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RESEARCH ARTICLE

An overview of complex boundary value problems

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ABSTRACT

In this overview we have pointed out some boundary value problems in a subset of complex plane. We start with Cauchy-Riemann operator and their conjugates. Then we introduce the Cauchy- Riemann operator and its conjugate. Firstly, we introduce the polyanalytic Pompeiu integral representation Tf and its conjugate which vanish at infinity. The basic polyanalytic Schwarz and polyanalytic Dirichlet problems are introduced. The later part is devoted to polyanalytic problems and discussions on polyharmonic problems. We have also summarized polyanalytic Neumann problem in the unit disk for $\partial_{\bar{z}} w = f$. In this case, we may have three types of boundary value problems. Those are polyharmonic Dirichlet problem, polyharmonic Neumann problem and polyharmonic Riquier (Navier) problem. In this later part we have given the iterated polyharmonic Green function.

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1. INTRODUCTION

Our aim is to discuss some boundary value problems related to Dirichlet problems in $\mathbb{C}$. This type of problems have started with Riemann and later on modified by Hilbert. The problems have been investigated over many different domains in $\mathbb{C}$, I.N. Vekua (1962); H. Begehr (2025); Aksoy at al. (2025).

1.1. Model representations

For a complex partial differential equation the variables are z and $\bar{z}$ and the operators $\partial_{\bar{z}}$ and ∂_z are known as Cauchy-Riemann operator and its conjugate. One of the main theorems in complex form is given in the following result.

Theorem 1.1. *(Gauss-Ostrogradskii theorem) In a regular domain $D \subset \mathbb{C}$, any function $w \in C(\overline{D};\mathbb{C}) \cap C^1(D;\mathbb{C})$ satisfies the following relations*

$$\int_D w_{\bar{z}}(z)\, dxdy = \frac{1}{2i}\int_{\partial D} w(z)\, dz \tag{1}$$

$$\int_D w_z(z)\, dxdy = -\frac{1}{2i}\int_{\partial D} w(z)\, d\bar{z}\,. \tag{2}$$

Now let us recall the Theorem 1.12 in I.N. Vekua (1962).

Theorem 1.2. *Let D be a bounded domain. If $f \in L_1(D)$ then the integrals*

$$Tf = -\frac{1}{\pi}\int_D \frac{f(\zeta)d\xi d\eta}{\zeta - z}$$

$$\bar{T}f = -\frac{1}{\pi}\int_D \frac{f(\zeta)d\xi d\eta}{\bar{\zeta} - \bar{z}}$$

exist for all points z outside $\overline{D}$, Tf and $\overline{T}f$ are holomorphic outside $\overline{D}$ with respect to z and $\bar{z}$, respectively, and vanish at infinity.

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Tf is called Pompeiu integral for Cauchy-Riemann operator and it satisfies $\partial_{\bar{z}} Tf = f$. This is a model equation for general first order complex differential equation.

2. POLYANALYTIC SCHWARZ AND DIRICHLET PROBLEMS

The basic representation formula for differential operator $\partial_{\bar{z}}$ is given below.

Theorem 2.1. *(Cauchy-Pompeiu integral representation)*
Any function $w \in C(\bar{D}; \mathbb{C}) \cap C^1(D; \mathbb{C})$ can be represented in the form

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial \mathbb{D}} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{\mathbb{D}} w_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta \tag{3}$$

for $z \notin \overline{D}$ the left-hand side must be replaced by 0.

Decomposing $\partial_{\bar{z}}^m w = f$ into a system of m Cauchy-Riemann equations we end up with the polyanalytic Cauchy-Pompeiu formula.

Theorem 2.2. *(Polyanalytic Cauchy-Pompeiu Integral formula)*
Any function $w \in C^{m-1}(\bar{D}; \mathbb{C}) \cap C^m(D; \mathbb{C})$ can be represented in the form

$$\begin{aligned} w(z) = & \sum_{\mu=0}^{m-1} \frac{1}{2\pi i} \int_{\partial D} \frac{(-1)^\mu (\overline{\zeta - z})^\mu}{\mu! (\zeta - z)} \partial_{\bar{\zeta}}^\mu w(\zeta) d\xi \\ & - \frac{1}{\pi} \int_D \frac{(-1)^{m-1} (\overline{\zeta - z})^{m-1}}{(m-1)! (\zeta - z)} \partial_{\bar{\zeta}}^m w(\zeta) d\xi dz, \quad z \in D. \end{aligned}$$

For $z \notin \bar{D}$ the left-hand side must be replaced by 0 .

2.1. Polyanalytic Schwarz problem

We start with the unit disc $\mathbb{D} = \{|z| < 1\}$. Then we have the following lemma for Schwarz problem.

Lemma 2.3. *The Schwarz problem for the Cauchy-Riemann operator*

$$\partial_{\bar{z}} w = f \quad \text{in} \quad \mathbb{D}, \quad \text{Re} w = \gamma \quad \text{on} \quad \partial \mathbb{D}, \quad \text{Im} w(0) = c,$$

$$f \in L_p(\mathbb{D}; \mathbb{C}), 2 < p, \quad \gamma \in C(\partial \mathbb{D}; \mathbb{R}), \quad c \in \mathbb{R},$$

is uniquely solvable by

$$\begin{aligned} w(z) &= ic + \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \\ & \quad - \frac{1}{2\pi} \int_{\mathbb{D}} \Big[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \Big] d\xi d\eta. \end{aligned}$$

Then we can give the Cauchy-Schwarz-Pompeiu representation for polyanalytic operators in $\mathbb{D}$.

Theorem 2.4. *(Cauchy-Schwarz-Pompeiu representation for the polyanalytic operator)*

Any $w \in C^{m-1}(\overline{D};\mathbb{C}) \cap C^m(D,\mathbb{C})$, $m \in \mathbb{N}$, is representable as

$$\begin{aligned} w(z) \; = \; & \sum_{\mu=0}^{m-1}\Bigg\{\frac{i\mathrm{Im}\partial_{\overline{z}}^{\mu} w(z_0)}{\mu!}(z-z_0+\overline{z-z_0})^{\mu} \\ & +\frac{(-1)^{\mu}}{2\pi i\mu!}\int_{\partial D}\mathrm{Re}\partial_{\overline{\zeta}}^{\mu} w(\zeta)(\zeta-z+\overline{\zeta-z})^{\mu} \\ & \times\left[\frac{\zeta-z_0}{(\zeta-z)(\zeta-z_0)}d\zeta+\left(h_{1\overline{\zeta}}(z,\zeta)-\frac{1}{\overline{\zeta-z_0}}\right)d\overline{\zeta}\right]\Bigg\} \\ & +\frac{(-1)^m}{2\pi(m-1)!}\int_D\Bigg\{\partial_{\overline{\zeta}}^{m} w(\zeta)\left[\frac{\zeta+z-2z_0}{(\zeta-z)(\zeta-z_0)}-h_{1\zeta}(z_0,\zeta)\right] \\ & -\overline{\partial_{\overline{\zeta}}^{m} w(\zeta)}\left[2h_{1\overline{\zeta}}(z,\zeta)-h_{1\overline{\zeta}}(z_0,\zeta)-\frac{1}{\overline{\zeta-z_0}}\right]\Bigg\} \\ & \times(\zeta-z+\overline{\zeta-z})^{m-1}d\xi d\eta. \end{aligned}$$

It is easy to observe that this formula is the solution of the polyanalytic problem

$$\begin{gathered} \partial_{\overline{z}}^{m} w = f \text{ in } \mathbb{D}, f \in L_p(\mathbb{D};\mathbb{C}), 2<p, \\ \mathrm{Re}\partial_{\overline{z}}^{\mu} w = \gamma_\mu \text{ on } \partial\mathbb{D}, \gamma_\mu \in C(\partial\mathbb{D};\mathbb{R}), 0 \le \mu \le m-1, \\ \mathrm{Im}\partial_{\overline{z}}^{\mu} w(0) = c_\mu, c_\mu \in \mathbb{R}, 0 \le \mu \le m-1. \end{gathered}$$

This solution may also be represented in terms of Green function $G(z,\zeta)$. We should note that a domain D whose Green function $G_1(z,\zeta)$ is such that

$$h_1(z,\zeta) = \log|\zeta-z|^2 + G_1(z,\zeta)$$

that satisfies for $\zeta \in \partial D, z \in D$,

$$\mathrm{Re}\left[\frac{d\zeta}{\zeta-z} - h_{1\overline{\zeta}}(z,\zeta)d\overline{\zeta}\right] = 0.$$

2.2. Polyanalytic Dirichlet problem

Firstly let us state the solution of the inhomogeneous polyanalytic equation

$$\partial_{\overline{z}}^{m} w = f \text{ in } D, f \in L_p(D;\mathbb{C}), \quad 2<p,$$

satisfying the conditions

$$\partial_{\overline{z}}^{\mu} w = \gamma_\mu \text{ on } \partial D, \quad \gamma_\mu \in C(\partial D;\mathbb{R}), \quad 0 \le \mu \le m-1.$$

Now let us start the case $m = 1$.

Lemma 2.5. *For $f \in L_p(D;\mathbb{C}), 2<p$, and $\gamma \in C(\partial D;\mathbb{C})$, the Dirichlet problem*

$$w_{\overline{z}} = f \text{ in } D, w = \gamma \text{ on } \partial D,$$

is uniquely solvable and the solution is given by the formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{d\zeta}{\zeta-z} - \frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta-z}$$

if and only if the following condition holds:

$$\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)h_{1\zeta}(z,\zeta)d\zeta = \frac{1}{\pi}\int_D f(\zeta)h_{1\zeta}(z,\zeta)d\xi d\eta\,.$$

Theorem 2.6. *The polyanalytic Dirichlet problem where $f \in L_p(D;\mathbb{C}), 2<p, \gamma_\mu \in C(\partial D;\mathbb{C}), 0 \le \mu \le m-1$, has the solution*

$$\begin{aligned} w(z) = & \sum_{\mu=0}^{m-1}\frac{1}{2\pi i}\int_{\partial D}\frac{(-1)^{\mu}(\overline{\zeta-z})^{\mu}}{\mu!(\zeta-z)}\gamma_\mu(\zeta)d\zeta \\ & -\frac{1}{\pi}\int_D\frac{(-1)^{m-1}(\overline{\zeta-z})^{m-1}}{(m-1)!(\zeta-z)}f(\zeta)d\xi d\eta \end{aligned}$$

if and only if

$$\frac{1}{2\pi i}\int_{\partial D}\gamma_\mu(\zeta)\partial_\zeta h_1(z,\zeta)d\zeta + \sum_{\nu=\mu+1}^{m-1}\frac{1}{2\pi i}\int_{\partial D}\gamma_\nu(\zeta_{\nu-\mu+1})\Big(\frac{1}{\pi}\int_D\Big)^{\nu-\mu}\partial_{\zeta_1}h_1(z,\zeta_1) \times\prod_{\lambda=1}^{\nu-\mu}\frac{d\zeta_\lambda d\eta_\lambda}{\zeta_\lambda-\zeta_{\lambda+1}}d\zeta_{\nu-\mu+1} = \frac{1}{\pi}\int_D f(\zeta_{m-\mu})\Big(\frac{1}{\pi}\int_D\Big)^{m-\mu-1}\partial_{\zeta_1}h_1(z,\zeta_1) \times\prod_{\lambda=1}^{m-\mu-1}\frac{d\xi_\lambda d\eta_\lambda}{\zeta_\lambda-\zeta_{\lambda+1}}d\xi_{m-\mu}d\eta_{m-\mu}\ \text{for } 0\le\mu\le m-1$$

holds.

3. POLYANALYTIC NEUMANN-N PROBLEM IN THE UNIT DISK

Neumann boundary value problem demands to find functions with prescribed behaviour of its normal derivative on the boundary. The problem is not a well-posed problem. But it is solvable under solvability conditions. For $n = 1$ the statement of the problem may be reduced to Dirichlet problem. For $1 < n$ then we employ an iteration for Cauchy-Riemann equation.

Theorem 3.1. *The iterated Neumann-n problem for the polyanalytic operator in the unit disk $\mathbb{D}$ The iterated Neumann-n problem for the polyanalytic operator in the unit disk $\mathbb{D}$*

$$\partial_{\bar z}^n w = f \ \text{in}\ \mathbb{D}, \partial_\nu\partial_{\bar z}^\mu w = \gamma_\mu \ \text{on}\ \partial\mathbb{D}, \partial_{\bar z}^\mu w(0) = c_\mu, 0\le\mu\le n-1$$

for $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C}), 0<\alpha<1, \gamma_\mu\in C(\partial\mathbb{D};\mathbb{C}), c_\mu\in\mathbb{C}$, is uniquely solvable if and only if for any $\mu, 0\le\mu\le n-1$,

$$\sum_{\varrho=\mu}^{n-1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\varrho(\zeta)\frac{(-1)^{\varrho-\mu}}{(\varrho-\mu)!}\frac{(\overline{\zeta-z})^{\varrho-\mu}}{1-\bar z\zeta}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{(-1)^{n-1-\mu}}{(n-1-\mu)!}\frac{(\overline{\zeta-z})^{n-1-\mu}}{1-\bar z\zeta}d\bar\zeta + \frac{\bar z}{\pi}\int_{\mathbb{D}}f(\zeta)\frac{(-1)^{n-1-\mu}}{(n-1-\mu)!}\frac{(\overline{\zeta-z})^{n-1-\mu}}{(1-\bar z\zeta)^2}d\xi\, d\eta = 0 \tag{1}$$

are satisfied. The solution then is

$$w(z) = \sum_{\mu=0}^{n-1}\left[\frac{c_\mu}{\mu!}\bar z^\mu - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\mu(\zeta)\frac{(-1)^\mu}{\mu!}(\overline{\zeta-z})^\mu\log(1-z\bar\zeta)\frac{d\zeta}{\zeta}\right] - \frac{1}{2\pi i}\int_{\partial\mathbb{D}}f(\zeta)\frac{(-1)^{n-1}}{(n-1)!}(\overline{\zeta-z})^{n-1}\log(1-z\bar\zeta)d\bar\zeta - \frac{z}{\pi}\int_{\mathbb{D}}f(\zeta)\frac{(-1)^{n-1}}{(n-1)!}\frac{(\overline{\zeta-z})^{n-1}}{\zeta(\zeta-z)}d\xi\, d\eta. \tag{2}$$

4. POLYHARMONIC PROBLEMS

The poly-Poisson equation of order n

$$(\partial_z\partial_{\bar z})^n w = f \ \text{ in } \ D.$$

will assume different names depending on its boundary conditions

(i) If the boundary condition is

$$\partial_\nu^\mu w = \gamma_\mu, 0\le\mu\le n-1, \ \text{ on } \ \partial D,$$

we get the the classical polyharmonic Dirichlet problem.

(ii) If

$$\partial_\nu^\mu w = \gamma_\mu, 1 \leq \mu \leq n, \quad \text{on} \quad \partial D,$$

we get the the polyharmonic Neumann problem.

(iii) If

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \leq \mu \leq n-1, \quad \text{on} \quad \partial D,$$

we get the the polyharmonic Riquier (Navier) problem.

We may also present the following boundary value problems for the polyharmonic n-Poisson equation:

Problem I

$$\partial_\nu (\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \leq \mu \leq n-1, \quad \text{on} \quad \partial D,$$

Problem II

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_{0\mu}, 0 \leq 2\mu \leq n-1,$$

$$\partial_\nu (\partial_z \partial_{\overline{z}})^\mu w = \gamma_{1\mu}, 0 \leq 2\mu \leq n-2, \quad \text{on} \quad \partial D.$$

4.1. Iterated polyharmonic Green functions

We rewrite the n-Poisson equation with the Riquier conditions which we can decompose it into Dirichlet problems for the Poisson equation

$$\partial_z \partial_{\overline{z}} w_\mu = w_{\mu+1} \text{ in } D, \quad w_\mu = \gamma_\mu \text{ on } \partial D, \quad 0 \leq \mu \leq n-1.$$

Naturally we assume $w_0 = w$ and $w_n = f$. Using iteration technique we start with solutions

$$w_\mu(z) = -\frac{1}{4\pi} \int_{\partial D} \gamma_\mu(\zeta) \partial_{\nu_\zeta} G_1(z,\zeta) ds_\zeta - \frac{1}{\pi} \int_D w_{\mu+1}(\zeta) G_1(z,\zeta) d\xi d\eta,$$

and find

$$w(z) = -\frac{1}{4\pi} \sum_{\mu=0}^{n-1} \int_{\partial D} \gamma_\mu(\zeta) \partial_{\nu_\zeta} G_{\mu+1}(z,\zeta) ds_\zeta - \frac{1}{\pi} \int_D f(\zeta) G_n(z,\zeta) d\xi d\eta$$

where

$$G_\mu(z,\zeta) = -\frac{1}{\pi} \int_D G_1(z,\widetilde{\zeta}) G_{\mu-1}(\widetilde{\zeta},\zeta) d\widetilde{\xi} d\widetilde{\eta}, \quad 2 \leq \mu \leq n.$$

Thus we have obtained the following theorem.

Theorem 4.1. *If $f \in L_p(D;\mathbb{C})$, $2 < p$, $\gamma_\mu \in C^{0,\alpha}(\partial D;\mathbb{C})$, $0 < \alpha < 1$, then the Riquier boundary value problem*

$$(\partial_z \partial_{\overline{z}})^n w = f \quad \textit{in} \quad D,$$
$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu \quad \textit{on} \quad \partial D, \quad 0 \leq \mu \leq n-1,$$

is uniquely solvable and its solution has the form

$$w(z) = -\frac{1}{4\pi} \int_{\partial D} \sum_{\mu=0}^{n-1} \partial_{\nu_z} G_{\mu+1}(z,\zeta) \gamma_\mu(\zeta) ds_\zeta - \frac{1}{\pi} \int_D G_n(z,\zeta) f(\zeta) d\xi d\eta.$$

The iterated polyharmonic Green function $G_n(z,\zeta)$ has the following properties

- $G_n(\cdot,\zeta)$ is polyharmonic of order n in $D\backslash\{\zeta\}$,
- $G_n(z,\zeta) + \dfrac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log|\zeta - z|^2$ is polyharmonic of order n for $z \in D, \zeta \in D$,
- $(\partial_z \partial_{\overline{z}})^\mu G_n(z,\zeta) = 0$ for $0 \leq \mu \leq n-1$ and $z \in \partial D, \zeta \in D$.
- $G_n(z,\zeta) = G_n(\zeta,z)$ for $z,\zeta \in D, z \neq \zeta$,
- $(\partial_z \partial_{\overline{z}}) G_n(z,\zeta) = G_{n-1}(z,\zeta)$ in D
- For any $\zeta \in D$, $G_n(z,\zeta) = 0$ on ∂D.

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