tag{23}$$

Considering that the origin is at the central layer of the rod, the expression for S and F_c in reactors with fast neutrons has the form:

$$\dot{S} = K\dot{D}, \quad F_c = \left[\frac{1}{2h} N(x,t) + \frac{3z}{2h^3} M(x,t)\right]\Omega\,, \tag{24}$$

for one-sided irradiation:

$$\dot{S} = KJ_0 e^{-\mu(h-z)}, \ \ \Omega = B\overline{U}J_0 e^{-\mu(h-z)};$$

Considering (21) and (22) in (20), we get:

$$\begin{aligned}\int_V \Bigg[& \left(\frac{\dot{N}}{2h} + \frac{3z\dot{M}}{2h^3}\right)\delta(\dot{\varepsilon} + \dot{\theta}z) + (\dot{\varepsilon} + \dot{k}z)\delta\left(\frac{\dot{N}}{2h} + \frac{3z\dot{M}}{2h^3}\right) + \left(\frac{N}{2h} + \frac{3zM}{2h^3}\right)\frac{\partial \dot{w}}{\partial x}\delta\frac{\partial \dot{w}}{\partial x} - \\ & - \left(\frac{N}{2hE} + \frac{3zM}{2h^3 E} + \frac{S}{3}\right)^{*} \delta\left(\frac{\dot{N}}{2h} + \frac{3z\dot{M}}{2h^3}\right) - F_c\delta\left(\frac{\dot{N}}{2h} + \frac{3z\dot{M}}{2h^3}\right)\Bigg]dv = 0.\end{aligned} \tag{25}$$

Assuming that the rod's size in the y-axis direction is equal to one, and by expanding the integral over z in (25), we get:

$$\begin{aligned}\int_{-l}^{l} \Bigg\{ & \dot{N}\delta\dot{\varepsilon} + \dot{M}\delta\dot{\theta} + \dot{\varepsilon}\delta\dot{N} + \dot{\theta}\delta\dot{M} + \frac{\partial \dot{w}}{\partial x}\delta\frac{\partial \dot{w}}{\partial x}\cdot N - \left[\frac{Np_0}{4h^2 E_0} + \frac{3Mp_1}{4h^4 E_0} + S_1\right]^{\bullet} \delta\dot{N} - \left[\frac{3Np_1}{4h^4 E_0} + \right. \\ & \left. + \frac{9Mp_2}{4h^4 E_0} + S_2\right]^{\bullet} \delta\dot{M} - \Omega_0 N\delta\dot{N} - \Omega_1 M\delta\dot{N} - \Omega_1 N\delta\dot{M} - \Omega_2 M\delta\dot{M}\Bigg\}dx = 0.\end{aligned} \tag{26}$$

$$p_0 = \int_{-h}^{h}\left[\frac{\tau+1}{k\tau+1} - \frac{\mu\tau(k-1)}{(k\tau+1)^2}\cdot z\right]dz = \frac{\tau+1}{k\tau+1}\cdot 2h\,; \quad p_1 = \int_{-h}^{h} z\cdot\left[\frac{\tau+1}{k\tau+1} - \frac{\mu\tau(k-1)}{(k\tau+1)^2}\cdot z\right]dz = \frac{2h^3}{3}\cdot\frac{\mu\tau(1-k)}{(k\tau+1)^2}\,;$$

$$p_2 = \int_{-h}^{h} z^2\cdot\left[\frac{\tau+1}{k\tau+1} - \frac{\mu\tau(k-1)}{(k\tau+1)^2}\cdot z\right]dz = \frac{2h^3}{3}\cdot\frac{\tau+1}{k\tau+1}\,;$$

$$\dot{S}_1 = \int_{-h}^{h}\frac{1}{6h}\dot{S}dz = \frac{1}{6h\mu}KJ_0\left(1 - e^{-2\mu h}\right); \qquad \dot{S}_2 = \int_{-h}^{h}\frac{1}{2h^3}\dot{S}zdz = \frac{1}{2h^3\mu}KJ_0\left(h - \frac{1}{\mu} + he^{-2\mu h} + \frac{1}{\mu}e^{-2\mu h}\right);$$

$$\Omega_0 = \int_{-h}^{h} \frac{1}{4h^2} \Omega dz = \frac{B\overline{U}J_0}{4h^2\mu}\left(1 - e^{-2\mu h}\right); \quad \Omega_1 = \int_{-h}^{h} \frac{3z}{4h^4} \Omega dz = \frac{3B\overline{U}J_0}{4h^3\mu}\left(1 + e^{-2\mu h} + \frac{1}{\mu h} e^{-2\mu h} - \frac{1}{\mu h}\right);$$

$$\Omega_2 = \int_{-h}^{h} \frac{9z^2}{4h^6} \Omega dz = \frac{9B\overline{U}J_0}{4h^4\mu}\left(1 - e^{-2\mu h} - \frac{2}{\mu h} - \frac{2}{\mu h} e^{-2\mu h} + \frac{2}{\mu^2 h^2} - \frac{2}{\mu^2 h^2} e^{-2\mu h}\right).$$

By the condition, the rod is rigidly fixed at both ends, i.e., the boundary conditions can be written as:

$$w(x,\tau)\Big|_{x=-l,l} = 0; \quad \frac{\partial w}{\partial x}\Big|_{x=-l,l} = 0 . \tag{27}$$

Based on the boundary conditions (27) and physical considerations, for bending $w(x,\tau)$, we accept the following approximation:

$$w(x,\tau) = \varphi(\tau)\cos^2 \frac{\pi x}{2l}, \tag{28}$$

where $\varphi(\tau)$ is the unknown function of time.

Then the components of the strain rate tensor of the middle surface have the following form:

$$\dot{\varepsilon} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2\right]^{\bullet} = \frac{\pi^2}{4l^2} \cdot \varphi(\tau) \cdot \dot{\varphi}(\tau) \sin^2 \frac{\pi x}{l}, \tag{29}$$

$$\dot{\theta} = \left[-\frac{\partial^2 w}{\partial x^2}\right]^{\bullet} = \frac{\pi^2}{2l^2} \dot{\varphi}(\tau) \cos\frac{\pi x}{l} . \tag{30}$$

Based on the equation of state (19) for the force and moments, we obtain the following approximations:

$$N(x,\tau) = N_1(\tau) + N_2(\tau)\sin^2 \frac{\pi x}{l} + N_3(\tau)\cos\frac{\pi x}{l} , \tag{31}$$

$$M(x,\tau) = M_1(\tau) + M_2(\tau)\sin^2 \frac{\pi x}{l} + M_3(\tau)\cos\frac{\pi x}{l} . \tag{32}$$

By substituting the accepted approximations for the components of deformations, forces, and moments (29), (30), (31), and (32) into (26), and expanding the integrals over *x*, and equating the coefficients of the variations of $\delta\dot{\varphi}, \delta\dot{N}_1, \delta\dot{N}_2, \delta\dot{N}_3, \delta\dot{M}_1, \delta\dot{M}_2, \delta\dot{M}_3$ to zero, we obtain a system of differential equations. When obtaining the system of differential equations, we introduced the following dimensionless quantities:

$$\overline{\varphi} = \frac{\varphi}{l}, \ \overline{N}_i = \frac{N_i}{lE_0}, \overline{M}_i = \frac{M_i}{l^2 E_0}, \ (i = 1,2,3),$$

$$\overline{p}_j = \frac{p_j}{h^{j+1}}, \ \overline{\Omega}_j = l^{j+1} E_0 \Omega_j, \ (j = 0,1,2) \ ; \ \dot{\overline{S}}_1 = \dot{S}_1, \ \dot{\overline{S}}_2 = l\dot{S}_2, \ \gamma = \frac{l}{h}, \ \overline{\mu} = \mu h .$$

For the obtained system of differential equations, the initial conditions are taken in the following form:

$$N_1 = 0, \ N_2 = 0, \ N_3 = 0, \ M_1 = 0, \ M_2 = 0, \ M_3 = 0, \ \overline{\varphi} = \overline{\varphi}_0 \ \text{ at } \ \tau = 0 .$$

The resulting system was solved on a computer using the fourth-order Runge-Kutta method. In this case, Poisson's ratios were taken as $\nu = 0{,}5$, and it was assumed that Young's modulus does not depend on radiation dose, i.e., $E = E_0 = const$, and for radiation creep, experimental data obtained in reactors with fast neutrons were used. In this case, we have:

$$\bar{p}_0 = 2,\ \bar{p}_1 = 0,\ \bar{p}_2 = \frac{2}{3};\quad \dot{\bar{S}}_1 = \frac{1}{3\bar{\mu}} K J_0 \left(1 - e^{-2\bar{\mu}}\right);\quad \dot{\bar{S}}_2 = 0;\quad \bar{\Omega}_0 = \frac{B\bar{U} J_0 E_0}{2\gamma\bar{\mu}}\left(1 - e^{-2\bar{\mu}}\right);\quad \bar{\Omega}_1 = 0;$$

$$\bar{\Omega}_2 = \frac{9B\bar{U} J_0 E_0}{2\gamma^3 \bar{\mu}}\left(1 - e^{-2\bar{\mu}} - \frac{2}{\bar{\mu}} - \frac{2}{\bar{\mu}} e^{-2\bar{\mu}} + \frac{2}{\bar{\mu}^2} - \frac{2}{\bar{\mu}^2} e^{-2\bar{\mu}}\right);\quad \bar{\theta}_1 = -\frac{4\gamma K}{3B\bar{U} E_0}\left(1 - e^{-2B\bar{U} J_0 E_0 e^{-\bar{\mu}}\tau}\right);\quad \bar{\theta}_2 = 0.$$

The results obtained are shown in Figures 3-5. In the numerical solution of this problem, we used experimental data obtained for SW 316 steels.

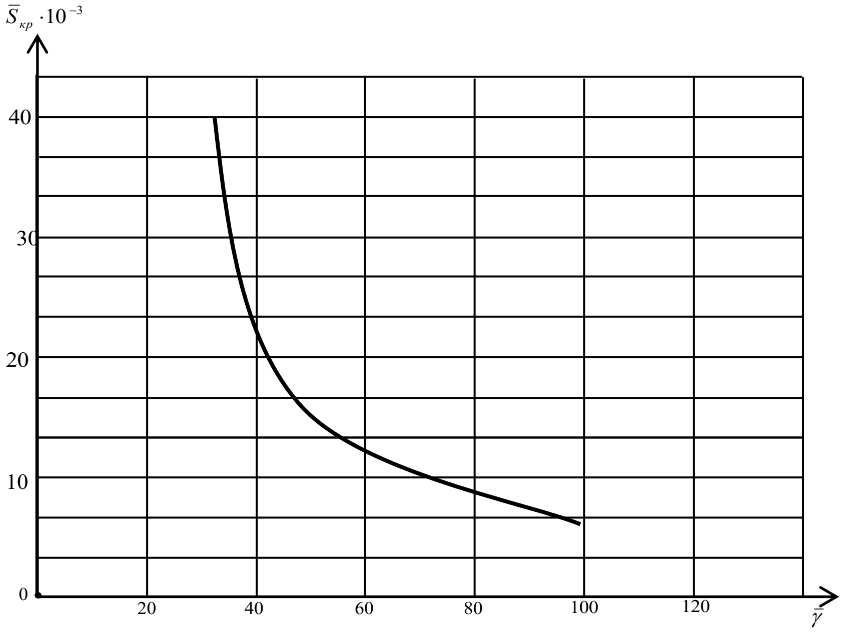

Figure 3. The dependence of the critical volumetric swelling on the relative thickness for a rod.

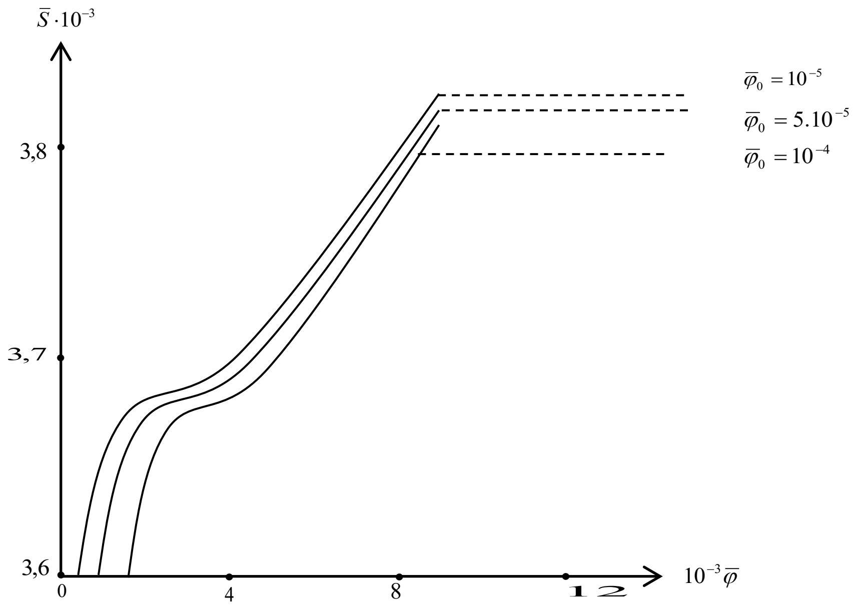

Figure 4. The dependence of $\bar{S} = \bar{S}(\bar{\varphi})$ at different values of the initial bend $\bar{\varphi}_0$ (SW 316).

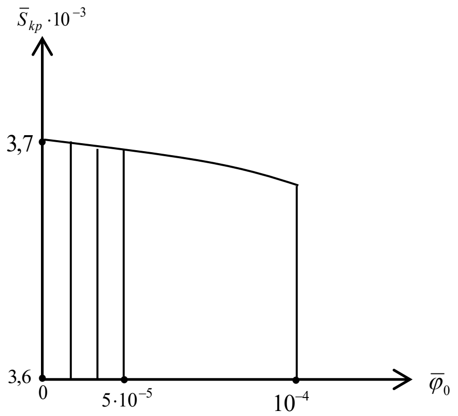

Figure 5. $\bar{S}_{\kappa p} = \bar{S}_{\kappa p}(\bar{\varphi}_0)$ for a straight rod made of steel SW 316 and for $\gamma = 30$

The diagrams show that the critical volumetric swelling S_{kp} and, accordingly, the critical time t_{kp} decrease with an increase in the initial deflection and relative thickness γ.

Results and Discussion

In those areas of technology where neutron radiation and high temperatures are present, it is necessary to take into account the effect of radiation and temperature on the mechanical properties of the material when designing structures. The problem is further complicated when geometric nonlinearity is taken into account. A functionality for solving such problems using the variational method is proposed.

The advantages of the proposed functional (4) are that:

1. A functionality has been developed for studying the stress-strain state of a structure subjected to neutron irradiation. This functionality takes into account geometric nonlinearity, creep, and changes in the mechanical properties of the material depending on the irradiation dose.
2. It has been proven that the Euler equations obtained on the basis of this functional are equations describing the stress-strain state (SSS) of structural elements. Using this functional, it is possible to analyze the behavior of structures subjected to neutron irradiation.
3. In practical application of this functional, we obtain not a system of nonlinear algebraic equations, as is the case when the stresses and strains themselves change independently, but a system of nonlinear ordinary differential equations.
4. The Euler equation of this functional, which gives us the equilibrium equation, explicitly includes creep and plastic deformation components, and it is taken into account that all components depend on the temperature and radiation dose. The temperature and radiation dose themselves depend on time. Therefore, using this functional, it is possible to study the behavior of structural materials both under the influence of radiation and temperature, and separately.
5. The proposed functional was used to study the long-term stability of rods. In the numerical solution of this problem, experimental data obtained for SW316 steels were used.

The results are shown in Figures 3-5.

Thus, using this functional, it is possible to study the stress-strain state of three-dimensional structural elements under the influence of radiation and temperature, both jointly and separately.

Scientific Ethics Declaration

* The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM Journal belongs to the authors.

Conflict of Interest

* The authors declare that they have no conflicts of interest

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