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RESEARCH ARTICLE

Some remarks on the moment curves in $\mathbb{R}^4$ and $\mathbb{R}^3$

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ABSTRACT

The moment curves and their normalizations are key tools in obtaining the famous Kac formula from the theory of random polynomials. We study here the normalized moment curves $\Gamma_n \in S^n$ in the low dimensions n where S^n is the Euclidean n-dimensional unit sphere; more precisely we consider $n = 3$ and $n = 2$. First, we compute the image of the normalized moment curve Γ_3 under the well-known Hopf fibre map and show that this remarkable map reduces the length of Γ_3. Second, we analyze the curve Γ_2 using the theory of spherical Legendre curves. An image of Γ_2 is included.

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1. INTRODUCTION

The setting of this paper is provided by the space $\mathbb{R}^{n+1}$, $n \in \mathbb{N}^* = \{1, 2, ...\}$, which is an Euclidean vector space with respect to the canonical inner product:

$$\begin{cases} \langle u, v\rangle := u^1v^1 + ... + u^{n+1}v^{n+1}, u = (u^1, ..., u^{n+1}), v = (v^1, ..., v^{n+1}) \in \mathbb{R}^{n+1}, \\ 0 \leq \|u\|^2 := \langle u, u\rangle. \end{cases} \tag{1.1}$$

A special curve, called *moment curve*, is defined in (Edelman et al. 1995, p. 5-6) as:

$$\gamma_n : \mathbb{R} \to \mathbb{R}^{n+1}, \quad \gamma_n(t) := (1, t, ..., t^n). \tag{1.2}$$

It is a regular curve since the norm of its derivative is strictly positive: $\|\gamma_n'(t)\| > 0$ for all $t \in \mathbb{R}$. Also, *the normalized moment curve* is:

$$\Gamma_n : \mathbb{R} \to \mathbb{S}^n, \quad \Gamma_n(t) := \frac{\gamma_n(t)}{\|\gamma_n(t)\|}. \tag{1.3}$$

where S^n is the unit sphere of $\mathbb{E}^{n+1} := (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$. A main result of Edelman et al. (1995) is that the expected number of real zeros E_n of a random polynomial of degree n if the coefficients are independent and distributed normally is given by:

$$E_n = \frac{L(\Gamma_n)}{\pi} \tag{1.4}$$

where $L(\Gamma_n)$ is the Euclidean length of Γ_n. The first few values of E_n are: 1, 1.29702, 1.49276, 1.64049. Also, it is easy to express the image of the curve Γ_n through the stereographic projection φ_N from the North pole $N(0, ..., 0, 1) \in S^n$:

$$\varphi_N(\Gamma_n(t)) = \frac{\gamma_{n-1}(t)}{\sqrt{1 + t^2 + ... + t^{2n}} - t^n} \in \mathbb{R}^n. \tag{1.5}$$

The present work concerns with the normalized moment curve Γ_n for the low values $n = 3$ and $n = 2$. More precisely, when $n = 3$ we use the well-known Hopf bundle and as result we obtain a lower length. The second case is treated in the framework of spherical Legendre curves since Γ_2 appears as a frontal curve in this theory. For both values of n we study the image of the normalized moment curve through the Veronese map. Other spherical curves in the case $n = 2$, as the Clelia curve and the spherical nephroid, are studied in Crasmareanu (2024).

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2. THE MOMENT CURVE IN $\mathbb{R}^4$ AND THE HOPF MAP

The setting of this section is provided by $n = 3$ since the sphere $S^3 = SU(2)$ is the total space of the famous Hopf bundle $H : S^3 \subset \mathbb{C}^2 \to S^2\left(\frac{1}{2}\right) \subset \mathbb{R}\times\mathbb{C}$:

$$H(z,w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.1}$$

We express the curve Γ_3 as:

$$\Gamma_3(t) = (z(t), w(t)), \quad z(t) = \frac{1+it}{\sqrt{(1+t^2)(1+t^4)}}, \quad w(t) = t^2 z(t), \quad |z(t)|^2 = \frac{1}{1+t^4} \tag{2.2}$$

and then a straightforward computation gives:

$$H(\Gamma_3(t)) = \frac{1}{2}\left(\frac{1-t^4}{1+t^4}, \frac{2t^2}{1+t^4}, 0\right) \in S^2\left(\frac{1}{2}\right) \subset \mathbb{R}\times\mathbb{R}\times\{0\}. \tag{2.3}$$

Returning to the expectation numbers it is very easy to see that $E_1 = 1$. Indeed, for $\Gamma_1(t) = \frac{1}{\sqrt{1+t^2}}(1,t) \in S^1$ we consider the change of parameter $t = \tan\varphi$. It follows:

$$\Gamma_1(\varphi) = (\cos\varphi, \sin\varphi) \in S^1$$

and the condition $\cos\varphi = \frac{1}{\sqrt{1+t^2}} > 0$ yields the domain $\varphi \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. Then we have the length $L(\gamma_1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ and it results $E_1 = 1$.

For the case $n = 3$ we perform the change of parameter $t^2 = \tan\varphi \geq 0$ and then:

$$H(\Gamma_3(\varphi)) = \frac{1}{2}(\cos 2\varphi, \sin 2\varphi, 0) \in S^2\left(\frac{1}{2}\right) \tag{2.4}$$

but now $\varphi \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$. It follows that $2\varphi \in (0,\pi) \cup (2\pi, 3\pi)$ which implies that:

$$L(H \circ \Gamma_3) = \frac{1}{2}(\pi + \pi) = \pi < L(\Gamma_3) \simeq 1.49\pi. \tag{2.5}$$

Hence, the first conclusion of this section is that the Hopf map reduces the length of the normalized moment curve Γ_3.

Expressing $z = z_1 = q_1 + ip_1$, $w = z_2 = q_2 + ip_2$ the standard *contact form of* S^3 is the restriction of the 1-form:

$$\lambda_0 := \frac{1}{2}(p_1 dq_1 - q_1 dp_1 + p_2 dq_2 - q_2 dp_2) \tag{2.6}$$

to S^3. The tangent vector field of the normalized moment curve is:

$$\Gamma_3'(t) = \frac{1}{(1+t^2+t^4)^{\frac{3}{2}}}(-t-t^3, t, 2t+t^3+t^5, 3t^2+2t^4+2t^6) \tag{2.7}$$

and then:

$$\lambda_0(\Gamma_3'(t)) = \frac{-t(1+t+2t^3+t^5+t^7)}{(1+t^2+t^4)^2}. \tag{2.8}$$

As element in the Lie group $SU(2)$ the normalized moment curve Γ_3 is:

$$\Gamma_3(t) = \frac{1}{\sqrt{(1+t^2)(1+t^4)}}\begin{pmatrix} t^2(t+i) & t+i \\ -(t-i) & t^2(t-i)\end{pmatrix} \in SU(2), \quad Tr\Gamma_3(t) = 2Im(w(t)). \tag{2.9}$$

Secondly, we recall the complex Veronese map $V : S^3 \subset \mathbb{C}^2 \to S^5 \subset \mathbb{C}^3$:

$$V(z = x+iy, w = u+iv) := (z^2, \sqrt{2}zw, w^2) =$$

$$= (x^2 - y^2, 2xy, \sqrt{2}(xu - yv), \sqrt{2}(xv + yu), u^2 - v^2, 2uv). \tag{2.10}$$

We obtain the new spherical curve:

$$V \circ \Gamma_3(t) = \frac{1}{(1+t^2)(1+t^4)}(1-t^2, 2t, \sqrt{2}t^2(1-t), \sqrt{2}t^2(1+t), t^4(1-t^2), 2t^5) \in S^5. \tag{2.11}$$

3. THE MOMENT CURVE IN $\mathbb{R}^3$ AND THE SPHERICAL LEGENDRE CURVES

Now we consider $n = 2$ and recall that the unit spherical bundle is a compact 3-dimensional contact metric manifold given by:

$$T_1S^2 := \{(u, v) \in S^2 \times S^2; \langle u, v\rangle = 0\} \tag{3.1}$$

see for example Crasmareanu (2016). There is a natural action of $O(3)$ on T_1S^2:

$$(A, (u, v)) \in O(3) \times T_1S^2 \to (Au, Av) \in T_1S^2, \quad \langle Au, Av\rangle = \langle u, v\rangle = 0. \tag{3.2}$$

For example, the complex Veronese map applied on a pair $(z, w) \in S^3$ with $|z| = |w| = \frac{1}{\sqrt{2}}$ gives the symmetric orthogonal matrix:

$$A = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \in Sym(3) \cap O^-(3),$$

where $O^-(3) := \{\Gamma \in O(3); \det\Gamma = -1\}$ and $Sym(3)$ is the linear subspace of $M_3(\mathbb{R})$ consisting in symmetric matrices. Considering this matrix as representing a conic in the plane $\mathbb{E}^2$ (see the formalism of Crasmareanu (2021)) it is a hyperbola.

The general notion of Legendre curves associated to a contact form is well-known, but we will work directly in our framework using the approach of Takahashi (2016) (see also CrasmareanuChapter (2024)):

Definition 3.1 The smooth map $LC := (\gamma, v) : I \subseteq \mathbb{R} \to T_1S^2$, $t \in I \to (\gamma(t), v(t))$ is *a spherical Legendre curve* if $\langle\gamma'(t), v(t)\rangle = 0$ for all t in the open interval I. The map γ is called *the frontal* and v is *the dual of* γ.

Since $\mathbb{R}^3$ is endowed also with the cross product $\times$ we define $\mu = \gamma \times v$ and hence the triple $\mathcal{F} := \{\gamma, v, \mu\}^t$ is an positive oriented *moving frame* along the frontal γ; here t means the transposition, so $\mathcal{F}$ is a column matrix. Its moving equation is provided by the Proposition 2.2. of **?**:

$$\frac{d}{dt}\mathcal{F}(t) = \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix}\mathcal{F}(t), \quad \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix} \in o(3). \tag{3.3}$$

The pair of smooth functions (k_1, k_2) is called *the curvature* of the spherical Legendre curve $LC = (\gamma, v)$. Sometimes, it is more useful to denote a given LC with all its elements as $LC = (\gamma, v; \mu)$.

Let us consider now the Example 2.8. of Takahashi (2016); equivalently the Example 2.6 of CrasmareanuChapter (2024). Starting with the natural numbers (k, m, n) satisfying $m = k + n$ the LC is defined as:

$$\gamma(t) = \frac{1}{\sqrt{1 + t^{2n} + t^{2m}}}(1, t^n, t^m), \; v(t) = \frac{1}{\sqrt{n^2 + m^2t^{2k} + k^2t^{2m}}}(kt^m, -mt^k, n), \; t \in \mathbb{R} \tag{3.4}$$

with the associated curvature pair:

$$k_1(t) = -\frac{t^{n-1}\sqrt{n^2 + m^2t^{2k} + k^2t^{2m}}}{1 + t^{2n} + t^{2m}}, \quad k_2(t) = \frac{kmnt^{k-1}\sqrt{1 + t^{2n} + t^{2m}}}{n^2 + m^2t^{2k} + k^2t^{2m}}. \tag{3.5}$$

In fact, we have:

$$v = \frac{\gamma \times \gamma'}{\|\gamma \times \gamma'\|} = \frac{\gamma \times \gamma'}{\|\gamma'\|} = \gamma \times T \tag{3.6}$$

where $\{T, N, B\}$ is the Frenet frame of the γ as bi-regular space curve. We point out that the general normalized moment curve Γ_n is exactly the tangent vector field T of the curve:

$$t \in \mathbb{R} \to (\int \gamma_n)(t) = \left(t, \frac{t^2}{2}, \ldots, \frac{t^{n+1}}{n+1}\right).$$

Returning to our normalized moment curve Γ_2 it results that it corresponds exactly to the curve γ for $k = n = 1 < m = 2$; therefore v can be called *the Legendre dual* of the normalized moment curve Γ_3. We have:

$$\Gamma_2'(t) = \frac{1}{(1 + t^2 + t^4)^{\frac{3}{2}}}(-(t + 2t^3), 1 - t^4, 2t + t^3), \quad v(t) = \frac{1}{\sqrt{1 + 4t^2 + t^4}}(t^2, -2t, 1) \tag{3.7}$$

and the curvature pair:

$$k_1(t) = -\frac{\sqrt{1 + 4t^2 + t^4}}{1 + t^2 + t^4} < 0, \quad k_2(t) = \frac{2\sqrt{1 + t^2 + t^4}}{1 + 4t^2 + t^4} > 0. \tag{3.8}$$

The length of the curve Γ_2 is approximately $4.07472 < 2\pi$.

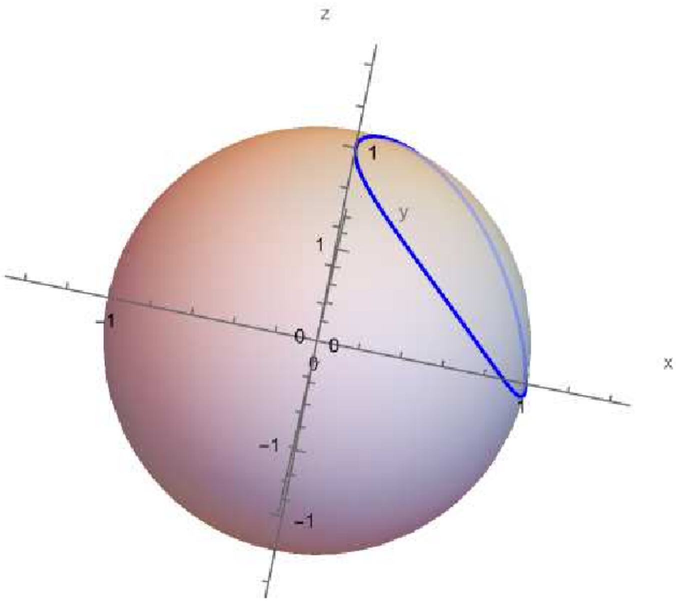

Figure 1. The curve Γ_2.

Recall the standard parametrization of S^2 as regular surface:

$$S^2 : \bar{r}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \quad u \in (0, 2\pi), \quad v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.9}$$

The normalized moment curve $\Gamma_2 \in S^2$ is given then by $u = u(t)$, $v = v(t)$ with:

$$\begin{cases} \cos u(t) = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2+t^4}} > 0, \quad \sin u(t) = \frac{t^2}{\sqrt{1+t^2+t^4}} \geq 0, \\ \cos v(t) = \frac{1}{\sqrt{1+t^2}} > 0, \quad \sin v(t) = \frac{t}{\sqrt{1+t^2}}. \end{cases} \tag{3.10}$$

With the change of parameter $t = \tan \varphi$ we have also:

$$\begin{cases} \Gamma_2(\varphi) = \frac{1}{\sqrt{\cos^2 \varphi + \sin^4 \varphi}}(\cos^2 \varphi, \sin \varphi \cos \varphi, \sin^2 \varphi), \\ \nu(\varphi) = \frac{1}{\sqrt{1+2\cos^2 \varphi \sin^2 \varphi}}(\sin^2 \varphi, -\sin 2\varphi, \cos^2 \varphi). \end{cases} \tag{3.11}$$

We point out that Γ_2 is not a Viviani curve being the intersection of the sphere S^2 not with a cylinder but with the elliptic cone $EC : xz = y^2$.

Moreover, the above spherical curves can be studied through the Veronese map:

$$V : S^2 \subset \mathbb{R}^3 \to S^4 \subset \mathbb{R}^5, V(u, v, w) := \left(\sqrt{3}vw, \sqrt{3}wu, \sqrt{3}uv, \frac{\sqrt{3}}{2}(u^2 - v^2), w^2 - \frac{u^2 + v^2}{2}\right). \tag{3.12}$$

Then:

$$\begin{cases} V \circ \Gamma_2(t) = \frac{1}{1+t^2+t^4}\left(\sqrt{3}t^3, \sqrt{3}t^2, \sqrt{3}t, \frac{\sqrt{3}}{2}(1 - t^2), t^4 - \frac{1+t^2}{2}\right), \\ V \circ \nu(t) = \frac{1}{1+2t^2+t^4}\left(-2\sqrt{3}t, \sqrt{3}t^2, -2\sqrt{3}t^3, \frac{\sqrt{3}t^2}{2}(t^2 - 4), 1 - \frac{t^4+4t^2}{2}\right). \end{cases} \tag{3.13}$$

An important remark is that the new curves, $V \circ \Gamma_2$ and $V \circ \nu$, are not orthogonal in $\mathbb{E}^6$.

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