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RESEARCH ARTICLE

Dirihlet problem for the generalized Beltrami equation

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ABSTRACT

In this article, we investigate the Dirichlet problem for the generalized Beltrami equation. Firstly, we introduce the solutions of the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc. Secondly, we state the properties of the integral operators for regular domains. Then, by using Banach fixed point theorem, we obtain the existence of the unique solution of the Dirichlet problem for the generalized Beltrami equation in the unit disc.

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1. INTRODUCTION AND PRELIMINARIES

Many researchers studied Dirichlet problem in different domains Begehr, H. (2005a,b); Vaitsiakhovich, T. (2008a); Vaitekhovich, T.S. (2008b); Vaitekhovich, T. (2007); Gökgöz, P.A. (2024a,b); Begehr, H. and Shupeyeva, B. (2021); Wang, Y. and Du, J. (2015); Aksoy, Ü. and Çelebi, A.O. (2012); Begehr, H.G.W. (1994); Vekua, I.N. (1962); Aksoy, Ü. and Çelebi, A.O. (2010); Begehr, H. and Vaitekhovich, T., (2012); Begehr, H. and Gaertner, E. (2007). In this paper, we study the Dirichlet problem for the generalized Beltrami equation in the unit disc. The rest of the paper is structured as follows: Section 2 is reserved for an overview of the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc. In Section 3, we examine the properties of the integral operators. In the last section, we obtain the existence of the unique solution of the Dirichlet problem for the generalized Beltrami equation in the unit disc using Banach fixed point theorem.

Now we provide the necessary background and fundamental concepts required for the development of the main results in this paper.

One of the main tools in solving complex boundary value problems is the complex analogue of Gauss's theorems Begehr, H. (2005a).

Theorem 1.1. *Gauss Theorems (complex form)*
Let $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$ in a regular domain D of the complex plane $\mathbb{C}$ then

$$\int_D w_{\bar{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz$$

and

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\bar{z}$$

From the Gauss theorems in complex form, the following representation formulas can be deduced Begehr, H. (2005a,b); Begehr, H.G.W. (1994). The following formulas provide an explicit representation of the solution of boundary value problems.

Theorem 1.2. *Cauchy-Pompeiu representations*
Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$. Then using $\zeta = \xi + i\eta$ in for $z \in D$

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

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and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_{D} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}$$

hold.

For the proof of Theorem 1.2, see for instance Begehr, H. (2005a). We can observe that if $w_{\bar{\zeta}}$ is given in D and the values of w along the boundary is known, we can identify a unique function $w(z)$. This representation is an example of the solution of the Dirichlet problem. This highlights how integral representation formulas facilitate the solution of boundary value problems.

In connection with the Hölder continuity of Cauchy integrals we mention a result from Begehr, H.G.W. (1994).

Theorem 1.3. *Let $w = u + iv$ be analytic in the unit disc $\mathbb{D}$, where v is continuous in the closure $\overline{\mathbb{D}}$ and Hölder continuous on the boundary $\partial\mathbb{D}$ satisfying*

$$|v(\zeta) - v(\tau)| \leq H|\zeta - \tau|^{\alpha}, \quad \zeta, \tau \in \partial\mathbb{D}.$$

Then w is Hölder continuous in $\overline{\mathbb{D}}$ with the same exponent and the constant kH where k only depends on α, i.e.

$$|w(z) - w(z_0)| \leq kH|z - z_0|^{\alpha}, \quad z, z_0 \in \overline{\mathbb{D}}.$$

2. DIRICHLET PROBLEM IN THE UNIT DISC

Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc has been studied by Begehr, H. (2005b). Now we state this problem as follows.

Theorem 2.1. *Begehr, H. (2005b) The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\bar{z}}^{n} w = f \text{ in } \mathbb{D},\ \partial_{\bar{z}}^{\nu} w = \gamma_{\nu} \text{ on } \partial\mathbb{D},\ 0 \leq \nu \leq n-1,$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}),\ \gamma_\nu \in C(\partial\mathbb{D};\mathbb{C}),\ 0 \leq \nu \leq n-1$, if and only if for $0 \leq \nu \leq n-1$

$$\sum_{\lambda=\nu}^{n-1} \frac{\bar{z}}{2\pi i}\int_{|\zeta|=1} (-1)^{\lambda-\nu}\frac{\gamma_\lambda(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!}d\zeta$$
$$+\frac{(-1)^{n-\nu}\bar{z}}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!}d\xi d\eta = 0.$$

The solution then is

$$w(z) = \sum_{\nu=0}^{n-1}\frac{(-1)^{\nu}}{2\pi i}\int_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\frac{(\overline{\zeta - z})^{\nu}}{\zeta - z}d\zeta$$
$$+\frac{(-1)^{n}}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{(\overline{\zeta - z})^{n-1}}{\zeta - z}d\xi d\eta.$$

Proof. For the proof of this theorem, we may use induction. In the case of $n = 1$, we obtain the Dirichlet problem for the Cauchy-Riemann equation in the unit disc as

$$\partial_{z} w = f \quad \text{in } \mathbb{D},\ w = \gamma_0 \text{ on } \partial\mathbb{D}.$$

The solution is given by the classical Cauchy-Pompeiu formula:

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\gamma_0(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_{|\zeta|<1}\frac{f(\zeta)}{\zeta - z}d\xi d\eta.$$

if the solvability condition holds.

Assume that the result holds for order $n-1$. In the case of n can be structured by decomposing the problem into the following system of equations:

$$\partial_{\bar{z}}^{n-1} w = \omega \text{ in } \mathbb{D},\ \partial_{\bar{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D},\ 0 \leq \nu \leq n-2,$$
$$\partial_{\bar{z}}\omega = f \text{ in } \mathbb{D},\ \partial_{\bar{z}}\omega = \gamma_{n-1} \text{ on } \partial\mathbb{D},$$

where

$$\omega(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_{n-1}(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

Each of these two problems is solved explicitly using known integral representations for lower-order cases. ω is substituted into the expression for w, thereby completing the proof.

3. PROPERTIES OF THE INTEGRAL OPERATORS

In this section, we state the properties of the integral operators for regular domains. The properties of the integral operators have been extensively investigated in Begehr, H. (2005a); Begehr, H. and Hile, G.N., (1997); Vekua, I.N. (1962).

Definition 3.1. Begehr, H. (2005a) For $f \in L_1(D;\mathbb{C})$ the integral operator

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z},\ z \in \mathbb{C}$$

is called Pompeiu operator.

Theorem 3.2. *Begehr, H. (2005a) If $f \in L_1(D;\mathbb{C})$ then for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy + \int_D f(z)\varphi(z)dxdy = 0.$$

Proof. We use the Cauchy-Pompeiu representation formula and the fact that the boundary values of φ vanish at the boundary. We obtain

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\varphi(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \varphi_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \left(T\varphi_{\bar{\zeta}}\right)(z).$$

We can interchange the order of integration

$$\int_D Tf(z)\varphi_{\bar{z}}(z)dxdy = -\frac{1}{\pi}\int_D f(\zeta)\int_D \varphi_{\bar{z}}(z)\frac{dxdy}{\zeta - z}d\xi d\eta = -\int_D f(\zeta)\varphi(\zeta)d\xi d\eta.$$

This means that

$$\partial_{\bar{z}}Tf = f$$

in distributional sense.

Theorem 3.3. *Vekua, I.N. (1962) If $f \in L^1(D)$ then Tf has generalized first order derivative with respect to $\bar{z}$ equal to f, i.e.,*

$$\frac{\partial}{\partial\bar{z}}Tf = f.$$

Proof. This theorem is a consequence of Theorem 3.2.

Also, we can compute the z derivative of $Tf(z)$.

Remark 3.4. For $z \in \mathbb{C}\backslash\bar{D}$, Tf is analytic and its derivative with respect to z is

$$\partial_z Tf(z) = \Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}.$$

Theorem 3.5. *Vekua, I.N. (1962) For $f \in L^p(\mathbb{C})$ we have $\Pi f \in L^p(\mathbb{C})$ and*

$$\|\Pi f\|_{L^p(\mathbb{C})} \leq \Lambda_p \|f\|_{L^p(\mathbb{C})} \quad (p > 1)$$

with $\|\Pi\|_{L^2(\mathbb{C})} = 1$.

Proof. For the proof of this theorem, see for instance Vekua, I.N. (1962), p. 66-72.

4. DIRICHLET PROBLEM FOR THE GENERALIZED BELTRAMI EQUATION IN THE UNIT DISC

Some of the boundary value problems for the Beltrami equation are studied by several researchers see for instance Begehr, H. and Harutyunyan, G. (2009); Begehr, H. and Obolashvili, E. (1994); Harutyunyan, G. (2007); Tutschke, W. (1983); Begehr, H. and Vaitekhovich, T. (2007); Yüksel, U. (2010).

In this section, we prove the existence and uniqueness of the solution to the Dirichlet problem for the generalized Beltrami equation in the unit disc using Banach fixed point theorem. Let $\mathbb{D}$ be the unit disc, $\partial\mathbb{D}$ its boundary and $C^\alpha(\overline{\mathbb{D}})$ the space of Hölder continuous functions in $\overline{\mathbb{D}}$ with the Hölder exponent α, where $0 < \alpha < 1$. Based on Begehr, H. and Vaitekhovich, T. (2007), the problem under consideration is as follows:

"Find the unique solution of the complex differential equation

$$w_{\bar{z}} = F(z, w, w_z) := q_1(z)w_z + q_2(z)\overline{w_z} + a(z)w + b(z)\bar{w} + c(z) \text{ in } \mathbb{D} \tag{1}$$

satisfying the boundary condition

$$w = \gamma \text{ on } \partial\mathbb{D} \tag{2}$$

where

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1 \tag{3}$$

and $q_1, q_2, a, b, c \in C^{\alpha}(\overline{\mathbb{D}})$, $\gamma \in C^{\alpha}(\partial\mathbb{D})$, $1 < p$."

We need some assumptions on the function $F(z, w, w_z)$.

- The function $F(z, w, w_z)$ is Hölder continuous with respect to z.

For $z_1, z_2 \in \overline{\mathbb{D}}$, we obtain

$$\begin{aligned}|F(z_1, w, w_z) - F(z_2, w, w_z)| &= |(q_1(z_1) - q_1(z_2))w_z + (q_2(z_1) - q_2(z_2))\overline{w_z} \\ &\quad + (a(z_1) - a(z_2))w + (b(z_1) - b(z_2))\overline{w} + c(z_1) - c(z_2)| \\ &\leq |w_z||q_1(z_1) - q_1(z_2)| + |w_z||q_2(z_1) - q_2(z_2)| \\ &\quad + |w||a(z_1) - a(z_2)| + |w||b(z_1) - b(z_2)| + |c(z_1) - c(z_2)|.\end{aligned}$$

So

$$|F(z_1, w, w_z) - F(z_2, w, w_z)| \leq C(1 + |w| + |w_z|)|z_1 - z_2|^{\alpha}.$$

- The function $F(z, w, w_z)$ satisfies the Lipschitz conditions with respect to z and w_z.

For $z_1, z_2 \in \overline{\mathbb{D}}$ and $w_{z1}, w_{z2} \in \mathbb{C}$, we obtain

$$\begin{aligned}|F(z_1, w, w_{z1}) - F(z_2, w, w_{z2})| &= |q_1(z_1)w_{z1} - q_1(z_2)w_{z2} + q_2(z_1)\overline{w_{z1}} - q_2(z_2)\overline{w_{z2}} \\ &\quad + (a(z_1) - a(z_2))w + (b(z_1) - b(z_2))\overline{w} + c(z_1) - c(z_2)| \\ &\leq |q_1(z_1)||w_{z1} - w_{z2}| + |q_1(z_1) - q_1(z_2)||w_{z2}| \\ &\quad + |q_2(z_1)||w_{z1} - w_{z2}| + |q_2(z_1) - q_2(z_2)||w_{z2}| \\ &\quad + |a(z_1) - a(z_2)||w| + |b(z_1) - b(z_2)||w| + |c(z_1) - c(z_2)|\end{aligned}$$

We know that all coefficients of (1) are in $C^{\alpha}(\overline{\mathbb{D}})$. Also we have the condition (3). Using these, the result follows.

$$|F(z_1, w, w_{z1}) - F(z_2, w, w_{z2})| \leq q_0|w_{z1} - w_{z2}| + C(1 + |w| + |w_{z2}|)|z_1 - z_2|^{\alpha}.$$

Our aim is to transform the Dirichlet problem for the generalized Beltrami equation (1) with (2) to a fixed point problem.
We prove the following theorem.

Theorem 4.1. *A function $w \in C^{1,\alpha}(\bar{\Omega})$ is a solution to the Dirichlet problem (1) with the boundary condition (2) if and only if w can be written as the integral equation*

$$w(z) = \varphi(z) + T\left[F(z, w, w_z)\right](z), \tag{4}$$

where $\varphi \in C^{\alpha}(\bar{\Omega})$ is holomorphic in the domain $\mathbb{D}$ satisfying the Dirichlet boundary condition

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}. \tag{5}$$

Proof. *We assume that $w \in C^{1,\alpha}(\bar{\Omega})$ is a solution to the Dirichlet problem (1) with the boundary condition (2). We define a function φ by*

$$\varphi(z) = w(z) - T\left[F(z, w, w_z)\right](z).$$

Differentiating φ with respect to $\bar{z}$, we obtain

$$\varphi_{\bar{z}} = w_{\bar{z}} - \left[F(z, w, w_z)\right] = 0.$$

That is, φ is a holomorphic function in $\mathbb{D}$. The Dirichlet condition (2) becomes

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}.$$

for the holomorphic function φ. We have $\gamma \in C^{\alpha}(\partial\mathbb{D})$. Further, $F(z, w, w_z) \in C^{\alpha}(\bar{\mathbb{D}})$ and therefore $T_{\Omega}[F(z, w, w_z)] \in C^{\alpha}(\bar{\mathbb{D}})$. Then, by Theorem 1.3, $\varphi(z) \in C^{\alpha}(\bar{\Omega})$ and therefore we have provided that w can be written as the integral equation (4) and Dirichlet boundary condition (5) is satisfied.

Conversely, suppose that w can be written as the integral equation (4), where φ is a holomorphic function satisfying (5). Differentiating (4) with respect to $\bar{z}$, we obtain

$$w_{\bar{z}} = F(z, w, w_z).$$

This shows that $w \in C^{1,\alpha}(\bar{\mathbb{D}})$ is a solution of the Dirichlet problem (1) with (2).

We use the representation

$$w(z) = \varphi(z) + T[F(z, w, w_z)](z) \tag{6}$$

where $\varphi(z)$ is an analytic function in $\mathbb{D}$. Differentiating (6) with respect to z and using the properties of the Pompeiu operators, we obtain

$$w_z(z) = \varphi' + \Pi[F(z, w, w_z)](z). \tag{7}$$

The boundary condition becomes

$$\varphi(z) = \gamma(z) - T[F(z, w, w_z)] \text{ on } \partial\mathbb{D}. \tag{8}$$

The equations (6) and (7) form the following system of integro-differential equations.

$$\begin{aligned} w(z) &= \varphi(z) + T[F(z, w, w_z)](z) \\ w_z(z) &= \varphi'(z) + \Pi[F(z, w, w_z)](z) \end{aligned}$$

For the simplicity, we denote w_z by w^*. Let w and w^* be any functions in $C^{\alpha}(\bar{\mathbb{D}})$.

We define an operator Q by

$$Q : (w, w^*) \to (W, W^*)$$

where

$$\begin{aligned} W &= \varphi_{(w,w^*)} + T[F(\cdot, w, w^*)] \\ W^* &= \varphi'_{(w,w^*)} + \Pi[F(\cdot, w, w^*)], \end{aligned}$$

$\varphi_{(w,w^*)}$ is holomorphic in $\mathbb{D}$ and satisfies the Dirichlet boundary condition

$$\varphi_{(w,w^*)} = \gamma(z) - T[F(z, w, w^*)] \text{ on } \partial\mathbb{D}.$$

The fixed point of the operator Q provides the solution of the (1) satisfying Dirichlet boundary condition (2). We will obtain the conditions on the coefficients of F under which the operator Q has a fixed point.

We introduce function space

$$S = \left\{(w, w^*) : w, w^* \in C^{\alpha}(\bar{\mathbb{D}})\right\}$$

equipped with the norm

$$\|(w, w^*)\|_{*,\alpha} = \max\left(\|w\|_{\alpha}, \|w^*\|_{\alpha}\right),$$

where $\|\cdot\|_{\alpha}$ denotes the Hölder norm in $C^{\alpha}(\bar{\Omega})$. Since $C^{\alpha}(\bar{\Omega})$ is a Banach space, S is a Banach space.

We pick (w_1, w_1^*) and (w_2, w_2^*) from S. Now let $Q(w_1, w_1^*) = (W_1, W_1^*)$ and $Q(w_2, w_2^*) = (W_2, W_2^*)$. Then we have

$$\begin{aligned} W_1 &= \varphi_{(w_1,w_1^*)} + T\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ W_1^* &= \varphi'_{(w_1,w_1^*)} + \Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] \end{aligned}$$

and

$$\begin{aligned} W_2 &= \varphi_{(w_2,w_2^*)} + T\left[F\left(\cdot, w_2, w_2^*\right)\right] \\ W_2^* &= \varphi'_{(w_2,w_2^*)} + \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]. \end{aligned}$$

where

$$\varphi_{(w_1,w_1{}^*)} = \gamma(z) - T[F(z, w_1, w_1{}^*)] \text{ on } \partial\mathbb{D}.$$

and

$$\varphi_{(w_2,w_2^*)} = \gamma(z) - T[F(z,w_2,w_2{}^*)] \ on\ \partial\mathbb{D}$$

respectively.

We obtain

$$\begin{aligned}\|W_1 - W_2\|_\alpha \leq \left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha + \|T\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha\end{aligned}$$

$$\begin{aligned}\left\|W_1^* - W_2^*\right\|_\alpha \leq \left\|\varphi'_{(w_1,w_1^*)} - \varphi'_{(w_2,w_2^*)}\right\|_\alpha + \|\Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] \\ - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha.\end{aligned}$$

For these computations, we set

$$C_1 := \|T\|_\alpha \left(C_{11}\left\|w_1^* - w_2^*\right\|_\alpha + C_{12}\|w_1 - w_2\|_\alpha\right)$$

$$C_2 := \|\Pi\|_\alpha \left(C_{11}\left\|w_1^* - w_2^*\right\|_\alpha + C_{12}\|w_1 - w_2\|_\alpha\right)$$

where $C_{11} := \|q_1\|_\alpha + \|q_2\|_\alpha$ and $C_{12} = \|a\|_\alpha + \|b\|_\alpha$.

Now, we use the fact that the operators T and Π are bounded in $C^\alpha(\bar{\Omega})$.

$$\left\|T\left[F\left(\cdot, w_1, w_1^*\right)\right] - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha \leq \|T\|_\alpha \|F\left(\cdot, w_1, w_1^*\right) - F\left(\cdot, w_2, w_2^*\right)\|_\alpha \leq C_1.$$

$$\|\Pi\left[F\left(\cdot, w_1, w_1^*\right)\right] - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\|_\alpha \leq \|\Pi\|_\alpha\| F\left(\cdot, w_1, w_1^*\right) - F\left(\cdot, w_2, w_2^*\right)\|_\alpha \leq C_2$$

and using Theorem 1.3, we have

$$\left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha \leq C_1 k$$

where k depends only on α. Similarly,

$$\left\|\varphi'_{(w_1,w_1^*)} - \varphi'_{(w_2,w_2^*)}\right\|_\alpha \leq C_2 \hat{k}$$

where $\hat{k}$ depends only on α.

Now we consider the distance $d\left(Q\left(w_1, w_1^*\right), Q\left(w_2, w_2^*\right)\right)$.

$$\begin{aligned}d\left(Q\left(w_1, w_1^*\right), Q\left(w_2, w_2^*\right)\right) &= \left\|\left(W_1 - W_2\right), \left(W_1^* - W_2^*\right)\right\| \\ &= \max\left(\|W_1 - W_2\|_\alpha, \left\|W_1^* - W_2^*\right\|_\alpha\right) \\ &\leq \max\left(\left\|\varphi_{(w_1,w_1^*)} - \varphi_{(w_2,w_2^*)}\right\|_\alpha + \right. \\ &\quad \left\|T\left[F\left(\cdot, w_1, w_1^*\right)\right] - T\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha, \|\varphi'_{(w_1,w_1^*)} - \\ &\quad \left.\varphi'_{(w_2,w_2^*)}\|_\alpha + \|\Pi\left[F\left(\cdot, \omega_1, w_1^*\right)\right] - \Pi\left[F\left(\cdot, w_2, w_2^*\right)\right]\right\|_\alpha \\ &\leq \max\left(C_1 k, C_2 \hat{k}\right) \\ &\leq \left(C_1 + C_2\right) d\left(\left(w_1, w_1^*\right), \left(w_2, w_2^*\right)\right) \max\left(k\|T\|_\alpha\right. \\ &\quad \left., \hat{k}\|\Pi\|_\alpha\right).\end{aligned}$$

The operator Q is contractive if

$$\|q_1\|_\alpha + \|q_2\|_\alpha + \|a\|_\alpha + \|b\|_\alpha < \frac{1}{\max\left(k\|T\|_\alpha + \hat{k}\|\Pi\|_\alpha\right)}$$

We obtain the existence of the unique solution (w, w^*) by using Banach fixed point theorem.

Remark 4.2. The methodology considered in this work suggests the potential for analogous studies regarding the solvability of the generalized Beltrami equation under alternative boundary conditions.

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