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Research Article

Exploring Generalized 2^k-Fibonacci Sequence: A New Family of the Fibonacci Sequence

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Abstract
The focus of this paper is to study the 2^k–Fibonacci sequence, which is defined for all integers $k > 0$, and its connections with both the Fibonacci and the Fibonacci-Lucas sequences. Among the main results, we highlight the expression of the 2^k-Fibonacci numbers as a linear combination of Fibonacci numbers and Fibonacci-Lucas numbers. Additionally, the paper presents several identities, such as Binet's formula, the Tagiuri-Vajda identity, d'Ocagne's identity, Catalan's identity, and the generating function. Furthermore, we explore some properties of these generalized sequences and establish formulas for sums of terms involving the 2^k-Fibonacci numbers.

Keywords: Binet's formula, Fibonacci-type sequences, 2^k-Fibonacci sequence, Generating function
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1. Introduction

The well-known integer Fibonacci sequence $\{f_n\}_{n\geq 0}$ is defined as $f_{n+2} = f_n + f_{n+1},\ n \geq 0$, with $f_0 = 0$ and $f_1 = 1$, the sequence A000045 in the OEIS [1]. The first terms of the Fibonacci sequence are: 0, 1, 1, 2, 3, 5, 8, 13, ... The Fibonacci recurrence, coupled with different initial terms, can be used to construct new number sequences or other similar sequences. For instance, let $\{l_n\}_{n\geq 0}$ be the n-th term of the sequence with $l_0 = 2$, $l_1 = 1$ and $l_{n+2} = l_n + l_{n+1},\ n \geq 0$. The resulting sequence 2, 1, 3, 4, 7, 11, 18, ... is the Fibonacci–Lucas sequence, the sequence A000032 catalogued in the OEIS [1]. Or again, let $\{m_n\}_{n\geq 0}$ be the n-th element of the sequence with $m_0 = 4, m_1 = 1$ and $m_{n+2} = m_n + m_{n+1},\ n \geq 0$. The resulting sequence 4, 1, 5, 6, 11, 17, ... is the Fibonacci–Mulatu sequence, this sequence is listed as A022095 in the OEIS [1]. There is a substantial corpus of literature that examines each of these sequences in isolation. Moreover, the connections between the Fibonacci-Mulatu sequence and the classical or ordinary Fibonacci and Fibonacci-Lucas sequences are investigated in [2], [3] and [4], among others.

The *shifted Fibonacci* sequence is again a variation of the classic Fibonacci sequence in which the recurrence relation remains unchanged, but the initial conditions or indexes are shifted. This modification introduces flexibility in defining new families of sequences while preserving the fundamental structure of Fibonacci-type sequences. In this study, we use the shifted Fibonacci sequence defined by the recursion: $f^*_n = f^*_{n-1} + f^*_{n-2}$, for $n \geq 2$, with initial values $f^*_0 = f^*_1 = 1$. It can be written as: $f^*_n = f_{n+1}$, where f_n represents the ordinary Fibonacci sequence.

Note that both $\{f_n\}_{n\geq 0}$, $\{f^*_n\}_{n\geq 0}$, $\{l_n\}_{n\geq 0}$ and $\{m_n\}_{n\geq 0}$ satisfy the same relation. Both Fibonacci, shifted Fibonacci,

Fibonacci–Lucas, and Fibonacci–Mulatu numbers satisfy numerous identities, many were discovered some centuries ago.

Normally, a new generalization of the Fibonacci sequence involves extending its structure by introducing additional parameters, terms, or mathematical rules. So, here, let us denote $\{F_{(k,n)}\}_{n\geq 1}$ by the Fibonacci-type sequence that satisfies

$$F_{(k,n+2)} = F_{(k,n)} + F_{(k,n+1)},\ n \geq 0\ , \tag{1.1}$$

with initial values $F_{(k,0)} = 2^k$ and $F_{(k,1)} = 1$, for all non-negative integers k. From now on, we refer to this sequence as the 2^k-Fibonacci sequence, and Table 1 shows some examples of this sequence.

k	sequence	name	
0	1, 1, 2, 3, 5, 8, 13, 21, ...	shifted Fibonacci	$A000045^*$ [1]
1	2, 1, 3, 4, 7, 11, 18, 29, ...	Fibonacci–Lucas	A000032 [1]
2	4, 1, 5, 6, 11, 17, 28, 45, ...	Fibonacci–Mulatu	A022095 [1]
3	8, 1, 9, 10, 19, 29, 48, ...	2^3-Fibonacci	
4	16, 1, 17, 18, 35, 53, 88, ...	2^4-Fibonacci	
		...	
t	2^t, 1, 2^t+1, 2^t+2, $2^{(t+1)}+3$, ...	2^t-Fibonacci	

Table 1.1. 2^k-Fibonacci for some values of k.

Throughout the text, unless stated otherwise, k is a non-negative integer. For instance, in Table 1, we can see that when $k = 0$ we obtain the shifted Fibonacci numbers. Similarly, setting $k = 1$ yields the Fibonacci–Lucas numbers. Finally, with $k = 2$, we derive the Fibonacci–Mulatu numbers.

For a study of the history and some applications of the Fibonacci sequence, as well as some generalizations or extensions, see [5] and [6]. In particular, there is an approach to the generating functions of this sequence in [7]. Many sequences have been defined in the mathematical literature by generalizing these number sequences. In some examples, the second-order recurrence sequences have been generalized by maintaining the recurrence relation while modifying the first two terms of the sequence. For instance, [8] introduced and demonstrated properties of a generalized Fibonacci sequence, [9] observed that several other known sequences are also of the Horadam sequence type by choosing appropriate initial terms and constants of the recurrence relation. Specifically, what Horadam studied in [8] was the general Fibonacci sequence. It was in [10] that Horadam studied what is now known as the Horadam sequence. Similarly, [11] contributed to the process of extending the Fibonacci sequence into the domain of complex numbers, which made it possible to extend other sequences as well. The second-order recurrence sequences were generalized by maintaining the recurrence relation while modifying the first two terms of the sequence. For example, [12, 13] and [14] presented the generalized k-Fibonacci numbers for a real number k, and found that these sequences were found by investigating the recursive application of two geometric transformations. Then [15] studied the k-Fibonacci generating matrices and an extension of the (k,t)-Fibonacci numbers in [16] or [17], for real numbers k,t. Similarly, [18, 19] and [20], have done some research on the sequences of numbers that can be derived from these sequences. Other researchers have generalized the Fibonacci sequence by keeping the first two terms of the sequence, but changing the recurrence relation slightly. Refs. [21] and [22] define the bi-periodic Fibonacci sequences as a generalization of some Fibonacci sequences; [23] studied the sequence $\{F_n^{(k)}\}_{n\geq 0}$, the k-generalized Fibonacci number, as a generalization of the tribonacci, tetranacci, ..., k-nacci sequences; and gave general expressions for this type of sequence; while [24] describes the various structures, mathematical beauty and identities associated with such sequences and numbers. In refs. [25, 26] and [27] generalized Fibonacci numbers and their associated polynomial extensions are presented. The Fibonacci sequence has many applications, whether in nature or not, as can be seen in [5] or [6] using a Fibonacci sequence to analyze the mathematical structure of the genetic code. It also explores some important applications of Fibonacci sequences.

This study is organized as follows: the introduction and background, and five additional sections. In Section 2, we derive Binet's formula for the 2^k-Fibonacci sequence and explore its applications, including connections with the classical Fibonacci and Fibonacci-Lucas sequences. Additionally, we extend the 2^k-Fibonacci numbers to include negative indexes. In Section 3, we present both the exponential generating function and the ordinary generating function for the 2^k-Fibonacci sequence, along with their applications. In Section 4, we establish several identities for the 2^k-Fibonacci sequence that hold for all integers k. Classical identities, such as those of Tagiuri-Vajda and Catalan, are derived, and the limits of certain quotients are discussed. In Section 5, we present results related to the partial sums and alternating partial sums of the terms in the 2^k-Fibonacci sequence. Finally, in Section 6, we discuss the connections between the 2^k-Fibonacci sequences and the classical Fibonacci and Fibonacci-Lucas sequences, and we outline a direction for future research.

The 2^k-Fibonacci sequence, like all Fibonacci-type sequences, has many captivating properties. Here we will prove almost all identities by the application of the Binet formula. This examination reveals the relationships and distinctive properties of these sequences, providing deeper insights into their structural patterns and mathematical significance.

2. Binet's Formula and Applications

In this section, we explore the first connections between Fibonacci-type sequences –specifically the 2^k-Fibonacci sequence– and the classical Fibonacci and Fibonacci-Lucas sequences, analyzing a set of key identities, most of them derived from or obtained using Binet's formula.

2.1 Background and preliminary results

The Fibonacci-type sequence is known for being associated with the characteristic equation

$$r^2 = r + 1\,. \tag{2.1}$$

Solving this equation for r yields two distinct roots α and β, that is, α is the golden ratio $(1+\sqrt{5})/2$, and β is its conjugate $(1-\sqrt{5})/2$, which play a central role in deriving properties of the sequence. As noted in sources such as [5] and [28], among others, the Binet formula provides a direct and elegant method for computing the n-th Fibonacci-type number without the need to iterate through the sequence. This formula enhances the roots of the characteristic equation to express the n-th term explicitly, highlighting the sequence's intrinsic mathematical beauty and efficiency.

The next auxiliary result gives the Binet formula for the classical Fibonacci sequence $\{f_n\}_{n\geq 0}$, and can be found in [6, Equation 58], which is also derived from [5, Theorem 7.4].

Lemma 2.1. *For every non-negative integer n, the general formula for the terms of the Fibonacci sequence $\{f_n\}_{n\geq 0}$ is*

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \tag{2.2}$$

where α and β are the distinct roots of Equation (2.1).

The following fact can be regarded as a specific example of a more general result, namely Theorem 7.4, as presented in [5]. This particular case aligns with the broader principles outlined in that theorem, providing a focused application within this context.

Theorem 2.2 (Binet-like Formula)**.** *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence, $c = 1-2^k\beta$ and $d = 1-2^k\alpha$. Then we have that*

$$F_{(k,n)} = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}\,, \tag{2.3}$$

where α and β are the distinct roots of Equation (2.1).

Proof. We need to find $c_1 = c_1(k)$ and $c_2 = c_2(k)$, $k \geq 0$, such that

$$F_{(k,n)} = c_1\alpha^n + c_2\beta^n\,.$$

By the definition

$$\begin{cases} F_{(k,0)} = c_1\alpha^0 + c_2\beta^0 = c_1 + c_2 = 2^k \\ F_{(k,1)} = c_1\alpha^1 + c_2\beta^1 = c_1\alpha + c_2\beta = 1 \end{cases}$$

which implies that

$$c_1 = \frac{1-2^k\beta}{\alpha-\beta} \quad \text{and} \quad c_2 = \frac{2^k\alpha - 1}{\alpha - \beta}\,.$$

Then,

$$F_{(k,n)} = \frac{(1-2^k\beta)\alpha^n - (1-2^k\alpha)\beta^n}{\alpha-\beta}\,,$$

as we wanted to demonstrate. ☐

As a consequence of Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. Then the Binet formula for the shifted Fibonacci, Fibonacci–Lucas, and Fibonacci–Mulatu numbers are given respectively by*

$$\begin{aligned}
&(a) \quad F_{(0,n)} &&= f_n^* = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\ ,\\
&(b) \quad F_{(1,n)} &&= l_n = \alpha^n+\beta^n\ ,\\
&(c) \quad F_{(2,n)} &&= m_n = \frac{10-\sqrt{5}}{5}\alpha^n+\frac{10+\sqrt{5}}{5}\beta^n\ ,
\end{aligned}$$

furthermore,

$$\begin{aligned}
&(d) \quad F_{(0,n)} &&= f_n^* = f_{n+1}\ ,\\
&(e) \quad F_{(1,n)} &&= l_n = f_{n-1}+f_{n+1}\ ,\\
&(f) \quad F_{(2,n)} &&= m_n = 2l_n - f_n\ ,
\end{aligned} \tag{2.4}$$

where α and β are the distinct roots of Equation (2.1), $\{f_n\}_{n\geq 0}$ is the ordinary Fibonacci sequence, $\{l_n\}$ is the Fibonacci–Lucas numbers, and $\{m_n\}$ is the Fibonacci–Mulatu numbers.

Proof. (a), (b), (c) and (d) are straightforward calculations.

(e) For $k=1$, then $c=1-2\beta$ and $d=1-2\alpha$, so

$$F_{(1,n)} = \frac{(1-2\beta)\alpha^n-(1-2\alpha)\beta^n}{\alpha-\beta} = \frac{\alpha^n-\beta^n}{\alpha-\beta}+2\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\ ,$$

since $\alpha\beta=-1$. According to Equation (2.2) we have

$$F_{(1,n)} = f_n+2f_{n-1} = f_{n-1}+f_{n+1}\ .$$

Following Table 1 for $k=1$, we have $F_{(1,n)}=l_n$.

(f) Combining Equations (2.2) and (2.3), we have

$$2l_n-f_n = 2\frac{(1-2\beta)\alpha^n-(1-2\alpha)\beta^n}{\alpha-\beta}-\frac{\alpha^n-\beta^n}{\alpha-\beta} = F_{(2,n)} = m_n\ ,$$

and we get the result, using Table 1 for $k=2$. □

Let $\{F_{(k,n)}\}_{n\geq 1}$ be the 2^k-Fibonacci sequence with initial values $F_{(k,0)}=2^k$ and $F_{(k,1)}=1$. Consider the constants $c=1-2^k\beta$ and $d=1-2^k\alpha$, where α and β represent the distinct roots of Equation (2.1). According to [5], the constant cd occurs in many of the formulas for Fibonacci-type numbers. It is called the characteristic of the Fibonacci-type sequence.

$$cd = [1-2^k\beta][1-2^k\alpha] = 1-2^k-4^k\ ,$$

since $\alpha+\beta=1$ and $\alpha\beta=-1$. We will denote it by the Greek letter μ (mu), making $\mu=-cd$.

Building on the previous discussion, the following result is presented:

Proposition 2.4. *The characteristic μ_k of the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ is given by*

$$\mu_k = 4^k+2^k-1\,.$$

For instance, the characteristic of both the shifted-Fibonacci sequence and the classical Fibonacci sequence is $\mu_0=1$. The Fibonacci-Lucas sequence, on the other hand, is characterized by $\mu_1=5$. Similarly, for the Fibonacci-Mulatu numbers, the characteristic is $\mu_2=19$.

Again using Theorem 2.2, specifically Corollary 2.3, we get the next two sets of results.

Proposition 2.5. *For all non-negative integers n, we have*

$$f_{n+1}l_n = f_{2n+1}+(-1)^n \tag{2.5}$$

$$f_nl_{n+1} = f_{2n+1}-(-1)^n \tag{2.6}$$

$$5f_nf_{n+1} = l_{2n+1}-(-1)^n \tag{2.7}$$

$$l_nl_{n+1} = l_{2n+1}+(-1)^n \tag{2.8}$$

$$f_{n+1}f_{n-1}+f_n^2=2f_n^2+(-1)^n \tag{2.9}$$

$$l_{n+4}=3l_{n+2}-l_n \tag{2.10}$$

where $\{f_n\}_n$ is the Fibonacci sequence and $\{l_n\}_n$ is the Fibonacci–Lucas sequence.

Proof. By Binet's formula for Fibonacci and Fibonacci–Lucas sequences, we have

$$\begin{aligned} f_n l_{n+1} &= \frac{\alpha^n-\beta^n}{\sqrt{5}}(\alpha^{n+1}+\beta^{n+1}) \\ &= \frac{1}{\sqrt{5}}\left[\alpha^{2n+1}-\beta^{2n+1}\right]-(-1)^n = f_{2n+1}-(-1)^n, \end{aligned}$$

where we use that $\alpha\beta=-1$ and $\alpha-\beta=\sqrt{5}$. Then we have obtained Equation (2.6). The Equations (2.5), (2.7), and (2.8) are obtained in a similar way.

Equation (2.9) follows from Cassini's formula, [5, Theorem 7.5], by summing $2f_n^2$ on both sides.

$$l_{n+4}=l_{n+3}+l_{n+2}=l_{n+2}+l_{n+1}+l_{n+2}=l_{n+2}+l_{n+2}-l_n+l_{n+2}=3l_{n+2}-l_n,$$

and Equation (2.10) follows. □

Proposition 2.6. *For the non-negative integers m and n. The following identities hold true*

$$l_{n-1}+l_{n+1}=5f_n \tag{2.11}$$

$$f_{n+4}=3f_{n+2}-f_n \tag{2.12}$$

$$f_{m-1}f_n+f_mf_{n+1}=f_{m+n} \tag{2.13}$$

$$l_nf_{n+1}+l_{n+1}f_n=2f_{2n+1} \tag{2.14}$$

$$m_n=f_n+4f_{n-1} \tag{2.15}$$

where $\{f_n\}_{n\geq 0}$ is the classical Fibonacci sequence, $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence, and $\{m_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

These results will be used throughout the paper, and their proofs are omitted for brevity. Verification of each statement can be achieved by applying the corresponding item of Corollary 2.3 or by consulting the references. Specifically, Equations (2.11) and (2.12) were established in [5] (Equations 5.18 and 5.16), Equation (2.13) was derived in [28, Equation 1.8], Equation (2.14) is presented in [6, Equation 16a], and Equation (2.15) is found in [29, Theorem 2].

2.2 Connections between 2^k-Fibonacci and Fibonacci numbers

In this subsection, we explain the relationship between the classical Fibonacci and Fibonacci–Lucas sequences with any 2^k-Fibonacci sequence.

It follows from Binet's formula that the following results hold.

Theorem 2.7. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. Then we have that*

$$F_{(k,n)}=F_{(0,n-1)}+2^kF_{(0,n-2)}, \tag{2.16}$$

for all $n\geq 2$ and $k\geq 0$.

Proof. According to Equation (2.3), we have

$$F_{(k,n)}=\frac{(1-2^k\beta)\alpha^n-(1-2^k\alpha)\beta^n}{\alpha-\beta}=\frac{\alpha^n-\beta^n}{\alpha-\beta}+2^k\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta},$$

and the result follows from Equation (2.2). □

As a direct consequence of Theorem 2.7, we have the following corollary.

Corollary 2.8. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. For all $n \geq 2$ and $k \geq 0$, we have*

$$F_{(k,n)} = f_n + 2^k f_{n-1}\,, \tag{2.17}$$

where $\{f_n\}_n$ is the Fibonacci sequence.

Corollary 2.9. *For non-negative integers n, we have the following corollary.*

(a) $F_{(0,n)} = f_{n+1}\,;$

(b) $l_n = f_n + f_{n+1}\,;$

(c) $m_n = l_n + 2f_n\,;$

where $\{f_n\}_n$ is the Fibonacci sequence, $\{l_n\}_n$ is the Fibonacci–Lucas sequence, and $\{m_n\}_n$ is the Fibonacci–Mulatu sequence.

The next result establishes a connection between the Fibonacci-type sequences of order 2^{k+1} and 2^k with the classical Fibonacci sequence.

Proposition 2.10. *For all fixed $n \geq 1$, the sequence $\{F_{(k,n)}\}_{k\geq 0}$ satisfies*

$$F_{(k+1,n)} - F_{(k,n)} = 2^k f_{n-1}\,,$$

where $\{f_n\}_n$ is the Fibonacci sequence.

Proof. Fix $n \geq 1$. Then, by Equation (2.17), we have

$$F_{(k+1,n)} - F_{(k,n)} = (2^{k+1} - 2^k) f_{n-1} = 2^k f_{n-1}\,,$$

as required. □

Corollary 2.11. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. Then we have that*

$$F_{(k+1,n)} = 2F_{(k,n)} - f_n\,, \tag{2.18}$$

for all $n \geq 1$ and $k \geq 0$, and where $\{f_n\}_n$ is the Fibonacci sequence.

Proof. By Equation (2.16) we have

$$2F_{(k,n)} = 2F_{(0,n-1)} + 2\cdot 2^{k-1} F_{(0,n-2)}\,.$$

Then,

$$F_{(k+1,n)} - 2F_{(k,n)} = F_{(0,n-1)} - 2F_{(0,n-1)} + 2^k F_{(0,n-2)} - 2^k F_{(0,n-2)} = -F_{(0,n-1)}\,,$$

as we wanted to demonstrate. □

Theorem 2.12. *Consider that k, n and t are non-negative integers. Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. For all $n \geq 1$, then we have that*

$$F_{(k+t,n)} = 2^t F_{(k,n)} - (2^t - 1) f_n\,, \tag{2.19}$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci.

Proof. We will use mathematical induction on t. For $t = 1$, the result follows from Equation (2.18). Suppose that the result is valid for some $t \geq 1$. Then

$$F_{((k+1)+t,n)} = 2^t F_{(k+1,n)} - (2^t - 1) f_n = 2^{t+1} F_{(k,n)} - (2^{t+1} - 1) f_n\,,$$

where we use Equation (2.18) once again in the second equation, and this completes the proof. □

When substituting $k = t = 1$ in Equation (2.19), we obtain the following corollary.

Corollary 2.13. *For non-negative integers n, we have*

$$m_n = 2l_n - f_n;$$

where $\{f_n\}_n$ *is the Fibonacci sequence,* $\{l_n\}_n$ *is the Fibonacci–Lucas sequence, and* $\{m_n\}_n$ *is the Fibonacci–Mulatu sequence.*

The following results allow us to express the 2^k- Fibonacci numbers as a linear combination of the Fibonacci–Lucas numbers and Fibonacci numbers, and is our main result.

Theorem 2.14. *Let* $\{F_{(k,n)}\}_{n\geq 0}$ *be the generalized* 2^k*-Fibonacci sequence. Then we have that*

$$F_{(k,n)} = 2^{k-1}l_n - (2^{k-1}-1)f_n\,, \tag{2.20}$$

for all $n \geq 1$ *and* $k \geq 1$*, where* $\{f_n\}_{n\geq 0}$ *and* $\{l_n\}_{n\geq 0}$ *are the Fibonacci and Lucas sequences, respectively.*

Proof. For a fixed k, let us apply the principle of mathematical induction to n. Note that

$$F_{(k,0)} = 2^{k-1}l_0 - (2^{k-1}-1)f_0 = 2^k\,,$$

and

$$F_{(k,1)} = 2^{k-1}l_1 - (2^{k-1}-1)f_1 = 1\,.$$

Assume that the property is true for all values less than or equal to n, that is, $F_{(k,m)} = 2^{k-1}l_m - (2^{k-1}-1)f_m$ for $m \leq n$. We need to prove that the property holds for $m+1$. According to Equation (1.1)

$$F_{(k,m+1)} = F_{(k,m)} + F_{(k,m-1)},\ m \geq 1\ .$$

Using the inductive hypothesis, we have:

$$F_{(k,m+1)} = F_{(k,m)} + F_{(k,m-1)} = [2^{k-1}l_m - (2^{k-1}-1)f_m] + [2^{k-1}l_{m-1} - (2^{k-1}-1)f_{m-1}] = 2^{k-1}l_{m+1} - (2^{k-1}-1)f_{m+1}\,.$$

Therefore, by the principle of mathematical induction, the property from Equation (2.20) is true for all non-negative integers n. □

For instance, when $k = 1$ then

$$F_{(1,n)} = 2^{1-1}l_n - (2^{1-1}-1)f_n = l_n,$$

and for $k = 2$, then

$$F_{(2,n)} = 2^{2-1}l_n - (2^{2-1}-1)f_n = 2l_n - f_n = m_n\ .$$

Proposition 2.15. *Let* $\{F_{(k,n)}\}_{n\geq 0}$ *be the generalized* 2^k*-Fibonacci sequence. Then we have that*

$$F_{(k,n)} = f_n + 2^k f_{n-1}\,, \tag{2.21}$$

where $\{f_n\}_{n\geq 0}$ *is the Fibonacci sequence.*

Proof. By combining Equations (2.4) and (2.20), we have

$$F_{(k,n)} = 2^{k-1}l_n - (2^{k-1}-1)f_n = 2^k f_{n-1} + f_n\,,$$

as required. □

As an immediate consequence of Proposition 2.15, we obtain the following corollary.

Corollary 2.16. *For non-negative integers n, we have*

(a) $l_n = f_n + 2f_{n-1};$

(b) $m_n = f_n + 2^2 f_{n-1};$

where $\{f_n\}_n$ *is the Fibonacci sequence,* $\{l_n\}_n$ *is the Fibonacci–Lucas sequence, and* $\{m_n\}_n$ *is the Fibonacci–Mulatu sequence.*

We will see below that the addition of two terms of the same parity of the 2^k-Fibonacci sequence is a combination of Fibonacci–Lucas terms.

Proposition 2.17. *For non-negative integers n and k, and for the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$, the following identity holds*

$$F_{(k,n-1)}+F_{(k,n+1)}=5\cdot 2^{k-1}f_n-(2^{k-1}-1)l_n\,, \tag{2.22}$$

where $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence and $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. It follows from Equations (2.4), (2.20), and (2.11) that

$$F_{(k,n-1)}+F_{(k,n+1)}=2^{k-1}(l_{n-1}+l_{n+1})-(2^{k-1}-1)(f_{n-1}+f_{n+1})=2^{k-1}(5\cdot f_n)-(2^{k-1}-1)l_n\,,$$

as required. □

2.3 Negative subscript

In this subsection, we will present an extension of the negative index for generalized 2^k-Fibonacci numbers; that is, we will show how to extend the 2^k-Fibonacci sequence to negative subscripts.

The Fibonacci and Fibonacci-Lucas numbers with negative subscripts, according to [5, Equations 5.19 and 5.20], have the relations

$$f_{-n}=(-1)^{n+1}f_n; \tag{2.23}$$

$$l_{-n}=(-1)^n l_n; \tag{2.24}$$

where $\{f_n\}_{n\geq 0}$ is the classical Fibonacci sequence and $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence.

We will give meaning to the sequence $F_{(k,-n)}$ for every integer n, and for the recurrence to remain valid. The generalized 2^k-Fibonacci sequence is characterized by a recurrence relation, which can be expressed as follows

$$F_{(k,n)}=F_{(k,n-1)}+F_{(k,n-2)},$$

with initial conditions $F_{(k,0)}=2^k$ and $F_{(k,1)}=1$. To extend the sequence to negative indexes, we use the modified recurrence relation

$$F_{(k,n-2)}=F_{(k,n)}-F_{(k,n-1)}.$$

Using this, we compute the 2^k-Fibonacci terms for the negative indexes. The calculations are as follows:

$$\begin{array}{lclcc}
\mathbf{F}_{(\mathbf{k},-\mathbf{1})} & = & F_{(k,1)}-F_{(k,0)} & = & 1-2^k \\
 & = & -1(-1+1\cdot 2^k) & = & -\mathbf{1}(-\mathbf{f_1}+\mathbf{f_2}\cdot\mathbf{2^k})\ , \\
\mathbf{F}_{(\mathbf{k},-\mathbf{2})} & = & F_{(k,0)}-F_{(k,-1)} & = & 2^k-(1-2^k) \\
 & = & 1(-1+2\cdot 2^k) & = & \mathbf{1}(-\mathbf{f_2}+\mathbf{f_3}\cdot\mathbf{2^k})\ , \\
\mathbf{F}_{(\mathbf{k},-\mathbf{3})} & = & F_{(k,-1)}-F_{(k,-2)} & = & 1-2^k-(-1+2\cdot 2^k) \\
 & = & -1(-2+3\cdot 2^k) & = & -\mathbf{1}(-\mathbf{f_3}+\mathbf{f_4}\cdot\mathbf{2^k})\ , \\
\mathbf{F}_{(\mathbf{k},-\mathbf{4})} & = & F_{(k,-2)}-F_{(k,-3)} & = & -1+2\cdot 2^k-(2-3\cdot 2^k) \\
 & = & 1(-3+5\cdot 2^k) & = & \mathbf{1}(-\mathbf{f_4}+\mathbf{f_5}\cdot\mathbf{2^k})\ , \\
\mathbf{F}_{(\mathbf{k},-\mathbf{5})} & = & F_{(k,-3)}-F_{(k,-4)} & = & 2-3\cdot 2^k-(-3+5\cdot 2^k) \\
 & = & -1(-5+8\cdot 2^k) & = & -\mathbf{1}(-\mathbf{f_5}+\mathbf{f_6}\cdot\mathbf{2^k})\ ,
\end{array}$$

and so on, and we can see the following pattern in the next result, which is a counterpart of Equation (2.17) for negative subscripts.

Proposition 2.18. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. For all integers $n\geq 1$, then the negative index n-th numbers satisfy*

$$F_{(k,-n)}=(-1)^n\left(2^k f_{n+1}-f_n\right), \tag{2.25}$$

where $\{f_n\}_{n\geq 1}$ is the Fibonacci sequence.

Proof. Let $U_{(k,-n)} = (-1)^n \left(2^k f_{n+1} - f_n\right)$ for all integers $n \geq 1$. We want to show that $U_{(k,-n)}$ verifies the recurrence relation $U_{(k,-n)} = U_{(k,-(n+2))} + U_{(k,-(n+1))}$. Indeed,

$$\begin{aligned} U_{(k,-(n+2))} + U_{(k,-(n+1))} &= (-1)^{n+2} \left(2^k f_{n+3} - f_{n+2}\right) + (-1)^{n+1} \left(2^k f_{n+2} - f_{n+1}\right) \\ &= (-1)^n \left(2^k f_{n+1} - f_n\right) = U_{(k,-n)} . \end{aligned}$$

Moreover, $U_{(k,-1)} = 2^k f_2 - f_1 = 2^k - 1$ and $U_{(k,-2)} = 2^k f_3 - f_2 = 2 \cdot 2^k - 1$. So, since $U_{(k,-n)}$ satisfies the recurrence that defines $F_{(k,-n)}$ with the same initial condition, we conclude $U_{(k,-n)} = F_{(k,-n)}$. □

The sequence $\{F_{(k,n)}\}_{n\geq 0}$ can be extended to negative subscripts by the following result.

Proposition 2.19. *Let $\{F_{(k,n)}\}_{n\geq 0}$ be the generalized 2^k-Fibonacci sequence. For integer $n \geq 1$, then the negative index n-th numbers $F_{(k,n)}$ satisfy the following*

$$F_{(k,-n)} = (-1)^n \left(F_{(k,n)} + 2(2^{k-1} - 1) f_n\right) , \tag{2.26}$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Combining Equations (2.20), (2.23) and (2.24) we have

$$\begin{aligned} F_{(k,-n)} &= 2^{k-1} l_{-n} - (2^{k-1} - 1) f_{-n} \\ &= (-1)^n \left(F_{(k,n)} + 2(2^{k-1} - 1) f_n\right) , \end{aligned}$$

which verifies the result. □

It follows from the previous result that the Fibonacci–Mulatu sequence with a negative index has the following property:

Corollary 2.20. *For all negative indexes, the Fibonacci–Mulatu sequence satisfies the following identity*

$$m_{-n} = (-1)^n (2 l_n + f_n) ,$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence, $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence, and $\{m_n\}_{n\geq 0}$ is the Fibonacci–Mulatu sequence.

Proof. At the beginning of the proof, we take $k = 2$ in Proposition 2.19. Then

$$F_{(2,-n)} = (-1)^n \left(F_{(2,n)} + 2(2^{2-1} - 1) f_n\right) = (-1)^n \left(F_{(2,n)} + 2 f_n\right) .$$

By Equation (2.18), we have

$$F_{(2,-n)} = (-1)^n \left(2F_{(1,n)} - f_n + 2 f_n\right) = (-1)^n \left(2F_{(1,n)} + f_n\right) ,$$

as required. □

3. Generating Functions

In this section, we present both the exponential generating function and the ordinary generating function for the 2^k-Fibonacci sequence.

In mathematical literature, the ordinary generating function for a sequence $\{a_n\}_{n\geq 0}$, denoted as $G_{a_n}(x)$, is defined as:

$$G_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots . \tag{3.1}$$

The following result provides the explicit form of the generating function for the 2^k-Fibonacci sequence.

Proposition 3.1. *The generating function for the generalized 2^k-Fibonacci sequence numbers $\{F_{(k,n)}\}_{n\geq 0}$, denoted by $G_{F_{(k,n)}}(x)$, is given by*

$$G_{F_{(k,n)}}(x) = \frac{2^k + (1 - 2^k) x}{1 - x - x^2} .$$

Proof. According to Equation (3.1), the ordinary generating function for the 2^k-Fibonacci sequence is given by

$$G_{F_{(k,n)}}(x) = F_{(k,0)} + F_{(k,1)}x + F_{(k,2)}x^2 + F_{(k,3)}x^3 + \cdots + F_{(k,n)}x^n + \cdots .$$

Using the relationships $xG_{F_{(k,n)}}(x)$ and $x^2G_{F_{(k,n)}}(x)$, we derive the following results

$$G_{F_{(k,n)}}(x)(1-x-x^2) = F_{(k,0)} + [F_{(k,1)} - F_{(k,0)}]x = 2^k + (1-2^k)x ,$$

as $F_{(k,0)} = 2^k$ and $F_{(k,1)} = 1$, we have

$$(1-x-x^2)G_{F_{(k,n)}}(x) = 2^k + (1-2^k)x,$$

or equivalently,

$$G_{F_{(k,n)}}(x) = \frac{2^k + (1-2^k)x}{1-x-x^2} ,$$

since $1-x-x^2 \neq 0$, and this completes the proof. □

The exponential generating function $E_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is represented as a power series given by

$$E_{a_n}(x) = a_0 + a_1x + \frac{a_2x^2}{2!} + \cdots + \frac{a_nx^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_nx^n}{n!}.$$

In the following result, we focus on the case where $a_n = F_{(k,n)}$, using the Binet formula for the 2^k-Fibonacci sequence. By doing so, we derive the exponential generating function for this specific sequence.

Proposition 3.2. *For all $n \geq 0$ the exponential generating function for the 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ is*

$$E_{F_{(k,n)}}(x) = \sum_{n=0}^{\infty} F_{(k,n)} \frac{x^n}{n!} = \frac{(1-2^k\beta)e^{\alpha x} - (1-2^k\alpha)e^{\beta x}}{\alpha - \beta} ,$$

where α and β are the distinct roots of Equation (2.1).

Proof. For each non-negative integer n, by Binet's formula (Equations (2.3)), we have

$$\frac{x^n}{n!}F_{(k,n)} = \frac{1-2^k\beta}{\alpha-\beta}\frac{(\alpha x)^n}{n!} - \frac{1-2^k\alpha}{\alpha-\beta}\frac{(\beta x)^n}{n!} .$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{x^n}{n!}F_{(k,n)} &= \frac{1-2^k\beta}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} - \frac{1-2^k\alpha}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} \\ &= \frac{1-2^k\beta}{\alpha-\beta}e^{\alpha x} - \frac{1-2^k\alpha}{\alpha-\beta}e^{\beta x}, \end{aligned}$$

and follows the result. □

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is defined as

$$P_{a_n}(x) = \sum_{n=0}^{\infty} \frac{a_nx^n}{n!}e^{-x}.$$

This function encodes the sequence $\{a_n\}_{n\geq 0}$ in terms of the parameter x. A significant relationship exists between the exponential generating function $E_{a_n}(x)$ and the Poisson generating function $P_{a_n}(x)$, given by

$$P_{a_n}(x) = e^{-x}E_{a_n}(x).$$

This relationship establishes a direct connection between the two generating functions, enabling the derivation of the Poisson generating function from its exponential counterpart.

As a specific case, considering the exponential generating function for the 2^k-Fibonacci sequence given in Proposition 3.2 by $E_{F_{(k,n)}}(x)$, we get the following result.

Corollary 3.3. *For all $n \geq 0$, the Poisson generating function for the 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ is*

$$P_{F_{(k,n)}}(x) = \frac{(1-2^k\beta)e^{(\alpha-1)x} - (1-2^k\alpha)e^{(\beta-1)x}}{\alpha-\beta},$$

where α and β are the distinct roots of Equation (2.1).

This result highlights the explicit form of the Poisson generating function for the Fibonacci–Mulatu sequence, emphasizing the roles of α and β in its structure.

4. Some Properties

In this section, we establish some identities for the 2^k-Fibonacci sequence for all integers n. The classical identities are studied, and finally, the limit of some quotients is presented.

4.1 Identities for the 2^k-Fibonacci sequence

The first two results establish the multiplication formula for two consecutive terms of the 2^k-Fibonacci sequence.

Proposition 4.1. *For all non-negative integers n and $k \geq 1$, the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following identity:*

$$10F_{(k,n)}F_{(k,n+1)} = \left(3\mu_k + 5 - 5\cdot 2^k\right) l_{2n+1} - \left(5\mu_k + 5 - 15\cdot 2^k\right) f_{2n+1} + 2\mu_k(-1)^n,$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence and $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence, and μ_k is the characteristic of 2^k-Fibonacci sequence.

Proof. By Equations (2.7), (2.8), (2.14), and (2.20) we obtain

$$\begin{aligned} 5F_{(k,n)}F_{(k,n+1)} &= 5[2^{k-1}l_n - (2^{k-1}-1)f_n][2^{k-1}l_{n+1} - (2^{k-1}-1)f_{n+1}] \\ &= 5\cdot 2^{2k-2}l_nl_{n+1} + 5(2^{k-1}-1)^2 f_nf_{n+1} - 5\cdot 2^{k-1}(2^{k-1}-1)(l_nf_{n+1} + l_{n+1}f_n) \\ &= 5\cdot 2^{2(k-1)}\left(l_{2n+1} + (-1)^n\right) + \left(2^{2(k-1)} - 2^k + 1\right)\left(l_{2n+1} - (-1)^n\right) - 5\cdot 2^{k-1}(2^{k-1}-1)(2f_{2n+1}) \\ &= \left(\frac{3}{2}\mu_k + \frac{5}{2} - \frac{5}{2}2^k\right) l_{2n+1} - 5\left(\frac{1}{2}\mu_k + \frac{1}{2} - \frac{3}{2}2^k\right) f_{2n+1} + \mu_k(-1)^n, \end{aligned}$$

then, by multiplying 2 on both sides and using that $\mu_k = 2^{2k} + 2^k - 1$, this completes the proof. □

The next result follows directly from Proposition 4.1 making $k = 2$.

Corollary 4.2. *Let $\{m_n\}_{n\geq 0}$ be the Fibonacci–Mulatu sequence. Then, we have*

$$5m_nm_{n+1} = 21l_{2n+1} - 20f_{2n+1} + 19(-1)^n,$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence and $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence.

Proof. Taking $k = 2$, and as $\mu_2 = 19$, then

$$\begin{aligned} 3\mu_2 + 5 - 5\cdot 2^2 &= 3\cdot 19 + 5 - 5\cdot 4 = 42 \\ 5\mu_2 + 5 - 15\cdot 2^2 &= 5\cdot 19 + 5 - 15\cdot 4 = 40 \\ 2\mu_2 &= 38. \end{aligned}$$

Therefore,

$$\begin{aligned} 10m_nm_{n+1} &= \left(3\mu_2 + 5 - 5\cdot 2^2\right) l_{2n+1} - \left(5\mu_2 + 5 - 15\cdot 2^2\right) f_{2n+1} + 2\mu_2(-1)^n \\ &= 42l_{2n+1} - 40f_{2n+1} + 38(-1)^n, \end{aligned}$$

Then, just divide by 2 both sides and the result follows. □

An alternative way to express the product of two consecutive 2^k-Fibonacci numbers is given by:

Proposition 4.3. *For all non-negative integers n and $k \geq 1$, the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following identity*

$$5F_{(k,n)}F_{(k,n+1)} = l_{2n+1} + 10\cdot 2^k f_n^2 + 4^k l_{2n-1} + (-1)^n(\mu_k + F_{(k+2,0)}),$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence and $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence.

Proof. By Equations (2.7), (2.9), and (2.21) we have

$$\begin{aligned}5F_{(k,n)}F_{(k,n+1)} &= 5[(f_n + 2^k f_{n-1})(f_{n+1} + 2^k f_n)]\\ &= l_{2n+1} + 10\cdot 2^k f_n^2 + 4^k l_{2n-1} + (4^k + 2^k - 1 + 4\cdot 2^k)(-1)^n,\end{aligned}$$

as $\mu_k = 4^k + 2^k - 1$, Proposition 2.4, it follows that

$$\begin{aligned}5F_{(k,n)}F_{(k,n+1)} &= l_{2n+1} + 10\cdot 2^k f_n^2 + 4^k l_{2n-1} + (\mu_k + 2^{k+2})(-1)^n\\ &= l_{2n+1} + 10\cdot 2^k f_n^2 + 4^k l_{2n-1} + (\mu_k + F_{(k+2,0)})(-1)^n,\end{aligned}$$

which concludes the proof. □

In the 2^k-Fibonacci sequence, the term of order $n+4$ can be represented as a linear combination of the terms of orders $n+2$ and n. Specifically, it is expressed as:

Proposition 4.4. *For all integers n, the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ sequence satisfies the following identity*

$$F_{(k,n+4)} = 3F_{(k,n+2)} - F_{(k,n)}.$$

Proof. It follows from Equations (2.10), (2.12), and (2.20) that

$$\begin{aligned}F_{(k,n+4)} &= 2^{k-1}l_{n+4} - (2^{k-1}-1)f_{n+4} = 2^{k-1}(3l_{n+2} - l_n) - (2^{k-1}-1)(3f_{n+2} - f_n)\\ &= 3\cdot 2^{k-1}l_{n+2} - 2^{k-1}l_n - 3(2^{k-1}-1)f_{n+2} + (2^{k-1}-1)f_n\\ &= 3(2^{k-1}l_{n+2} - (2^{k-1}-1)f_{n+2}) - (2^{k-1}l_n - (2^{k-1}-1)f_n)\\ &= 3F_{(k,n+2)} - F_{(k,n)}.\end{aligned}$$

which verifies the result. □

The following result is derived directly from Proposition 4.4 by setting $k=0$, $k=1$, and $k=2$, respectively.

Corollary 4.5. *Let $\{f_n^*\}_{n\geq 0}$, $\{l_n\}_{n\geq 0}$, and $\{m_n\}_{n\geq 0}$ be, respectively, the shifted Fibonacci sequence, the Fibonacci–Lucas sequence, and the Fibonacci–Mulatu sequence. Then, we have that*

(a) $f_n^* = 3f_{n+3} - f_{n+1}$,
(b) $l_n = 3l_{n+2} - l_n$,
(b) $m_n = 3m_{n+2} - m_n$,

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

The next auxiliary result establishes the addition formula for two terms in the index of the 2^k-Fibonacci numbers.

Lemma 4.6. *For non-negative integers n and m, and for the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$, we have*

$$F_{(k,n+m)} = f_{m-1}F_{(k,n)} + f_m F_{(k,n+1)}, \tag{4.1}$$

$$F_{(k,n-m)} = (-1)^m(f_{m+1}F_{(k,n)} - f_m F_{(k,n+1)}). \tag{4.2}$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

Equation (4.1) in Lemma 4.6 is a particular case of the general result due to Vajda [6, Equation (8)]. Namely, for any Fibonacci-type sequence $\{G_n\}_{n\geq 0}$ holds

$$G_{n+m} = f_{m-1}G_n + f_m G_{n+1}\,,$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence. When $k=1$, we have $F_{(1,n)} = l_n$ the Fibonacci–Lucas numbers, so

$$l_{n+m} = f_{m-1}l_n + f_m l_{n+1}\,,$$

and, when $k=2$, we have $F_{(2,n)} = m_n$ the Fibonacci–Mulatu number, see [29, Theorem 5]. So

$$m_{n+q} = f_{q-1}m_n + f_q m_{n+1}\ .$$

To finish the section, we will show that the double of all 2^k-Fibonacci is a linear combination of Fibonacci and Fibonacci-Lucas sequences.

Proposition 4.7. *For non-negative integers n, m, k, and for the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$, the following identity holds*

$$2F_{(k,n+m)} = l_m F_{(k,n)} + f_m(5\cdot 2^{k-1}f_n - (2^{k-1}-1)l_n)\,, \tag{4.3}$$

where $\{l_n\}_{n\geq 0}$ is the Fibonacci–Lucas sequence and $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. Combining Equations (4.1) and (4.2) we have

$$F_{(k,n+m)} + (-1)^m F_{(k,n-m)} = f_{m-1}F_{(k,n)} + f_m F_{(k,n+1)} + f_{m+1}F_{(k,n)} - f_m F_{(k,n+1)} = l_m F_{(k,n)}$$

and

$$F_{(k,n+m)} - (-1)^m F_{(k,n-m))} = f_{m-1}F_{(k,n)} + f_m F_{(k,n+1)} - f_{m+1}F_{(k,n)} + f_m F_{(k,n+1)} = f_m(F_{(k,n-1)} + F_{(k,n+1)})\,.$$

Now, by summing the last two equations, we obtain

$$2F_{(k,n+m)} = l_m F_{(k,n)} + f_m(F_{(k,n-1)} + F_{(k,n+1)})\,.$$

By the use of Equation (2.22), we obtain the result. □

4.2 Some classical identities

The following auxiliary result involves two Fibonacci sequences and can be found at [6, Equation 18].

Lemma 4.8. *[6] For any Fibonacci-type sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$, the following identity holds*

$$G_{n+h}H_{n+q} - G_n H_{n+h+q} = (-1)^n(G_h H_q - G_0 H_{h+q})\,, \tag{4.4}$$

where n,h,q are any natural numbers.

The Tagiuri-Vajda identity for the 2^k-Fibonacci sequence is presented as follows, which we get using the previous result.

Theorem 4.9. *Let n,h,q be any natural number. We have*

$$F_{(k,n+h)}F_{(k,n+q)} - F_{(k,n)}F_{(k,n+h+q)} = (-1)^{n+1}\mu_k f_h f_q\ , \tag{4.5}$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence, and $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence, and μ_k is the characteristic of $\{F_{(k,n)}\}_{n\geq 0}$.

Proof. We specialize in Equation (4.4) taking $G_i = H_i = F_{(k,i)}$, and we have

$$F_{(k,n+h)}F_{(k,n+q)} - F_{(k,n)}F_{(k,n+h+q)} = (-1)^n[(F_{(k,h)}F_{(k,q)} - F_{(k,0)}F_{(k,h+q)}]\,.$$

As $F_{(k,0)} = 2^k$ and by use of the Equations (2.20) and (4.3) we have

$$\begin{aligned}
F_{(k,h)}F_{(k,q)} - F_{(k,0)}F_{(k,h+q)} &= F_{(k,h)}F_{(k,q)} - 2^{k-1}2F_{(k,h+q)} \\
&= F_{(k,h)}F_{(k,q)} - 2^{k-1}l_qF_{(k,h)} + 5\cdot 2^{2(k-1)}f_qf_h - 2^{k-1}(2^{k-1}-1)f_ql_h \\
&= [F_{(k,h)}(F_{(k,q)} - 2^{k-1}l_q) - 2^{k-1}f_q(5\cdot 2^{k-1}f_h - (2^{k-1}-1)l_h)] \\
&= [-(2^{k-1}-1)F_{(k,h)}f_q - 2^{k-1}f_q(5\cdot 2^{k-1}f_h - (2^{k-1}-1)l_h)] \\
&= (-1)f_hf_q[4\cdot 2^{2(k-1)} + 2\cdot 2^{k-1} - 1] \\
&= (-1)f_hf_q[4^k + 2^k - 1]\,,
\end{aligned}$$

as $\mu_k = 4^k + 2^k - 1$, which establishes the proof. □

As a direct consequence of the Tagiuri-Vajda identity, the following results are generalized to establish d'Ocagne's identity, Catalan's identity, and Cassini's identity specifically for the 2^k-Fibonacci sequence.

Proposition 4.10 (d'Ocagne's identity)**.** *Let m, n be non-negative integers, then*

$$F_{(k,m)}F_{(k,n+1)} - F_{(k,n)}F_{(k,m+1)} = (-1)^{n+1}\mu_k f_{m-n}\,,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence, and $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence, and μ_k is the characteristic of $\{F_{(k,n)}\}_{n\geq 0}$.

Proof. Consider $h = m - n$ and $q = 1$ in Equation (4.5), then

$$F_{(k,m)}F_{(k,n+1)} - F_{(k,n)}F_{(k,m+1)} = (-1)^{n+1}\mu_k f_{m-n}f_1 = (-1)^{n+1}\mu_k f_{m-n}\,.$$

which proves the result. □

Proposition 4.11 (Catalan's identity)**.** *Let n, q be non-negative integers, then*

$$F_{(k,n+q)}F_{(k,n-q)} - (F_{(k,n)})^2 = (-1)^{n+q}\mu_k f_q^2\,, \tag{4.6}$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence, and $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence, and μ_k is the characteristic of $\{F_{(k,n)}\}_{n\geq 0}$.

Proof. Taking $h = -q$ in Equation (4.5), we have that

$$F_{(k,n-q)}F_{(k,n+q)} - F_{(k,n)}F_{(k,n)} = (-1)^{n+1}\mu_k f_{-q}f_q\,.$$

As $f_{-q} = (-1)^{q+1}f_q$, we have

$$F_{(k,n+q)}F_{(k,n-q)} - (F_{(k,n)})^2 = (-1)^{n+q+2}\mu_k (f_q)^2\,,$$

and the result follows. □

As a consequence of Catalan's identity, as $f_1 = 1$ and by doing $q = 1$ in Equation (4.6), we have the following result.

Corollary 4.12 (Cassini-Simson's identity)**.** *For all non-negative integers n, we have*

$$(F_{(k,n)})^2 - F_{(k,n+1)}F_{(k,n-1)} = (-1)^n\mu_k\,,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence, and μ_k is the characteristic of $\{F_{(k,n)}\}_{n\geq 0}$.

Other consequence is the Cassini-Simson identity for subscripts even:

Corollary 4.13. *For all non-negative integers n, we have*

$$(F_{(k,2n)})^2 - F_{(k,2n+1)}F_{(k,2n-1)} = \mu_k\,,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence, and μ_k is the characteristic of $\{F_{(k,n)}\}_{n\geq 0}$.

Now, we present the Convolution identity for generalized 2^k-Fibonacci sequence.

Proposition 4.14 (Convolution's identity)**.** *Let m,n be non-negative integers, then the generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following identity*

$$F_{(k,m-1)}F_{(k,n)}+F_{(k,m)}F_{(k,n+1)}=F_{(k,m+n)}+2^kF_{(k,m+n-1)}\,.$$

Proof. As $F_{(k,n)}=f_n+2^kf_{n-1}$, Equation (2.21), then

$$\begin{aligned}F_{(k,m-1)}F_{(k,n)}+F_{(k,m)}F_{(k,n+1)}&=(f_{m-1}+2^kf_{m-2})(f_n+2^kf_{n-1})+(f_m+2^kf_{m-1})(f_{n+1}+2^kf_n)\\&=f_{m-1}f_n+2^kf_{m-1}f_{n-1}+2^kf_{m-2}f_n+2^{2k}f_{m-2}f_{n-1}+f_mf_{n+1}\\&\qquad+2^kf_mf_n+2^kf_{m-1}f_{n+1}+2^{2k}f_{m-1}f_n\\&=F_{(k,n+m)}+2^kF_{(k,n+m-1)}\,,\end{aligned}$$

as required. □

The next result provides an alternative convolution identity for the generalized 2^k-Fibonacci sequence in terms of the Fibonacci sequence.

Proposition 4.15. *The generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following*

$$F_{(k,m-1)}F_{(k,n)}+F_{(k,m)}F_{(k,n+1)}=(2^{k+1}+1)f_{m+n}+2^{k+1}(2^{k-1}-1)f_{m+n-2}\,,$$

where $\{f_n\}_{n\geq 0}$ is the Fibonacci sequence.

Proof. It follows from Equation (2.21) that

$$\begin{aligned}F_{(k,m-1)}F_{(k,n)}+F_{(k,m)}F_{(k,n+1)}&=f_{m+n}+2^{k+1}f_{m+n-1}+2^{2k}f_{m+n-2}\\&=f_{m+n}+2^{k+1}(f_{m+n-1}+f_{m+n-2})+(2^{2k}-2^{k+1})f_{m+n-2}\\&=(2^{k+1}+1)f_{m+n}+2^{k+1}(2^{k-1}-1)f_{m+n-2}\,,\end{aligned}$$

as required. □

A direct calculation yields the Melham identity.

Proposition 4.16 (Melham's identity)**.** *The generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following identity*

$$F_{(k,n+1)}F_{(k,n+2)}F_{(k,n+6)}-(F_{(k,n+3)})^3=(-1)^{n+1}\mu_kF_{(k,n)}\,.$$

Proof. Indeed, we have

$$\begin{aligned}F_{(k,n+1)}F_{(k,n+2)}(5F_{(k,n+2)}+3F_{(k,n+1)})-(F_{(k,n+2)}+F_{(k,n+1)})^3&=2F_{(k,n+1)}(F_{(k,n+2)})^2-(F_{(k,n+2)})^3-(F_{(k,n+1)})^3\\&=(-1)^{n+1}\mu_kF_{(k,n)}\,,\end{aligned}$$

where we apply Cassini-Simson's identity, Corollary 4.12, the fact that $F_{(k,n+6)}=5F_{(k,n+2)}+3F_{(k,n+1)}$ and as $\mu_k=4^k+2^k-1$. This provides the result. □

This result follows directly from Catalan's identity.

Proposition 4.17 (Gelin-Cesàro's identity)**.** *The generalized 2^k-Fibonacci sequence $\{F_{(k,n)}\}_{n\geq 0}$ satisfies the following identity*

$$F_{(k,n+2)}F_{(k,n+1)}F_{(k,n-1)}F_{(k,n-2)}-(F_{(k,n)})^4=-\mu_k^2\,.$$

Proof. Using Catalan's identity (4.6) for $q=2$ and $q=1$ we obtain

$$F_{(k,n+2)}F_{(k,n+1)}F_{(k,n-1)}F_{(k,n-2)}-(F_{(k,n)})^4=(F_{(k,n)})^4-\mu_k^2(-1)^{2n}-(F_{(k,n)})^4=-\mu_k^2\,,$$

as $\mu_k=4^k+2^k-1$, which verifies the result. □

4.3 Some limit identities

The quotient between two successive terms of a sequence, $\{a_n\}_{n\geq 0}$, is given by $q_n = \frac{a_{n+1}}{a_n}$, where q_n is the ratio of the terms a_{n+1} and a_n. For example, in the classical Fibonacci sequence $\{f_n\}_{n\geq 0}$, we have $q_n = \frac{f_{n+1}}{f_n}$. And for sufficiently large n, q_n converges to the golden ratio $\alpha = (1+\sqrt{5})/2$. In [30] present a study of some generalized Fibonacci sequences and their relationship to the golden ratio ϕ.

The first result shows that the quotient $q_{k,n}$ for the sequence $\{F_{(k,n)}\}_{n\geq 0}$ also converges to α as n goes to infinity. Moreover, we analyze the behavior of $q_{k,n}$ as k goes to infinity after n goes to infinity.

Proposition 4.18. *If $F_{(k,n)}$ are the n-th term of 2^k-Fibonacci sequence, then*

$$\lim_{n\to\infty} \frac{F_{(k,n+1)}}{F_{(k,n)}} = \alpha\,, \tag{4.7}$$

and

$$\lim_{n\to\infty} \frac{F_{(k,-(n+1))}}{F_{(k,-n)}} = -\alpha\,, \tag{4.8}$$

where α is the golden ratio $(1+\sqrt{5})/2$.

Proof. According to Binet's formula (2.2), we have that

$$\frac{F_{(k,n+1)}}{F_{(k,n)}} = \frac{\frac{(1-2^k\beta)\alpha^{n+1}-(1-2^k\alpha)\beta^{n+1}}{\alpha-\beta}}{\frac{(1-2^k\beta)\alpha^{n}-(1-2^k\alpha)\beta^{n}}{\alpha-\beta}} = \frac{\alpha^{n+1}\left[\frac{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n+1}}{\alpha-\beta}\right]}{\alpha^{n}\left[\frac{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n}}{\alpha-\beta}\right]}$$
$$= \alpha\,\frac{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n+1}}{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n+1}}\,.$$

Since $|\beta/\alpha| < 1$, it follows that $(\beta/\alpha)^n \to 0$ as $n\to\infty$. Thus, we have

$$\lim_{n\to\infty} \frac{F_{(k,n+1)}}{F_{(k,n)}} = \alpha\,\frac{1-2^k\beta}{1-2^k\beta} = \alpha\,,$$

and thus (4.7) follows.

Using Equation (2.26), we can write

$$\frac{F_{(k,-(n+1))}}{F_{(k,-n)}} = \frac{(-1)^{n+1}\left(F_{(k,n+1)}+2(2^{k-1}-1)f_{n+1}\right)}{(-1)^{n}\left(F_{(k,n)}+2(2^{k-1}-1)f_{n}\right)} = -\frac{F_{(k,n+1)}+2(2^{k-1}-1)f_{n+1}}{F_{(k,n)}+2(2^{k-1}-1)f_{n}}\,.$$

It follows from Binet's formula that

$$\frac{F_{(k,n+1)}+2(2^{k-1}-1)f_{n+1}}{F_{(k,n)}+2(2^{k-1}-1)f_{n}} = \alpha\,\frac{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n+1}+2(2^{k-1}-1)(1-(\frac{\beta}{\alpha})^{n+1})}{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n}+2(2^{k-1}-1)(1-(\frac{\beta}{\alpha})^{n})}\,.$$

Therefore,

$$\lim_{n\to\infty}\left[\frac{F_{(k,n+1)}+2(2^{k-1}-1)f_{n+1}}{F_{(k,n)}+2(2^{k-1}-1)f_{n}}\right] = \alpha\,\frac{(1-2^k\beta)+2(2^{k-1}-1)}{(1-2^k\beta)+2(2^{k-1}-1)} = \alpha\,.$$

Hence,

$$\lim_{n\to\infty} \frac{F_{(k,-(n+1))}}{F_{(k,-n)}} = -\alpha\,.$$

□

In what follows, we can immediately establish the next result using fundamental tools from the calculus of limits, along with (4.7) and (4.8).

Corollary 4.19. *If $F_{(k,n)}$ are the n-th term of the 2^k-Fibonacci sequence, then*

$$\lim_{n\to\infty}\frac{F_{(k,n)}}{F_{(k,n+1)}}=\frac{1}{\alpha},$$

and

$$\lim_{n\to\infty}\frac{F_{(k,-n)}}{F_{(k,-(n+1))}}=-\frac{1}{\alpha},$$

where $\alpha=(1+\sqrt{5})/2$.

Proposition 4.20. *If $F_{(k,n)}$ are the n-th term of 2^k-Fibonacci sequence, then*

$$\lim_{n\to\infty}\frac{F_{(k+1,n+1)}}{F_{(k,n)}}=\frac{\alpha+2^{k+1}}{1-2^k\beta}, \tag{4.9}$$

and

$$\lim_{n\to\infty}\frac{F_{(k+1,-(n+1))}}{F_{(k,-n)}}=-\frac{2^{k+1}(1+\alpha)-\alpha}{2^k(1-\beta)-1}, \tag{4.10}$$

where α and β are the distinct roots of Equation (2.1).

Proof. By Binet's formula (2.2), we have

$$\lim_{n\to\infty}\frac{F_{(k+1,n+1)}}{F_{(k,n)}}=\alpha\lim_{n\to\infty}\frac{(1-2^{k+1}\beta)-(1-2^{k+1}\alpha)(\frac{\beta}{\alpha})^{n+1}}{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n+1}}=\frac{\alpha(1-2^{k+1}\beta)}{1-2^k\beta}=\frac{\alpha+2^{k+1}}{1-2^k\beta},$$

and (4.9) follows, where we use that $\alpha\beta=-1$ and $(\beta/\alpha)^n\to 0$ as $n\to\infty$.

Analogously,

$$\begin{aligned}\lim_{n\to\infty}\frac{F_{(k+1,-(n+1))}}{F_{(k,-n)}}&=\alpha\lim_{n\to\infty}\frac{(1-2^{k+1}\beta)-(1-2^{k+1}\alpha)(\frac{\beta}{\alpha})^{n+1}+2(2^k-1)(1-(\frac{\beta}{\alpha})^{n+1})}{(1-2^k\beta)-(1-2^k\alpha)(\frac{\beta}{\alpha})^{n}+2(2^{k-1}-1)(1-(\frac{\beta}{\alpha})^{n})}\\&=\frac{2^{k+1}(1+\alpha)-\alpha}{2^k(1-\beta)-1},\end{aligned}$$

which completes the proof. □

Corollary 4.21. *If $F_{(k,n)}$ are the n-th term of 2^k-Fibonacci sequence, then*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F_{(k+1,n+1)}}{F_{(k,n)}}=2\alpha,$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F_{(k+1,-(n+1))}}{F_{(k,-n)}}=-2\alpha,$$

where $\alpha=(1+\sqrt{5})/2$.

Proof. By Equation (4.9) we have that

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F_{(k+1,n+1)}}{F_{(k,n)}}=\lim_{k\to\infty}\frac{2^k(\frac{\alpha}{2^k}+2)}{2^k(\frac{1}{2^k}-\beta)}=\frac{2}{-\beta}=2\alpha.$$

By Equation (4.10) we have that

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{F_{(k+1,-(n+1))}}{F_{(k,-n)}}=-\lim_{k\to\infty}\frac{2^k[2(1+\alpha)-\frac{\alpha}{2^{k+1}}]}{2^k[(1-\beta)-\frac{1}{2^k}]}=-\frac{2(1+\alpha)}{1-\beta}=2(1+\alpha)\beta=2(\beta+\beta\alpha)=2(\beta-1)=-2\alpha,$$

since α and β are the distinct roots of Equation (2.1). □

Lemma 4.22. *Let μ_k the characteristic of $\{F_{(k,n)}\}_n$, the generalized 2^k-Fibonacci sequence. Then*

$$\lim_{k\to\infty} \frac{\mu_{k+1}}{\mu_k} = 4\,. \tag{4.11}$$

Proof. In fact,

$$\lim_{k\to\infty} \frac{\mu_{k+1}}{\mu_k} = \lim_{k\to\infty} \frac{4^{k+1}+2^{k+1}-1}{4^k+2^k-1} = \lim_{k\to\infty} \frac{4+\frac{2^{k+1}}{4^k}-\frac{1}{4^k}}{1+\frac{2^k}{4^k}-\frac{1}{4^k}} = 4\,,$$

as we wanted to demonstrate. □

Theorem 4.23. *For all non-negative integers n, we have*

$$\lim_{k\to\infty} \frac{(F_{(k+1,n)})^2 - F_{(k+1,n+1)}F_{(k+1,n-1)}}{(F_{(k,n)})^2 - F_{(k,n+1)}F_{(k,n-1)}} = 4\,,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the generalized 2^k-Fibonacci sequence.

Proof. It follows from Cassini-Simson's identity and Lemma 4.22. □

5. A Partial Sum Involving the 2^k-Fibonacci Numbers

This section explores results related to the partial sums of the terms in the 2^k-Fibonacci sequence, denoted as $\{F_{(k,n)}\}_{n\geq 0}$. The sum of the first $n+1$ terms of the sequence is expressed as:

$$\sum_{j=0}^{n} F_{(k,j)} = F_{(k,0)} + F_{(k,1)} + F_{(k,2)} + \cdots + F_{(k,n-1)} + F_{(k,n)}\,.$$

We begin by presenting three key results regarding the partial sums of the 2^k-Fibonacci sequence.

First, the sum of the first $n+1$ terms of the 2^k-Fibonacci sequence .

Theorem 5.1. *For all non-negative integers n, then*

$$\sum_{j=0}^{n} F_{(k,j)} = F_{(k,n+2)} - 1\ ,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ is the 2^k-Fibonacci sequence.

Proof. According to Equation (1.1) we have the following equations:

$$\begin{aligned}
F_{(k,0)} &= F_{(k,2)} - F_{(k,1)}\,,\\
F_{(k,1)} &= F_{(k,3)} - F_{(k,2)}\,,\\
&\cdots\\
F_{(k,n-1)} &= F_{(k,n+1)} - F_{(k,n)}\,,\\
F_{(k,n)} &= F_{(k,n+2)} - F_{(k,n+1)}\,.
\end{aligned}$$

By adding both sides of these equations, we arrive at the following result:

$$\sum_{j=0}^{n} F_{(k,j)} = F_{(k,n+2)} - F_{(k,1)}.$$

Since $F_{(k,1)} = 1$ for all k, we conclude the result. □

Note that the 2^k-Fibonacci sequence belongs to the class of Lucas(Horadam) type sequences. Therefore, the partial sum presented in Theorem 5.1 is a specific example of the fundamental summation rule recently published in [31].

Next, the sum of the terms in odd indexes of the 2^k-Fibonacci sequence is given by:

Proposition 5.2. *For all non-negative integers n, then*

$$\sum_{j=0}^{n} F_{(k,2j+1)} = F_{(k,2n+2)} - 2^k \ ,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ *is the* 2^k*-Fibonacci sequence.*

The sum of the terms at even indexes in the 2^k-Fibonacci sequence can be expressed as:

Proposition 5.3. *For all non-negative integers n, then*

$$\sum_{j=0}^{n} F_{(k,2j)} = F_{(k,2n+1)} + 2^k - 1 \ ,$$

where $\{F_{(k,n)}\}_{n\geq 0}$ *is the* 2^k*-Fibonacci sequence.*

The proof of Proposition 5.2 is similar to that of Theorem 5.1, while the proof of Proposition 5.3 relies on Proposition 5.1.

An consequence from previous results is the results presented below, these naturally arise from the established relationships and further reinforce the conclusions derived from Propositions 5.2 and 5.3.

Proposition 5.4. *Let* $\{F_{(k,n)}\}_{n\geq 0}$ *be the* 2^k*-Fibonacci sequence. For all non-negative integers m, we have the following formulas*

(a) $\sum_{k=0}^{2n+1} (-1)^j F_{(k,j)} = 2^{k+1} - 1 - F_{(k,2n)}$*; if n is odd,*

and

(b) $\sum_{k=0}^{2(n+1)} (-1)^k F_{(k,j)} = 2^{k+1} - 1 + F_{(k,2n+1)}$ *; if n is even.*

Proof. (a) As $2n+1$ is odd, that is, the last term is negative, so

$$\begin{aligned}\sum_{k=0}^{2n+1} (-1)^k F_{(k,j)} &= F_{(k,0)} - F_{(k,1)} + F_{(k,2)} - F_{(k,3)} + \cdots + F_{(k,2n)} - F_{(k,2n+1)} \\ &= \sum_{j=0}^{n} F_{(k,2j)} - \sum_{j=0}^{n} F_{(k,2j+1)} \ .\end{aligned}$$

According to Propositions 5.2 and 5.3, it follows that:

$$\sum_{k=0}^{2n+1} (-1)^j F_{(k,j)} = (F_{(k,2n+1)} + 2^k - 1) - (F_{(k,2n+2)} - 2^k) = F_{(k,2n+1)} - F_{(k,2n+2)} + 2^{k+1} - 1 \ .$$

We obtain the result, using Equation (1.1).

(b) In which case, $2(n+1)$ is even, so

$$\sum_{j=0}^{2(n+1)} (-1)^j F_{(k,j)} \quad = \quad F_{(k,0)} - F_{(k,1)} + F_{(k,2)} - \cdots - F_{(k,2n+1)} + F_{(k,2n+2)} = F_{(k,2n+1)} + 2^{k+1} - 1.$$

As in item (a), apply the Propositions 5.2 and 5.3. □

Finally, we derive an expression for the sum of the squares of the first n terms of the 2^k-Fibonacci sequence, relating it to the classical Fibonacci sequence.

Proposition 5.5. *Let* $\{F_{(k,n)}\}_{n\geq 0}$ *be the* 2^k*-Fibonacci sequence. The sum of the squares of the first* $n+1$ *terms of the* 2^k*-Fibonacci sequence is given by*

$F_{(k,0)}^2 + F_{(k,1)}^2 + F_{(k,2)}^2 + F_{(k,3)}^2 + \cdots + F_{(k,n-1)}^2 + F_{(k,n)}^2 = 2^k(2^k - 1) + F_{(k,n)} F_{(k,n+1)}$,

for all non-negative integers n.

Proof. To begin, observe that for $n \geq 2$, the following holds:

$$F_{(k,n)}F_{(k,n+1)} - F_{(k,n-1)}F_{(k,n)} = F_{(k,n)}(F_{(k,n+1)} - F_{(k,n-1)}) = F^2_{(k,n)}\,.$$

Thus, we find:

$$\begin{aligned} F^2_{(k,2)} &= F_{(k,2)}F_{(k,3)} - F_{(k,1)}F_{(k,2)}, \\ F^2_{(k,3)} &= F_{(k,3)}F_{(k,4)} - F_{(k,2)}F_{(k,3)}, \\ &\dots \\ F^2_{(k,n-1)} &= F_{(k,n-1)}F_{(k,n)} - F_{(k,n-2)}F_{(k,n-1)}, \\ F^2_{(k,n)} &= F_{(k,n)}F_{(k,n+1)} - F_{(k,n-1)}F_{(k,n)}. \end{aligned}$$

Adding both sides of these equations yields:

$$F^2_{(k,2)} + F^2_{(k,3)} + \cdots + F^2_{(k,n-1)} + F^2_{(k,n)} = F_{(k,n)}F_{(k,n+1)} - F_{(k,1)}F_{(k,2)}\,.$$

Since $F_{(k,0)} = 2^k$, $F_{(k,1)} = 1$, and $F_{(k,2)} = 2^k + 1$, the result follows. □

6. Final Considerations

In this work, we introduced the concept of 2^k-Fibonacci numbers, which is closely associated with the classical Fibonacci number. We presented and studied precisely a novel family of Fibonacci-type sequences. The main focus of this paper is on algebraic identities. So, the relationship between classical Fibonacci sequences and arbitrary 2^k-Fibonacci sequences is explained. The general applications of Fibonacci numbers in various fields have long attracted the interest of mathematicians, researchers, and other enthusiasts. Motivated by this, our study proposed the 2^k-Fibonacci numbers and provided a comprehensive analysis of their properties and relationships with other well-known sequences, such as the ordinary Fibonacci, Fibonacci-Lucas, and Fibonacci-Mulatu numbers. We established several new identities that encouraged the study of the patterns in these sequences, significantly improving the understanding of their mathematical properties. Additionally, we derived the generating function for this new class of sequences and formulated a generalized Binet-type formula to describe them explicitly. Furthermore, we have explored the classical identities associated with 2^k-Fibonacci numbers, including the Tagiuri-Vajda, d'Ocagne, Catalan, and Cassini identities, thereby demonstrating their deep connections to traditional results related to Fibonacci-type sequences. In future work, we intend to explore matrix methods, combinatorial interpretations, and asymptotic growth in as much detail as possible regarding generalized 2^k-Fibonacci numbers and their applications.

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