Convergence of the class of methods for solutions of certain sixth-order boundary value problems

K. Farajeyan *, J. Rashidinia †‡ and R. Jalilian §

Abstract

The Class of various order numerical methods based on non-polynomial spline have been developed for the solution of linear and non-linear sixth-order boundary value problems. We developed non-polynomial spline which contains a parameter $\rho$, act as the frequency of the trigonometric part of the spline function, when such parameter tends to zero the defined spline reduce into the septic polynomial spline, the consistency relation of non-polynomial spline derived in such a way that, to be fitted to approximate the solution of the given sixth-order boundary value problems. Boundary formulas are developed to associate with presented spline methods. Truncation errors are given, we developed the class of second, fourth, sixth and eight order methods. Convergence analysis has been proved. The obtained methods have been tested on nine examples, to illustrate practical usefulness of our approach. The results of our higher eight order method compare with the existing methods so far.

Keywords: Sixth-order boundary value problem, Non-polynomial spline, Boundary formulae, Convergence analysis.

2000 AMS Classification: Primary 65L10 ; Secondary 65D20

Received : 28.08.2015 Accepted : 29.11.2016 Doi : 10.15672/HJMS.2017.430

*Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran, Email: karim_faraj@yahoo.com
†Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran, Email: rashidinia@iust.ac.ir
‡Corresponding Author.
§Department of Mathematics, Razi University Tagh Bostan, Kermanshah P.O. Box 6714967346 Iran, Email: remjalilian72@gmail.com
1. Introduction

We consider non-linear sixth-order boundary value problem of type

\[ u^{(6)}(x) = f(x, u), \quad a < x < b, \quad a, b, x \in \mathbb{R}, \]

with the following boundary conditions

\[ u(a) = \lambda_1, u'(a) = \lambda_2, u''(a) = \lambda_3, \quad u(b) = \lambda_4, u'(b) = \lambda_5, u''(b) = \lambda_6, \]

or

\[ u(a) = \lambda_7, u''(a) = \lambda_8, u^{(4)}(a) = \lambda_9, \quad u(b) = \lambda_{10}, u''(b) = \lambda_{11}, u^{(4)}(b) = \lambda_{12}, \]

where \( \lambda_i \) for \( i = 1, 2, \ldots, 12 \), are finite real constants and \( u(x) \) and \( f(x, u) \) are continuous functions defined in the interval \([a, b]\).

The sixth-order boundary value problem occurs in several models of engineering and branches of physics, applied mathematics and astrophysics. For example in astrophysics, they believed that the narrow convecting layers border which have fix layers, compass A-type of stars may be modeled by sixth-order boundary value problems [7, 22].

The existence and uniqueness of solution of the sixth-order boundary value problem has been discussed by Agarwal [1]. Many attempts have been done to approximate the solution boundary value problems (1.1)-(1.2). Finite difference methods of various orders, for the solutions of such problems have been developed by Boutayeb and Twizell [5], Pandey [17], Twizell [23] and Twizell and Boutayeb [24].

Sinc-Galerkin method for the solution of sixth-order boundary value problems has been developed by El-Gamel et al. [6]. The spectral method based on Bernstein polynomials for solving high order non-linear boundary value problems have been developed by Behroozifar [4]. The Homotopy perturbation and Variational iteration methods for solving sixth-order boundary value problems have been given by Noor et al. [15, 16].

Adomian decomposition method for solving sixth-order boundary value problems developed by Wazwaz [27] and Hayani [8]. Daftarlar Jafari method (DJM) for solutions of fifth and sixth-orders boundary value problems presented by Ullah et al. [25]. The series solution method for higher-order boundary value problems has been developed by Aslanov [2].

The numerical solution based on polynomial and non-polynomial splines have been developed by many authors, to solve sixth-orders boundary value problems (1.1)-(1.2). Siddiqi and Twizell [21] derived the polynomial splines of degree six, also Siddiqi et al. [19] used quintic spline and later on Siddiqi and Akram [20] used septic spline to developed the numerical solution of (1.1)-(1.2).


The spline functions proposed in this paper have the form \( T_7 = \text{span} \{1, x, x^2, x^3, x^4, x^5, \cos(\rho x), \sin(\rho x)\} \) where \( \rho \) is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and such parameter will be used to developed the classes of the various order and raise the accuracy of the methods. Thus in each subinterval \( x_i \leq x \leq x_{i+1} \) we have

\[ \text{span} \{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}, \quad \text{when} \quad \rho \to 0, \]
which is the septic polynomial spline.

In this paper, in section 2, non-polynomial spline is developed. We derive the consistency relation of non-polynomial spline in such a way to be fitted with the solution of boundary value problems (1.1) along with boundary condition (1.2). The class of various order of second up to eight-order methods have been obtained. Development of the boundary formulas are considered to be associated with various order methods, in section 3. Convergence analysis proved in section 4 and finally in section 5, numerical examples are included to compare our results with the results obtained by other existing methods, and to show superiority of our approach.

2. Numerical methods

To develop the spline approximation to the sixth-order boundary value problem (1.1)-(1.2), the interval \([a, b]\) is divided in to \(n\) equal subintervals using the grid \(x_i = a + ih_i\), \(i = 0, 1, 2, ..., n\), where \(h = \frac{b-a}{n}\). We consider the following non-polynomial spline \(S_i(x)\), on each subinterval \([x_i, x_{i+1}]\), \(i = 0, 1, 2, ..., n-1\), \(x_0 = a, x_n = b\),

\[
S_i(x) = \sum_{j=0}^{5} a_{ij}(x - x_i)^j + b_i \cos \rho(x - x_i) + c_i \sin \rho(x - x_i),
\]

where \(a_{ij}, (j = 0, 1, 2, 3, 4, 5), b_i, \) and \(c_i,\) are real finite constants and \(\rho,\) is free parameter. The spline is defined in terms of its first, second and sixth derivatives, and we denote these values at knots as:

\[
S_i(x_i) = u_i, S'_i(x_i) = m_i, S''_i(x_i) = M_i, S''''_i(x_i) = p_i,
\]

\[
S_i(x_{i+1}) = u_{i+1}, S'_i(x_{i+1}) = m_{i+1}, S''_i(x_{i+1}) = M_{i+1}, S''''_i(x_{i+1}) = p_{i+1},
\]

\(i = 0, 1, 2, ..., n-1\).

Assuming \(u(x)\), to be the exact solution of the boundary value problem (1.1) and \(u_i\), be an approximation to \(u(x_i)\), using the continuity conditions of third, fourth and fifth \((S''''_i(x_i) = S''''(x_i))\) where \(m = 3, 4, 5\), and also by elimination of \(m_i, \) and \(M_i,\) we obtain the following relations between \(u_i\) and \(p_i: \)

\[
h^6(\alpha p_{i-3} + \beta p_{i-2} + \gamma p_{i-1} + \delta p_i + \gamma p_{i+1} + \beta p_{i+2} + \alpha p_{i+3}) = 
\]

\[
u_{i-3} - 6u_{i-2} + 15u_{i-1} - 20u_i + 15u_{i+1} - 6u_{i+2} + u_{i+3},
\]

\(i = 3, 4, ..., n - 3\).

where

\[
\alpha = \frac{1}{12000}(-120 + \theta(120 - 20\theta^2 + \theta^4)) \csc(\theta),
\]

\[
\beta = \frac{1}{6000}(360 - \theta(120 - 20\theta^2 + \theta^4)) \cot(\theta) + \theta(-240 - 20\theta^2 + 13\theta^4) \csc(\theta),
\]

\[
\gamma = \frac{1}{12000}(-1800 + (960 \theta + 800\theta^3 - 52\theta^5)) \cot(\theta) + \theta(840 + 100\theta^2 + 67\theta^4) \csc(\theta)),
\]

\[
\delta = \frac{1}{6000}(600 - \theta(240 + 20\theta^2 - 13\theta^4) + (360 + 60\theta^2 + 33\theta^4) \cos(\theta)) \csc(\theta)),
\]

If \(\rho \to 0, (\theta = \rho h), \theta \to 0\) then \((\alpha, \beta, \gamma, \delta) \to (\frac{1}{12000}, \frac{1}{6000}, \frac{1}{12000}, \frac{1}{6000})\), then we obtain the second-order method and also the relations defined by (2.3) reduce into septic polynomial spline function [20]. Now by using the spline relation (2.3) and discretize the given system (1.1) at the grid points \(x_i\), we obtain \((n-5)\) non-linear equation in the \((n-1)\), unknowns \(u_i, \ i = 1, 2, ..., n - 1,\) as:

\[
(u_{i+3} + u_{i+4}) - \alpha h^6(f(x_{i-3}, u_{i-3}) + f(x_{i+3}, u_{i+3})) - 
6(u_{i+2} + u_{i+4}) - 5h^6(f(x_{i-2}, u_{i-2}) + f(x_{i+2}, u_{i+2})) + 
15(u_{i+1} + u_{i+4}) - 5h^6(f(x_{i-1}, u_{i-1}) + f(x_{i+1}, u_{i+1})) - 
20u_i - 5h^6f(x_i, u_i) = 0, \quad i = 3, 4, ..., n - 3.
\]
We obtain the local truncation error corresponding to the method (2.3) as:

\begin{equation}
T_i = (1 - 2(\alpha + \beta + \gamma) - \delta)h^6 u_i^{(6)} + (\frac{1}{4} - 9\alpha - 4\beta - \gamma)h^8 u_i^{(8)} + \sum_{i=1}^{n} \left( \frac{\alpha}{n^2} - \frac{\alpha}{n^3} \right) - \frac{3}{16} \beta - \frac{1}{16} (\gamma)h^{12} u_i^{(12)} + \sum_{i=3}^{n} \left( \frac{1}{17} \frac{\alpha}{n^2} - \frac{1}{17} \beta - \frac{1}{17} \gamma \right)h^{16} u_i^{(16)} + \ldots,
\end{equation}

(2.5)

By using the above truncation error to eliminate the coefficients of various powers of \( h \), we can obtain classes of the methods in the following form.

**Second-order method**

If we choose \( \alpha = \frac{1}{120}, \beta = \frac{1}{12}, \gamma = \frac{2}{105}, \delta = \frac{1}{12} \), the coefficient of \( h^6 \) in (2.5) can be vanish, then the truncation error of method is \( T_i = \frac{1}{120} h^8 u_i^{(8)} + O(h^{10}) \).

**Fourth-order method**

For \( \alpha = 0, \beta = 0, \gamma = \frac{1}{4}, \delta = \frac{1}{4} \), the coefficient of \( h^6 \) and \( h^8 \) in (2.5) can be vanish, then the truncation error of method is \( T_i = \frac{1}{120} h^{10} u_i^{(10)} + O(h^{12}) \).

**Sixth-order method**

For \( \alpha = 0, \beta = \frac{1}{120}, \gamma = \frac{13}{60}, \delta = \frac{1}{20} \), the coefficient of \( h^6 \) up to \( h^{10} \) in (2.5) can be vanish simultaneously and then the truncation error of method is \( T_i = \frac{1}{30240} h^{12} u_i^{(12)} + O(h^{14}) \).

**Eighth-order method**

For \( \alpha = \frac{1}{30240}, \beta = \frac{1}{240}, \gamma = \frac{2189}{10080}, \delta = \frac{413}{560} \), the coefficient of \( h^6 \) up to \( h^{12} \) in (2.5) can be vanish simultaneously and then the truncation error of method is \( T_i = \frac{1}{30240} h^{14} u_i^{(14)} + O(h^{16}) \).

3. Development of the boundary formulas

System of equation (2.4) contains \((n-5)\) equations, with the \((n-1)\) number of unknown, so that to obtain the unique solution of the system we need four more equations to be associated with system (2.4).

By using boundary conditions (1.2) we can develop these equations to be associate with system (2.4), but here we obtained different class of methods so that we need to developed the boundary value formulas of various orders, in our knowledge so far in the literature most of the existing methods based on spline are suffer from boundary conditions, in this paper we need to develop the new class of boundary conditions of orders 4, 6 and 8, so that we define the following identity:

\begin{align}
&\sum_{i=0}^{4} \eta_i u_i + h \mu_1 u_0 + h^2 \lambda_1 u_0'' = h^5 \sum_{i=0}^{7} \nu_i u_i^{(6)} + t_1, \\
&\sum_{i=0}^{5} \kappa_i u_i + h \mu_2 u_0 + h^2 \lambda_2 u_0'' = h^6 \sum_{i=0}^{7} \omega_i u_i^{(6)} + t_2, \\
&\sum_{i=0}^{6} \kappa_i u_{n-i} - h \mu_2 u_n + h^2 \lambda_2 u_n'' = h^6 \sum_{i=0}^{7} \phi_i u_{n-i}^{(6)} + t_{n-2}, \\
&\sum_{i=0}^{4} \eta_i u_{n-i} - h \mu_1 u_n + h^2 \lambda_1 u_n'' = h^5 \sum_{i=0}^{7} \rho_i u_{n-i}^{(6)} + t_{n-1}, \\
&\sum_{i=0}^{4} \tau_i u_i + h^2 \varphi_1 u_0'' + h^4 g_1 u_0^{(4)} = h^6 \sum_{i=0}^{7} \sigma_i u_i^{(6)} + t_3, \\
&\sum_{i=0}^{5} \zeta_i u_i + h^2 \varphi_2 u_0'' + h^4 g_2 u_0^{(4)} = h^6 \sum_{i=0}^{7} \psi_i u_i^{(6)} + t_4, \\
&\sum_{i=0}^{5} \zeta_i u_{n-i} + h^2 \varphi_2 u_n'' + h^4 g_2 u_n^{(4)} = h^6 \sum_{i=0}^{7} \psi_i u_{n-i}^{(6)} + t_{n-4}, \\
&\sum_{i=0}^{4} \tau_i u_{n-i} + h^2 \varphi_1 u_n'' + h^4 g_1 u_n^{(4)} = h^6 \sum_{i=0}^{7} \sigma_i u_{n-i}^{(6)} + t_{n-3},
\end{align}

(3.1)

Using Taylor’s expansion we obtain the unknown coefficients in (3.1) and (3.2) as follows:

| \( (\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \mu_1, \lambda_1) \) | \((415, -576, 216, -64, 9, 300, 72)\), |
| \( (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \mu_2, \lambda_2) \) | \((2491, -375, 125, -220, 0, 1, 610, 50)\), |
| \( (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \varphi_1, \varphi_2) \) | \((-5, 14, -14, 6, -1, 2, -\frac{1}{6})\), |
Boundary equations of fourth order:

\[
(t_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6) = \left( \frac{4}{105}, \frac{12}{27}, \frac{4}{27}, 0, 0, 0, 0, 0 \right),
\]
\[
(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = \left( 25 \psi_{10}, 1025 \psi_{10}, 925 \psi_{10}, 425 \psi_{10}, 0, 0, 0, 0 \right),
\]
\[
(t_0 = \frac{9}{700} h^{10} u_{10} + O(h^{12}), t_2 = \frac{25}{288} h^{10} u_{10} + O(h^{12}),
\]
\[
(t_{n-1} = \frac{9}{700} h^{10} u_{10} + O(h^{12}), t_{n-2} = \frac{9}{700} h^{10} u_{10} + O(h^{12}),
\]
\[
(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = \left( -\frac{325}{12}, -\frac{29}{12}, -\frac{101}{12}, -\frac{25}{12}, -\frac{25}{12}, -\frac{25}{12}, 0, 0, 0, 0 \right),
\]
\[
(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = \left( \frac{120900}{12}, -\frac{1032}{12}, -\frac{1032}{12}, 0, 0, 0, 0, 0 \right),
\]
\[
(t_3 = \frac{6523}{1812400} h^{10} u_{10} + O(h^{12}), t_4 = \frac{3007}{3628800} h^{10} u_{10} + O(h^{12}),
\]
\[
(t_{n-3} = \frac{6523}{1812400} h^{10} u_{10} + O(h^{12}), t_{n-4} = \frac{3007}{3628800} h^{10} u_{10} + O(h^{12}),
\]

Boundary equations of sixth order:

\[
(t_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = \left( \frac{1301}{2160}, \frac{1523}{2160}, \frac{2461}{2160}, -\frac{332}{2160}, \frac{131}{2160}, -\frac{31}{2160}, 0, 0 \right),
\]
\[
(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = \left( \frac{166528}{2160}, \frac{166528}{2160}, \frac{13264}{2160}, \frac{1188}{2160}, \frac{166528}{2160}, -\frac{166528}{2160} \right),
\]
\[
(t_1 = \frac{37}{1980} h^{12} u_{10} + O(h^{14}), t_2 = \frac{205}{1958} h^{12} u_{10} + O(h^{14}),
\]
\[
(t_{n-1} = \frac{37}{1980} h^{12} u_{10} + O(h^{14}), t_{n-2} = \frac{205}{1958} h^{12} u_{10} + O(h^{14}),
\]
\[
(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = \left( -\frac{6493}{12}, -\frac{6493}{12}, -\frac{6493}{12}, -\frac{6493}{12}, -\frac{6493}{12}, -\frac{6493}{12}, -\frac{6493}{12}, \right),
\]
\[
(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = \left( \frac{120960}{12}, \frac{25920}{12}, \frac{25920}{12}, \frac{25920}{12}, 30320 \right),
\]
\[
(t_3 = \frac{160927}{39916800} h^{12} u_{10} + O(h^{14}), t_4 = \frac{499}{79833600} h^{12} u_{10} + O(h^{14}),
\]
\[
(t_{n-3} = \frac{160927}{39916800} h^{12} u_{10} + O(h^{14}), t_{n-4} = \frac{499}{79833600} h^{12} u_{10} + O(h^{14}),
\]

Boundary equations of eight order:

\[
(t_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = \left( 1484253 \frac{4726277}{23209}, -186409 \frac{44379}{23209}, -373 \frac{308}{23209} \right),
\]
\[
\left( \frac{1201090}{12}, \frac{1201090}{12}, \frac{1201090}{12}, \frac{1201090}{12}, \frac{1201090}{12}, \frac{1201090}{12}, \frac{1201090}{12} \right),
\]
\[
(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = \left( \frac{480985}{2465}, \frac{10029805}{2465}, \frac{314500}{2465}, \frac{381715}{2465}, -\frac{36875}{2465}, -\frac{314500}{2465}, \right),
\]
\[
(t_1 = \frac{15567}{6336900} h^{14} u_{10} + O(h^{16}), t_2 = \frac{9193}{26417000} h^{14} u_{10} + O(h^{16}),
\]
\[
(t_{n-1} = \frac{15567}{6336900} h^{14} u_{10} + O(h^{16}), t_{n-2} = \frac{9193}{26417000} h^{14} u_{10} + O(h^{16}),
\]
\[
(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = \left( -\frac{5357}{970567}, -\frac{8304601}{970567}, \frac{590767}{970567}, -\frac{761869}{970567}, \frac{693507}{970567}, \right),
\]
\[
(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = \left( \frac{531177}{970567}, \frac{17031}{970567}, -\frac{13093}{970567}, \right),
\]
\[
(t_3 = \frac{4446432060}{10801367} h^{14} u_{10} + O(h^{16}), t_4 = \frac{1018379}{5446432060} h^{14} u_{10} + O(h^{16}),
\]
\[
(t_{n-3} = \frac{4446432060}{10801367} h^{14} u_{10} + O(h^{16}), t_{n-4} = \frac{1018379}{5446432060} h^{14} u_{10} + O(h^{16}),
\]

4. Convergence analysis

In this section, we investigate the convergence analysis of the presented eight-order method, in the same manner we can prove the convergence analysis for the rest of other classes of methods. The system of equations (2.4) along with boundary conditions (3.1) or (3.2) yields the non-linear system of equations, and may be written in matrix form as

\[
A_0 U^{(1)} + h^6 B U^{(1)} = R^{(1)}, \quad (A_0 U^{(1)} + h^6 B U^{(1)} = R^{(1)}),
\]

in (4.1) the matrices $A_0, B$ and $B$ are order $n - 1$ and are given by $A_0 = P^4$. 
$P = (p_{ij})$ is monotone tri diagonal matrix defined by

\begin{equation}
(4.2) \quad p_{ij} = \begin{cases} 2, & i = j = 1, 2, 3, \ldots, n - 1, \\ -1, & |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases}
\end{equation}

By using Henrici [9] the matrix $P$ is a monotone matrix and we have

\begin{equation}
(4.3) \quad \|P^{-1}\| \leq \frac{(b - a)^2}{8h^2},
\end{equation}

and the matrix $B$ and $\overline{B}$ in case of eight-order method defined by

\begin{equation}
(4.4) \quad \overline{B} = \begin{pmatrix}
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 \\
\omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 & \omega_7 \\
\beta & \gamma & \delta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\
\omega_7 & \omega_6 & \omega_5 & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\
\nu_7 & \nu_6 & \nu_5 & \nu_4 & \nu_3 & \nu_2 & \nu_1
\end{pmatrix},
\end{equation}

\begin{equation}
(4.5) \quad B = \begin{pmatrix}
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 \\
\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\
\beta & \gamma & \delta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\
\psi_7 & \psi_6 & \psi_5 & \psi_4 & \psi_3 & \psi_2 & \psi_1 \\
\sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1
\end{pmatrix},
\end{equation}

We get that

\begin{equation}
(4.6) \quad A_0 = P^3,
\end{equation}

where $A_0$ is seven-diagonal matrix thus we have

\begin{equation}
(4.7) \quad \|A_0^{-1}\| \leq \frac{(b - a)^6}{512h^6}.
\end{equation}

The matrices $f^{(1)}$, $R^{(1)}$ and $\overline{R}^{(1)}$ each have $n - 1$ components and are given by

\begin{equation}
(4.8) \quad f^{(1)} = (f_1^{(1)}, \ldots, f_{n-1}^{(1)})^t,
\end{equation}
where \( f_l^{(1)}(U^{(1)}) = f(x_l, u_l^{(1)}), l = 1, 2, ..., n - 1, \) and

\[
\begin{pmatrix}
-\tau_0 u_0 - h^2 \vartheta_1 u_0'' - h^4 g_1 u_0^{(4)} + h^6 \sigma_0 u_0^{(6)} \\
-\zeta_0 u_0 - h^2 \vartheta_2 u_0'' - h^4 g_2 u_0^{(4)} + h^6 \psi_0 u_0^{(6)} \\
0
\end{pmatrix}.
\]

(4.9) \( R^{(1)} = \)

\[
\begin{pmatrix}
\vdots \\
0 \\
-\tau_0 u_n + h^2 \vartheta_1 u_n'' - h^4 g_1 u_n^{(4)} + h^6 \sigma_0 u_n^{(6)} \\
-\zeta_0 u_n - h^2 \vartheta_2 u_n'' - h^4 g_2 u_n^{(4)} + h^6 \psi_0 u_n^{(6)} \\
0
\end{pmatrix}.
\]

(4.10) \( \overline{R}^{(1)} = \)

\[
\begin{pmatrix}
-\nu_0 u_0 - h\mu_1 u_0' - h^2 \lambda_1 u_0^{(2)} + h^6 \nu_0 u_0^{(6)} \\
-\nu_0 u_n + h\mu_1 u_n' - h^2 \lambda_1 u_n^{(2)} + h^6 \nu_0 u_n^{(6)} \\
0 \\
\end{pmatrix}.
\]

where \( u_0^{(6)} = f(x_0, u_0), u_n^{(6)} = f(x_n, u_n). \) We suppose that

(4.11) \( A_0 \overline{U}^{(1)} + h^6 B f^{(1)}(U^{(1)}) = R^{(1)} + t^{(1)}, \)

where the vector \( \overline{U}^{(1)} = u(x_l), l = 1, 2, ..., n - 1, \) is the exact solution and \( t^{(1)} = [t_1^{(1)}, t_2^{(1)}, ..., t_{n-1}^{(1)}]^T \), is the vector of order \( n - 1 \) of local truncation errors. Also it is the same way we can prove the convergence analysis for \( \overline{B} \) and \( \overline{R}^{(1)} \), of the method. From (4.1) and (4.11) we have:

(4.12) \( [A] E^{(1)} = [A_0 + h^6 B F_k(U^{(1)})] E^{(1)} = t^{(1)}, \)

where

(4.13) \( E^{(1)} = \overline{U}^{(1)} - U^{(1)} = [e_1^{(1)}, e_2^{(1)}, ..., e_{n-1}^{(1)}]^T; \)

\( f^{(1)}(\overline{U}^{(1)}) - f^{(1)}(U^{(1)}) = F_k(U^{(1)}) E^{(1)}, \)

and \( F_k(U^{(1)}) = \text{diag} \{ \frac{\partial f^{(1)}}{\partial u_l^{(1)}} \}, l = 1, 2, ..., n - 1, \) is a diagonal matrix of order \( n - 1. \)

Lemma 4.1 If \( M \) is a square matrix of order \( N \) and \( \| M \| < 1 \), then \( (I + M)^{-1} \) exist and

\( \| (I + M)^{-1} \| \leq \frac{1}{\| I - M \|}. \)

Proof: [26]

Lemma 4.2 The matrix \( [A_0 + h^6 B F_k(U^{(1)})] \) in (4.12) is nonsingular, provided \( Y < \frac{255.1675200}{\| y_0 \| (b-a)^2} \), where \( Y = \max \{ |\frac{\partial f^{(1)}}{\partial u_l^{(1)}}|, l = 1, 2, ..., n - 1 \} \) (The norm referred to is the \( L_{\infty} \) norm).

Proof: We know that \( [A_0 + h^6 B F_k(U^{(1)})] = A_0[I + A_0^{-1} B F_k(U^{(1)})] \); we need to show that inverse of \( [I + h^6 A_0^{-1} B F_k(U^{(1)})] \) exist. By using lemma 4.1, we have

(4.14) \( h^6 \| A_0^{-1} B F_k(U^{(1)}) \| \leq h^6 \| A_0^{-1} \| ||B|| \| F_k(U^{(1)}) \| < 1, \)
by using \((4.5)\) we obtain \(\|B\| \leq \frac{5506027}{49999999} \) and also have \(\|F_k(U^{(1)})\| \leq Y = \max \left| \frac{\partial f_l^{(1)}}{\partial u_l^{(1)}} \right|, l = 1, 2, \ldots, n-1,\) and then by using \((4.7)\) and \((4.14)\) we obtain
\[
Y < \frac{2554675200}{5506027(b-a)^6}.
\]
As a consequence of Lemmas 4.2 and 4.1 the non-linear system \((4.1)\) has a unique solution if \(Y < \frac{2554675200}{5506027(b-a)^6}\).

**Theorem 4.3** Let \(u(x)\) be the exact solution of the boundary value problem \((1.1)\) with boundary conditions \((1.2)\) and we assume \(u_i, i = 1, 2, \ldots, n-1,\) be the numerical solution obtained by solving the non-linear system \((4.1)\). Then we have:
\[
\|E^{(1)}\| \equiv O(h^8), \text{ (provided) } Y < \frac{2554675200}{5506027(b-a)^6} \text{ for eight-order method.}
\]
**Proof:** We can write the error equation \((4.12)\) in the following form
\[
E^{(1)} = (A_0 + hB\frac{\partial F_k(U^{(1)})}{\partial u} - k^1) = (I + hA_0^{-1}B\frac{\partial F_k(U^{(1)})}{\partial u} - k^1)^{-1}d^{(1)},
\]
\[
\|E^{(1)}\| \leq \| (I + hA_0^{-1}B\frac{\partial F_k(U^{(1)})}{\partial u} - k^1)^{-1} \| \|d^{(1)}\|.
\]
It follows that
\[
\|E^{(1)}\| \leq \frac{\|A_0^{-1}\| \|d^{(1)}\|}{1 - h\|A_0^{-1}\| \|B\| \|\frac{\partial F_k(U^{(1)})}{\partial u}\|},
\]
provided that \(h\|A_0^{-1}\| \|B\| \|\frac{\partial F_k(U^{(1)})}{\partial u}\| < 1.\) Also we have
\[
\|d^{(1)}\| \leq \frac{108013637}{54486432000}h^{14}M_{14},
\]
where
\[
\alpha = \frac{1}{30240}, \beta = \frac{41}{5040}, \gamma = \frac{2189}{10080}, \delta = \frac{4153}{7560}.
\]
Substituting \(\|A_0^{-1}\|, \|\frac{\partial F_k(U^{(1)})}{\partial u}\|, \|B\|\) and \(\|d^{(1)}\|\) from above relations in \((4.15)\) and simplifying we obtain
\[
\|E^{(1)}\| \leq \frac{108013637(b-a)^6h^8M_{14}}{10920(2554675200 - 5506027(b-a)^6Y)} \equiv O(h^8),
\]
It is an eight-order convergent method provided
\[
Y < \frac{2554675200}{5506027(b-a)^6}.
\]

5. **Numerical illustration**

In this section for sake of briefness the eight-order presented method are applied to the following test problems. If we choose \(\alpha = \frac{1}{30240}, \beta = \frac{41}{5040}, \gamma = \frac{2189}{10080}, \delta = \frac{4153}{7560}\) we obtained the eight-order method. Examples 1-9 have been solved and also compared the obtained solution with the exact solution. The maximum absolute errors in solutions of eight-order method are tabulated in Tables 1-10. The maximum absolute errors in solutions of examples 1-9 are compared with methods in \([12, 13, 25, 11, 15, 18, 19, 20]\) moreover, in figures 1-3 we plot the graphs of exact and numerical solution for the nonlinear examples 5, 8 and 9, with different values of step size \(h.\)

**Example 1.** Consider the following linear problem \([12, 19, 20]\)
\[
u^{(6)}(x) = u(x) - 6e^x, \quad 0 \leq x \leq 1,
\]
\[
u(0) = 1, u'(0) = 0, u''(0) = -1, u(1) = 0, u'(1) = -e, u''(1) = -2e.
The exact solution for this problem is \( u(x) = (1 - x)e^x \).

**Example 2.** Consider the following linear problem [12, 19]

\[
u^{(6)}(x) = -u(x) + 6\cos(x), \quad 0 \leq x \leq 1,
\]

\[u(0) = 0, \quad u'(0) = -1, \quad u''(0) = 2, \quad u(1) = 0, \quad u'(1) = \sin(1), \quad u''(1) = 2\cos(1).
\]

The exact solution for this problem is \( u(x) = (x - 1)\sin(x) \).

**Example 3.** Consider the following linear problem [20]

\[
u^{(6)}(x) = -(5x + 1)u(x) + (185x - 25x^2 + 10x^4)\cos(x) + (270 - 36x^2)\sin(x),
\]

\[-1 \leq x \leq 1,
\]

\[u(-1) = 4\cos(1), \quad u'(-1) = \cos(1) + 4\sin(1), \quad u''(-1) = -16\cos(1) + 2\sin(1),
\]

\[u(1) = -2\cos(1), \quad u'(1) = \cos(1) + 2\sin(1), \quad u''(1) = 14\cos(1) - 2\sin(1).
\]

The exact solution for this problem is \( u(x) = (2x^3 - 5x + 1)\cos(x) \).

**Example 4.** Consider the following linear problem [20]

\[
u^{(6)}(x) = -u(x) + 6(2x\cos(x) + 5\sin(x)), \quad -1 \leq x \leq 1,
\]

\[u(-1) = 0, \quad u'(-1) = 2\sin(1), \quad u''(-1) = -4\cos(1) - 2\sin(1),
\]

\[u(1) = 0, \quad u'(1) = 2\sin(1), \quad u''(1) = 4\cos(1) + 2\sin(1).
\]

The exact solution for this problem is \( u(x) = (x^2 - 1)\sin(x) \).

**Example 5.** Consider the following non-linear problem [11, 15, 25]

\[
u^{(6)}(x) = e^x u^2(x), \quad 0 \leq x \leq 1,
\]

\[u(0) = -u'(0) = u''(0) = 1, \quad u(1) = -u'(1) = u''(1) = e^{-1}.
\]

The exact solution for this problem is \( u(x) = e^{-x} \).

**Example 6.** Consider the following linear problem [15]

\[
u^{(6)}(x) = u(x) - 6e^x, \quad 0 \leq x \leq 1,
\]

\[u(0) = 1, \quad u'(0) = -1, \quad u^{(4)}(0) = -3, \quad u(1) = 0, \quad u'(1) = -2e, \quad u^{(4)}(1) = -4e.
\]

The exact solution for this problem is \( u(x) = (1 - x)e^x \).

We solved this example with different order of methods and the computed results are tabulated in tables 6-7.

**Example 7.** Consider the following linear problem [18]

\[
u^{(6)}(x) = -u(x) + 6(2x\cos(x) + 5\sin(x)), \quad 0 \leq x \leq 1,
\]

\[u(0) = u''(0) = u^{(4)}(0) = u(1) = 0,
\]

\[u''(1) = 2\sin(1) + 4\cos(1), \quad u^{(4)}(1) = -12\sin(1) - 8\cos(1).
\]

The exact solution for this problem is \( u(x) = (x^2 - 1)\sin(x) \).

**Example 8.** Consider the following non-linear problem [11, 13, 15, 25]

\[
u^{(6)}(x) = e^{-x}u^2(x), \quad 0 \leq x \leq 1,
\]

\[u(0) = u''(0) = u^{(4)}(0) = 1, \quad u(1) = u''(1) = u^{(4)}(1) = e.
\]

The exact solution for this problem is \( u(x) = e^x \).
Table 1: Maximum absolute errors of Example 1

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>6.4170 × 10^{-16}</td>
<td>3.6463 × 10^{-6}</td>
<td>1.37 × 10^{-6}</td>
<td>6.64 × 10^{-9}</td>
</tr>
<tr>
<td>16</td>
<td>1.6230 × 10^{-18}</td>
<td>5.0209 × 10^{-8}</td>
<td>1.04 × 10^{-8}</td>
<td>7.06 × 10^{-11}</td>
</tr>
<tr>
<td>32</td>
<td>7.6442 × 10^{-21}</td>
<td>2.1369 × 10^{-16}</td>
<td>2.25 × 10^{-16}</td>
<td>2.38 × 10^{-11}</td>
</tr>
<tr>
<td>64</td>
<td>3.0025 × 10^{-23}</td>
<td>1.2289 × 10^{-9}</td>
<td>7.04 × 10^{-9}</td>
<td>9.39 × 10^{-11}</td>
</tr>
<tr>
<td>128</td>
<td>1.1822 × 10^{-25}</td>
<td>1.4521 × 10^{-9}</td>
<td>7.46 × 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>256</td>
<td>4.6183 × 10^{-28}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>1.8040 × 10^{-30}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: Maximum absolute errors of Example 2

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.5357 × 10^{-16}</td>
<td>1.8429 × 10^{-6}</td>
<td>2.95 × 10^{-9}</td>
</tr>
<tr>
<td>16</td>
<td>9.0187 × 10^{-19}</td>
<td>1.3951 × 10^{-7}</td>
<td>4.50 × 10^{-10}</td>
</tr>
<tr>
<td>32</td>
<td>4.1814 × 10^{-21}</td>
<td>9.4848 × 10^{-9}</td>
<td>3.65 × 10^{-11}</td>
</tr>
<tr>
<td>64</td>
<td>1.6501 × 10^{-23}</td>
<td>5.6293 × 10^{-10}</td>
<td>5.92 × 10^{-11}</td>
</tr>
<tr>
<td>128</td>
<td>6.4497 × 10^{-26}</td>
<td>6.4848 × 10^{-10}</td>
<td>-</td>
</tr>
<tr>
<td>256</td>
<td>2.5198 × 10^{-28}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>9.8430 × 10^{-31}</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3: Maximum absolute errors of Example 3

<table>
<thead>
<tr>
<th>n</th>
<th>Our method</th>
<th>Method in [20]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.2569 × 10^{-10}</td>
<td>1.17 × 10^{-7}</td>
</tr>
<tr>
<td>16</td>
<td>6.8147 × 10^{-14}</td>
<td>1.62 × 10^{-9}</td>
</tr>
<tr>
<td>32</td>
<td>9.2677 × 10^{-16}</td>
<td>3.80 × 10^{-11}</td>
</tr>
<tr>
<td>64</td>
<td>3.8581 × 10^{-18}</td>
<td>9.52 × 10^{-7}</td>
</tr>
<tr>
<td>128</td>
<td>1.5136 × 10^{-20}</td>
<td>8.68 × 10^{-7}</td>
</tr>
<tr>
<td>256</td>
<td>5.9141 × 10^{-23}</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>2.3102 × 10^{-25}</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4: Maximum absolute errors of Example 4

<table>
<thead>
<tr>
<th>n</th>
<th>Our method</th>
<th>Method in [20]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.8027 × 10^{-14}</td>
<td>8.25 × 10^{-10}</td>
</tr>
<tr>
<td>16</td>
<td>2.5527 × 10^{-15}</td>
<td>1.13 × 10^{-8}</td>
</tr>
<tr>
<td>32</td>
<td>3.5500 × 10^{-17}</td>
<td>2.64 × 10^{-7}</td>
</tr>
<tr>
<td>64</td>
<td>1.4796 × 10^{-19}</td>
<td>6.96 × 10^{-8}</td>
</tr>
<tr>
<td>128</td>
<td>5.8057 × 10^{-22}</td>
<td>7.17 × 10^{-8}</td>
</tr>
<tr>
<td>256</td>
<td>2.2684 × 10^{-24}</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>8.8614 × 10^{-27}</td>
<td>-</td>
</tr>
</tbody>
</table>

Example 9. Consider the following non-linear problem [11, 13, 18]

\[ u^{(6)}(x) = -u^2(x) + (x^2 - 1)^2 \sin^2(x) + (31 - x^2) \sin(x) + 12x \cos(x), \]

\[ 0 \leq x \leq 1, \]

\[ u(0) = u''(0) = u^{(4)}(0) = u(1) = 0, \]

\[ u''(1) = 2 \sin(1) + 4 \cos(1), \]

\[ u^{(4)}(1) = -12 \sin(1) - 8 \cos(1). \]

The exact solution for this problem is \( u(x) = (x^2 - 1) \sin(x) \).
Table 5: Maximum absolute errors of Example 5

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.459×10^{-20}</td>
<td>1.56×10^{-9}</td>
<td>3.1×10^{-14}</td>
<td>-2.347×10^{-7}</td>
</tr>
<tr>
<td>0.2</td>
<td>2.664×10^{-19}</td>
<td>3.62×10^{-9}</td>
<td>1.9×10^{-13}</td>
<td>-1.389×10^{-6}</td>
</tr>
<tr>
<td>0.3</td>
<td>7.225×10^{-19}</td>
<td>3.84×10^{-9}</td>
<td>4.8×10^{-13}</td>
<td>-3.307×10^{-6}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.473×10^{-19}</td>
<td>3.67×10^{-9}</td>
<td>8.0×10^{-13}</td>
<td>-5.203×10^{-6}</td>
</tr>
<tr>
<td>0.5</td>
<td>5.148×10^{-20}</td>
<td>2.34×10^{-9}</td>
<td>1.0×10^{-12}</td>
<td>-6.198×10^{-6}</td>
</tr>
<tr>
<td>0.6</td>
<td>6.907×10^{-20}</td>
<td>1.58×10^{-11}</td>
<td>1.0×10^{-12}</td>
<td>-5.780×10^{-6}</td>
</tr>
<tr>
<td>0.7</td>
<td>1.514×10^{-19}</td>
<td>2.58×10^{-9}</td>
<td>8.1×10^{-13}</td>
<td>-4.082×10^{-7}</td>
</tr>
<tr>
<td>0.8</td>
<td>1.663×10^{-19}</td>
<td>5.10×10^{-9}</td>
<td>4.3×10^{-13}</td>
<td>-1.902×10^{-6}</td>
</tr>
<tr>
<td>0.9</td>
<td>4.233×10^{-20}</td>
<td>5.02×10^{-9}</td>
<td>9.2×10^{-13}</td>
<td>-5.570×10^{-7}</td>
</tr>
</tbody>
</table>

Table 6: Maximum absolute errors of Example 6

<table>
<thead>
<tr>
<th>x</th>
<th>α = 0, β = 0, γ = \frac{7}{12}, δ = \frac{1}{4}</th>
<th>α = 0, β = \frac{1}{2}, γ = \frac{13}{20}, δ = \frac{415}{843}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.718×10^{-9}</td>
<td>1.979×10^{-12}</td>
</tr>
<tr>
<td>0.2</td>
<td>8.984×10^{-9}</td>
<td>3.755×10^{-12}</td>
</tr>
<tr>
<td>0.3</td>
<td>1.238×10^{-8}</td>
<td>5.162×10^{-12}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.458×10^{-8}</td>
<td>6.077×10^{-12}</td>
</tr>
<tr>
<td>0.5</td>
<td>1.536×10^{-8}</td>
<td>6.419×10^{-12}</td>
</tr>
<tr>
<td>0.6</td>
<td>1.463×10^{-8}</td>
<td>6.152×10^{-12}</td>
</tr>
<tr>
<td>0.7</td>
<td>1.247×10^{-8}</td>
<td>5.286×10^{-12}</td>
</tr>
<tr>
<td>0.8</td>
<td>9.070×10^{-9}</td>
<td>3.884×10^{-12}</td>
</tr>
<tr>
<td>0.9</td>
<td>4.770×10^{-9}</td>
<td>2.063×10^{-12}</td>
</tr>
</tbody>
</table>

Table 7: Maximum absolute errors of Example 6

For α = \frac{4}{10080}, β = \frac{415}{843}, γ = \frac{2189}{10080} and δ = \frac{316}{843}

<table>
<thead>
<tr>
<th>x</th>
<th>Our method</th>
<th>Method in [15]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.986×10^{-14}</td>
<td>4.093×10^{-4}</td>
</tr>
<tr>
<td>0.2</td>
<td>3.766×10^{-14}</td>
<td>7.782×10^{-4}</td>
</tr>
<tr>
<td>0.3</td>
<td>5.172×10^{-14}</td>
<td>1.070×10^{-3}</td>
</tr>
<tr>
<td>0.4</td>
<td>6.079×10^{-14}</td>
<td>1.258×10^{-3}</td>
</tr>
<tr>
<td>0.5</td>
<td>6.406×10^{-14}</td>
<td>1.322×10^{-3}</td>
</tr>
<tr>
<td>0.6</td>
<td>6.132×10^{-14}</td>
<td>1.258×10^{-3}</td>
</tr>
<tr>
<td>0.7</td>
<td>5.245×10^{-14}</td>
<td>1.070×10^{-3}</td>
</tr>
<tr>
<td>0.8</td>
<td>3.842×10^{-14}</td>
<td>7.782×10^{-4}</td>
</tr>
<tr>
<td>0.9</td>
<td>2.035×10^{-14}</td>
<td>4.093×10^{-4}</td>
</tr>
</tbody>
</table>

Table 8: Maximum absolute errors of Example 7

<table>
<thead>
<tr>
<th>n</th>
<th>Our method</th>
<th>Method in [18]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.521×10^{-12}</td>
<td>1.652×10^{-8}</td>
</tr>
<tr>
<td>16</td>
<td>2.718×10^{-12}</td>
<td>2.497×10^{-10}</td>
</tr>
<tr>
<td>32</td>
<td>4.075×10^{-14}</td>
<td>2.128×10^{-14}</td>
</tr>
<tr>
<td>64</td>
<td>9.857×10^{-21}</td>
<td>-</td>
</tr>
<tr>
<td>128</td>
<td>3.279×10^{-21}</td>
<td>-</td>
</tr>
<tr>
<td>256</td>
<td>1.226×10^{-25}</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>4.737×10^{-28}</td>
<td>-</td>
</tr>
</tbody>
</table>
### Table 9: Maximum absolute errors of Example 8

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.466 × 10^{-15}</td>
<td>8.34 × 10^{-10}</td>
<td>-4.8 × 10^{-11}</td>
<td>-1.233 × 10^{-4}</td>
</tr>
<tr>
<td>0.2</td>
<td>2.780 × 10^{-15}</td>
<td>8.26 × 10^{-10}</td>
<td>-3.1 × 10^{-11}</td>
<td>-2.354 × 10^{-4}</td>
</tr>
<tr>
<td>0.3</td>
<td>3.817 × 10^{-15}</td>
<td>4.15 × 10^{-10}</td>
<td>-1.3 × 10^{-10}</td>
<td>-3.277 × 10^{-4}</td>
</tr>
<tr>
<td>0.4</td>
<td>4.486 × 10^{-15}</td>
<td>7.90 × 10^{-11}</td>
<td>-1.5 × 10^{-10}</td>
<td>-3.857 × 10^{-4}</td>
</tr>
<tr>
<td>0.5</td>
<td>4.726 × 10^{-15}</td>
<td>4.49 × 10^{-10}</td>
<td>-9.1 × 10^{-10}</td>
<td>-2.354 × 10^{-4}</td>
</tr>
<tr>
<td>0.6</td>
<td>3.868 × 10^{-15}</td>
<td>4.15 × 10^{-10}</td>
<td>-1.3 × 10^{-10}</td>
<td>-3.277 × 10^{-4}</td>
</tr>
<tr>
<td>0.7</td>
<td>2.833 × 10^{-15}</td>
<td>2.51 × 10^{-10}</td>
<td>-9.9 × 10^{-11}</td>
<td>-2.459 × 10^{-4}</td>
</tr>
<tr>
<td>0.8</td>
<td>1.500 × 10^{-15}</td>
<td>8.87 × 10^{-12}</td>
<td>-5.2 × 10^{-11}</td>
<td>-1.299 × 10^{-4}</td>
</tr>
</tbody>
</table>

### Table 10: Maximum absolute errors of Example 9

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2.5172 × 10^{-12}</td>
<td>1.78 × 10^{-9}</td>
<td>1.6497 × 10^{-10}</td>
<td>9.69 × 10^{-9}</td>
</tr>
<tr>
<td>16</td>
<td>2.7143 × 10^{-15}</td>
<td>1.43 × 10^{-8}</td>
<td>2.5983 × 10^{-10}</td>
<td>2.64 × 10^{-10}</td>
</tr>
<tr>
<td>32</td>
<td>4.0661 × 10^{-18}</td>
<td>1.01 × 10^{-9}</td>
<td>6.8631 × 10^{-11}</td>
<td>5.43 × 10^{-11}</td>
</tr>
<tr>
<td>64</td>
<td>9.8406 × 10^{-21}</td>
<td>6.75 × 10^{-11}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>128</td>
<td>3.2740 × 10^{-23}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>256</td>
<td>1.2242 × 10^{-25}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>512</td>
<td>4.7294 × 10^{-28}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 1: The graph of exact and approximation solutions with \(0 \leq x \leq 1\) and \(h = \frac{1}{20}\) for Example 5.
Figure 2: The graph of exact and approximation solutions with $0 \leq x \leq 1$ and $h = \frac{1}{40}$ for Example 8.

Figure 3: The graph of exact and approximation solutions with $0 \leq x \leq 1$ and $h = \frac{1}{40}$ for Example 9.

**Conclusion**

We approximate solution of the sixth-order linear and non-linear boundary value problems by using non-polynomial spline, we developed the class of various order of 4, 6 and 8 methods. The new approach enable us to approximate the solution at every point
of the range of integration. Tables 1-10 show that our approach produced better in the sense that $\max |e_i|$ is minimum in comparison with the methods developed in [12, 13, 25, 11, 15, 18, 19, 20]. The results obtained by our methods are observed to be better than those obtained by Arshad Khan [12, 13], Ullah et al. [25], Navnit Jha et al. [11], Noor et al. [15], Ramadan et al. [18] and Siddiqi et al. [19, 20] as discussed in examples 1-9.

Acknowledgement

It should be mentioned that the above article has been derived from Ph.D thesis, at the Islamic Azad University Central Tehran Branch.

References


