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Convergence of the class of methods for solutions of certain sixth-order boundary value problems

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Abstract

The Class of various order numerical methods based on non-polynomial spline have been developed for the solution of linear and non-linear sixth-order boundary value problems. We developed non-polynomial spline which contains a parameter ρ, act as the frequency of the trigonometric part of the spline function, when such parameter tends to zero the defined spline reduce into the septic polynomial spline, the consistency relation of non-polynomial spline derived in such a way that, to be fitted to approximate the solution of the given sixth-order boundary value problems. Boundary formulas are developed to associate with presented spline methods. Truncation errors are given, we developed the class of second, fourth, sixth and eight order methods. Convergence analysis has been proved. The obtained methods have been tested on nine examples, to illustrate practical usefulness of our approach. The results of our higher eight order method compare with the existing methods so far.

Keywords: Sixth-order boundary value problem, Non-polynomial spline, Boundary formulae, Convergence analysis.

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1. Introduction

We consider non-linear sixth-order boundary value problem of type

$$u^{(6)}(x) = f(x,u), \quad a < x < b, \ \ a,b,x \in R, \tag{1.1}$$

with the following boundary conditions

$$u(a) = \lambda_1, u^{'}(a) = \lambda_2, u^{''}(a) = \lambda_3, \quad u(b) = \lambda_4, u^{'}(b) = \lambda_5, u^{''}(b) = \lambda_6,$$

or

$$u(a) = \lambda_7, u^{''}(a) = \lambda_8, u^{(4)}(a) = \lambda_9, \quad u(b) = \lambda_{10}, u^{''}(b) = \lambda_{11}, u^{(4)}(b) = \lambda_{12},$$

(1.2)

where λ_i for $i = 1, 2, ..., 12$, are finite real constants and $u(x)$ and$f(x,u)$ are continuous functions defined in the interval $[a,b]$.
The sixth-order boundary value problem occurs in several models of engineering and branches of physics, applied mathematics and astrophysics. For example in astrophysics, they believed that the narrow convecting layers border which have fix layers, compass A-type of stars may be modeled by sixth-order boundary value problems [7, 22].
The existence and uniqueness of solution of the sixth-order boundary value problem has been discussed by Agarwal [1]. Many attempts have been done to approximate the solution boundary value problems (1.1)-(1.2). Finite difference methods of various orders, for the solutions of such problems have been developed by Boutayeb and Twizell [5], Pandey [17], Twizell [23] and Twizell and Boutayeb [24]. Sinc-Galerkin method for the solution of sixth-order boundary value problems has been developed by El-Gamel et al. [6]. The spectral method based on Bernstein polynomials for solving high order non-linear boundary value problems have been developed by Behroozifar [4]. The Homotopy perturbation and Variational iteration methods for solving sixth-order boundary value problems have been given by Noor et al. [15, 16]. Adomian decomposition method for solving sixth-order boundary value problems developed by Wazwaz [27] and Hayani [8]. Daftardar Jafari method (DJM) for solutions of fifth and sixth-orders boundary value problems presented by Ullah et al. [25]. The series solution method for higher-order boundary value problems has been developed by Aslanov [2].
The numerical solution based on polynomial and non-polynomial splines have been developed by many authors, to solve sixth-orders boundary value problems (1.1)-(1.2). Siddiqi and Twizell [21] derived the polynomial splines of degree six, also Siddiqi et al. [19] used quintic spline and later on Siddiqi and Akram [20] used septic spline to developed the numerical solution of (1.1)-(1.2).
Non-polynomial spline has been used by Akram et al. [3] later on Ramadan et al. [18] used non-polynomial spline for the solution of sixth-order boundary value problems. Jalilian et al. [10] presented the solutions of non-linear sixth-order boundary value problems using nonic-spline method. Jha et al. [11] introduced an efficient algorithm based on non-polynomial spline approximations on a geometric mesh for the numerical solution of linear and non-linear two-point boundary value problems. Lang et al. [14] used quintic spline and Arshad Khan et al. [12, 13] applied parametric quintic spline and septic splines for the solution of sixth-order boundary value problems.

The spline functions proposed in this paper have the form $T_7 = \text{span}\{1, x, x^2, x^3, x^4, x^5, \cos(\rho x), \sin(\rho x)\}$ where ρ is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and such parameter will be used to developed the classes of the various order and raise the accuracy of the methods. Thus in each subinterval $x_i \leq x \leq x_{i+1}$ we have

$$\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}, \quad (\text{when} \ \ \rho \to 0),$$

which is the septic polynomial spline.

In this paper, in section 2, non-polynomial spline is developed. We derive the consistency relation of non-polynomial spline in such a way to be fitted with the solution of boundary value problems (1.1) along with boundary condition (1.2). The class of various order of second up to eight-order methods have been obtained. Development of the boundary formulas are considered to be associated with various order methods, in section 3. Convergence analysis proved in section 4 and finally in section 5, numerical examples are included to compare our results with the results obtained by other existing methods, and to show superiority of our approach.

2. Numerical methods

To develop the spline approximation to the sixth-order boundary value problem (1.1)-(1.2), the interval $[a, b]$ is divided in to n equal subintervals using the grid $x_i = a + ih$, $i = 0, 1, 2, ..., n$, where $h = \frac{b-a}{n}$. We consider the following non-polynomial spline $S_i(x)$, on each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, 2, ..., n-1$, $x_0 = a, x_n = b$,

$$S_i(x) = \sum_{j=0}^{5} a_{ij}(x - x_i)^j + b_i \cos\rho(x - x_i) + c_i \sin\rho(x - x_i), \tag{2.1}$$

where $a_{ij}, (j = 0, 1, 2, 3, 4, 5), b_i$, and c_i, are real finite constants and ρ, is free parameter. The spline is defined in terms of its first, second and sixth derivatives, and we denote these values at knots as:

$$\begin{gathered} S_i(x_i) = u_i, S_i^{'}(x_i) = m_i, S_i^{''}(x_i) = M_i, S_i^{(6)}(x_i) = p_i, \\ S_i(x_{i+1}) = u_{i+1}, S_i^{'}(x_{i+1}) = m_{i+1}, S_i^{''}(x_{i+1}) = M_{i+1}, S_i^{(6)}(x_{i+1}) = p_{i+1}, \\ i = 0, 1, 2, ..., n-1. \end{gathered} \tag{2.2}$$

Assuming $u(x)$, to be the exact solution of the boundary value problem (1.1) and u_i, be an approximation to $u(x_i)$, using the continuity conditions of third, fourth and fifth $(S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ where $\mu = 3, 4$ and 5), and also by elimination of m_i, and M_i, we obtain the following relations between u_i and p_i:

$$\begin{gathered} h^6(\alpha p_{i-3} + \beta p_{i-2} + \gamma p_{i-1} + \delta p_i + \gamma p_{i+1} + \beta p_{i+2} + \alpha p_{i+3}) = \\ u_{i-3} - 6u_{i-2} + 15u_{i-1} - 20u_i + 15u_{i+1} - 6u_{i+2} + u_{i+3}, \\ i = 3, 4, ..., n-3. \end{gathered} \tag{2.3}$$

where

$\alpha = \frac{1}{120\theta^6}(-120 + \theta(120 - 20\theta^2 + \theta^4)\csc(\theta))$,
$\beta = \frac{1}{60\theta^6}(360 - \theta(120 - 20\theta^2 + \theta^4)\cot(\theta) + \theta(-240 - 20\theta^2 + 13\theta^4)\csc(\theta))$,
$\gamma = \frac{1}{120\theta^6}(-1800 + (960\theta + 80\theta^3 - 52\theta^5)\cot(\theta) + \theta(840 + 100\theta^2 + 67\theta^4)\csc(\theta))$,
$\delta = \frac{1}{30\theta^6}(600 - \theta(240 + 20\theta^2 - 13\theta^4 + (360 + 60\theta^2 + 33\theta^4)\cos(\theta))\csc(\theta))$,

If $\rho \to 0, (\theta = \rho h), \theta \to 0$ then $(\alpha, \beta, \gamma, \delta) \to (\frac{1}{5040}, \frac{1}{42}, \frac{397}{1680}, \frac{151}{315})$, then we obtain the second-order method and also the relations defined by (2.3) reduce into septic polynomial spline function [20]. Now by using the spline relation (2.3) and discretize the given system (1.1) at the grid points x_i, we obtain $(n-5)$ non-linear equation in the $(n-1)$, unknowns u_i, $i = 1, 2, ..., n-1$, as:

$$\begin{aligned} &(u_{i-3} + u_{i+3}) - \alpha h^6(f(x_{i-3}, u_{i-3}) + f(x_{i+3}, u_{i+3}))- \\ &6(u_{i-2} + u_{i+2}) - \beta h^6(f(x_{i-2}, u_{i-2}) + f(x_{i+2}, u_{i+2}))+ \\ &15(u_{i-1} + u_{i+1}) - \gamma h^6(f(x_{i-1}, u_{i-1}) + f(x_{i+1}, u_{i+1}))- \\ &20u_i - \delta h^6 f(x_i, u_i) = 0, \qquad i = 3, 4, ..., n-3. \end{aligned} \tag{2.4}$$

We obtain the local truncation error corresponding to the method (2.3) as:

$$
\begin{aligned}
T_i = (1-2(\alpha+\beta+\gamma)-\delta)h^6u_i^{(6)} + (\tfrac{1}{4}-9\alpha-4\beta-\gamma)h^8u_i^{(8)}+ \\
(\tfrac{7}{240}-\tfrac{27}{4}\alpha-\tfrac{4}{3}\beta-\tfrac{1}{12}\gamma)h^{10}u_i^{(10)} + (\tfrac{2}{945}-\tfrac{81}{40}\alpha-\tfrac{8}{45}\beta-\tfrac{1}{360}\gamma)h^{12}u_i^{(12)}+ \\
(\tfrac{13}{120960}-\tfrac{729}{2240}\alpha-\tfrac{4}{315}\beta-\tfrac{1}{20160}\gamma)h^{14}u_i^{(14)}+ \\
(\tfrac{31}{7603200}-\tfrac{729}{22400}\alpha-\tfrac{8}{14175}\beta-\tfrac{1}{1814400}\gamma)h^{16}u_i^{(16)}+..., \\
i=3,4,...,n-3.
\end{aligned}
\tag{2.5}
$$

By using the above truncation error to eliminate the coefficients of various powers h, we can obtain classes of the methods in the following form.

Second-order method

If we choose $\alpha=\frac{1}{5040}, \beta=\frac{1}{42}, \gamma=\frac{397}{1680}$ and $\delta=\frac{151}{315}$, the coefficient of h^6 in (2.5) can be vanish, then the truncation error of method is $T_i=-\frac{1}{12}h^8u_i^{(8)}+O(h^{10})$.

Fourth-order method

For $\alpha=0, \beta=0, \gamma=\frac{1}{4}$ and $\delta=\frac{1}{2}$, the coefficient of h^6 and h^8 in (2.5) can be vanish, then the truncation error of method is $T_i=\frac{1}{120}h^{10}u_i^{(10)}+O(h^{12})$.

Sixth-order method

For $\alpha=0, \beta=\frac{1}{120}, \gamma=\frac{13}{60}$ and $\delta=\frac{11}{20}$, the coefficient of h^6 up to h^{10} in (2.5) can be vanish simultaneously and then the truncation error of method is $T_i=\frac{1}{30240}h^{12}u_i^{(12)}+O(h^{14})$.

Eighth-order method

For $\alpha=\frac{1}{30240}, \beta=\frac{41}{5040}, \gamma=\frac{2189}{10080}$ and $\delta=\frac{4153}{7560}$, the coefficient of h^6 up to h^{12} in (2.5) can be vanish simultaneously and then the truncation error of method is $T_i=\frac{-1}{57600}h^{14}u_i^{(14)}+O(h^{16})$.

3. Development of the boundary formulas

System of equation (2.4) contains $(n-5)$ equations, with the $(n-1)$ number of unknown, so that to obtain the unique solution of the system we need four more equations to be associated with system (2.4).

By using boundary conditions (1.2) we can develop these equations to be associate with system (2.4), but here we obtained different class of methods so that we need to developed the boundary value formulas of various orders, in our knowledge so far in the literature most of the existing methods based on spline are suffer from boundary conditions, in this paper we need to develop the new class of boundary conditions of orders 4, 6 and 8, so that we define the following identity:

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{4}\eta_iu_i+h\mu_1u_0^{'}+h^2\lambda_1u_0^{''}=h^6\sum_{i=0}^{7}\nu_iu_i^{(6)}+t_1, \\
&\textstyle\sum_{i=0}^{5}\kappa_iu_i+h\mu_2u_0^{'}+h^2\lambda_2u_0^{''}=h^6\sum_{i=0}^{7}\omega_iu_i^{(6)}+t_2, \\
&\textstyle\sum_{i=0}^{5}\kappa_iu_{n-i}-h\mu_2u_n^{'}+h^2\lambda_2u_n^{''}=h^6\sum_{i=0}^{7}\omega_iu_{n-i}^{(6)}+t_{n-2}, \\
&\textstyle\sum_{i=0}^{4}\eta_iu_{n-i}-h\mu_1u_n^{'}+h^2\lambda_1u_n^{''}=h^6\sum_{i=0}^{7}\nu_iu_{n-i}^{(6)}+t_{n-1},
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{4}\tau_iu_i+h^2\vartheta_1u_0^{''}+h^4\varrho_1u_0^{(4)}=h^6\sum_{i=0}^{7}\sigma_iu_i^{(6)}+t_3, \\
&\textstyle\sum_{i=0}^{5}\zeta_iu_i+h^2\vartheta_2u_0^{''}+h^4\varrho_2u_0^{(4)}=h^6\sum_{i=0}^{7}\psi_iu_i^{(6)}+t_4, \\
&\textstyle\sum_{i=0}^{5}\zeta_iu_{n-i}+h^2\vartheta_2u_n^{''}+h^4\varrho_2u_n^{(4)}=h^6\sum_{i=0}^{7}\psi_iu_{n-i}^{(6)}+t_{n-4}, \\
&\textstyle\sum_{i=0}^{4}\tau_iu_{n-i}+h^2\vartheta_1u_n^{''}+h^4\varrho_1u_n^{(4)}=h^6\sum_{i=0}^{7}\sigma_iu_{n-i}^{(6)}+t_{n-3},
\end{aligned}
\tag{3.2}
$$

by using Taylor's expansion we obtain the unknown coefficients in (3.1) and (3.2) as follows:

$(\eta_0,\eta_1,\eta_2,\eta_3,\eta_4,\mu_1,\lambda_1)=(415,-576,216,-64,9,300,72)$,

$(\kappa_0,\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5,\mu_2,\lambda_2)=(\frac{2491}{9},-375,125,\frac{-250}{9},0,1,\frac{610}{3},50)$,

$(\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\vartheta_1,\varrho_1)=(-5,14,-14,6,-1,2,-\frac{5}{6})$,

$(\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \vartheta_2, \varrho_2) = (4, -14, 20, -15, 6, -1, -1, -\frac{1}{12})$,

Boundary equations of fourth order:
$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = (\frac{4}{105}, \frac{19}{7}, 2, \frac{1}{21}, 0, 0, 0, 0)$,
$(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = (\frac{25}{1512}, \frac{1025}{504}, \frac{925}{504}, \frac{425}{1512}, 0, 0, 0, 0)$,
$(t_1 = \frac{9}{700} h^{10} u_0^{(10)} + O(h^{12}), t_2 = \frac{5}{288} h^{10} u_0^{(10)} + O(h^{12}))$,
$(t_{n-1} = \frac{9}{700} h^{10} u_n^{(10)} + O(h^{12}), t_{n-2} = \frac{5}{288} h^{10} u_n^{(10)} + O(h^{12}))$,
$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (-\frac{323}{5040}, -\frac{1133}{2016}, -\frac{101}{504}, -\frac{25}{2016}, 0, 0, 0, 0)$,
$(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = (\frac{29}{10080}, -\frac{1009}{4032}, -\frac{2519}{5040}, -\frac{5041}{20160}, 0, 0, 0, 0)$,
$(t_3 = \frac{6523}{1814400} h^{10} u_0^{(10)} + O(h^{12}), t_4 = \frac{-30073}{3628800} h^{10} u_0^{(10)} + O(h^{12}))$,
$(t_{n-3} = \frac{6523}{1814400} h^{10} u_n^{(10)} + O(h^{12}), t_{n-4} = \frac{-30073}{3628800} h^{10} u_n^{(10)} + O(h^{12}))$,

Boundary equations of sixth order:
$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = (\frac{1301}{23100}, \frac{15223}{5775}, \frac{24611}{11550}, -\frac{332}{5775}, \frac{131}{3300}, -\frac{31}{5775}, 0, 0)$,
$(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = (\frac{2515}{66528}, \frac{32345}{16632}, \frac{65815}{33264}, \frac{205}{1188}, \frac{2455}{66528}, -\frac{65}{16632}, 0, 0)$,
$(t_1 = \frac{37}{19800} h^{12} u_0^{(12)} + O(h^{14}), t_2 = \frac{305}{199584} h^{12} u_0^{(12)} + O(h^{14}))$,
$(t_{n-1} = \frac{37}{19800} h^{12} u_n^{(12)} + O(h^{14}), t_{n-2} = \frac{305}{199584} h^{12} u_n^{(12)} + O(h^{14}))$,
$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (\frac{-34697}{604800}, \frac{-76723}{129600}, \frac{-133901}{907200}, \frac{-1253}{21600}, \frac{4979}{259200}, \frac{-2833}{907200}, 0, 0)$,
$(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = (\frac{-6493}{1209600}, \frac{-56327}{259200}, \frac{-142327}{259200}, \frac{-65719}{302400}, \frac{-29303}{3628800}, \frac{-11}{259200},$
$0, 0)$,
$(t_3 = \frac{106927}{39916800} h^{12} u_0^{(12)} + O(h^{14}), t_4 = \frac{499}{79833600} h^{12} u_0^{(12)} + O(h^{14}))$,
$(t_{n-3} = \frac{106927}{39916800} h^{12} u_n^{(12)} + O(h^{14}), t_{n-4} = \frac{499}{79833600} h^{12} u_n^{(12)} + O(h^{14}))$,

Boundary equations of eight order:
$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = (\frac{184253}{3153150}, \frac{4726277}{1801800}, \frac{23209}{10725}, \frac{-186409}{1801800}, \frac{34379}{450450}, \frac{-373}{17160}, \frac{808}{225225},$
$\frac{-3097}{12612600})$,
$(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = (\frac{480985}{12108096}, \frac{10029805}{5189184}, \frac{3476285}{1729728}, \frac{221855}{1729728}, \frac{381715}{5189184}, \frac{-36875}{1729728},$
$\frac{2465}{576576}, \frac{-14255}{36324288})$,
$(t_1 = -\frac{33587}{63063000} h^{14} u_0^{(14)} + O(h^{16}), t_2 = -\frac{4793}{26417664} h^{14} u_0^{(14)} + O(h^{16}))$,
$(t_{n-1} = -\frac{33587}{63063000} h^{14} u_n^{(14)} + O(h^{16}), t_{n-2} = -\frac{4793}{26417664} h^{14} u_n^{(14)} + O(h^{16}))$,
$(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (\frac{-65357}{1247400}, \frac{-8304601}{13305600}, \frac{-590767}{9979200}, \frac{-7661869}{39916800}, \frac{697307}{4989600},$
$\frac{-538177}{7983360}, \frac{17011}{907200}, \frac{-13093}{5702400})$,
$(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7) = (\frac{-5941}{1108800}, \frac{-1928191}{8870400}, \frac{-10955801}{19958400}, \frac{-5789947}{26611200}, \frac{-9799}{1247400},$
$\frac{-12409}{79833600}, \frac{19}{604800}, \frac{-41}{11404800})$,
$(t_3 = \frac{108013637}{54486432000} h^{14} u_0^{(14)} + O(h^{16}), t_4 = \frac{1018379}{54486432000} h^{14} u_0^{(14)} + O(h^{16}))$,
$(t_{n-3} = \frac{108013637}{54486432000} h^{14} u_n^{(14)} + O(h^{16}), t_{n-4} = \frac{1018379}{54486432000} h^{14} u_n^{(14)} + O(h^{16}))$.

4. Convergence analysis

In this section, we investigate the convergence analysis of the presented eight-order method, in the same manner we can prove the convergence analysis for the rest of other classes of methods. The system of equations (2.4) along with boundary conditions (3.1) or (3.2) yields the non-linear system of equations, and may be written in matrix form as

$$A_0 U^{(1)} + h^6 B \mathbf{f}^{(1)}(U^{(1)}) = R^{(1)}, (A_0 U^{(1)} + h^6 \overline{B} \mathbf{f}^{(1)}(U^{(1)}) = \overline{R}^{(1)}), \tag{4.1}$$

in (4.1) the matrices A_0, B and $\overline{B}$ are order $n-1$ and are given by

$$A_0 = P^3,$$

$P = (p_{ij})$ is monotone tri diagonal matrix defined by

$$(4.2) \qquad p_{ij} = \begin{cases} 2, & i = j = 1, 2, 3, ..., n-1, \\ -1, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

By using Henrici [9] the matrix P is a monotone matrix and we have

$$(4.3) \qquad \|(P)^{-1}\| \leq \frac{(b-a)^2}{8h^2},$$

and the matrix B and $\overline{B}$ in case of eight-order method defined by

$$(4.4) \qquad \overline{B} = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & & & & \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 & \omega_7 & & & & \\ \beta & \gamma & \delta & \gamma & \beta & \alpha & & & & & \\ \alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\ & & & & \alpha & \beta & \gamma & \delta & \gamma & \beta \\ & & & \omega_7 & \omega_6 & \omega_5 & \omega_4 & \omega_3 & \omega_2 & \omega_1 \\ & & & \nu_7 & \nu_6 & \nu_5 & \nu_4 & \nu_3 & \nu_2 & \nu_1 \end{pmatrix},$$

$$(4.5) \qquad B = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & & & & \\ \psi_1 & \psi_2 & \psi_3 & \omega_4 & \psi_5 & \psi_6 & \psi_7 & & & & \\ \beta & \gamma & \delta & \gamma & \beta & \alpha & & & & & \\ \alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \alpha & \beta & \gamma & \delta & \gamma & \beta & \alpha \\ & & & & \alpha & \beta & \gamma & \delta & \gamma & \beta \\ & & & \psi_7 & \psi_6 & \psi_5 & \omega_4 & \psi_3 & \psi_2 & \psi_1 \\ & & & \sigma_7 & \sigma_6 & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \end{pmatrix},$$

We get that

$$(4.6) \qquad A_0 = P^3,$$

where A_0 is seven-diagonal matrix thus we have

$$(4.7) \qquad \|A_0^{-1}\| \leq \frac{(b-a)^6}{512h^6}.$$

The matrixs $\mathbf{f}^{(1)}, R^{(1)}$ and $\overline{R}^{(1)}$ each have $n-1$ components and are given by

$$(4.8) \qquad \mathbf{f}^{(1)} = (f_1^{(1)}, ..., f_{n-1}^{(1)})^t,$$

where $\mathbf{f}_l^{(1)}(U^{(1)}) = f(x_l, u_l^{(1)}), l = 1, 2, ..., n-1$, and

$$R^{(1)} = \begin{pmatrix} -\tau_0 u_0 - h^2\vartheta_1 u_0'' - h^4\varrho_1 u_0^{(4)} + h^6\sigma_0 u_0^{(6)}, \\ -\zeta_0 u_0 - h^2\vartheta_2 u_0'' - h^4\varrho_2 u_0^{(4)} + h^6\psi_0 u_0^{(6)}, \\ -u_0 + \alpha h^6 u_0^{(6)}, \\ 0 \\ \vdots \\ 0 \\ -u_n + \alpha h^6 u_n^{(6)}, \\ -\zeta_0 u_n - h^2\vartheta_2 u_n'' - h^4\varrho_2 u_n^{(4)} + h^6\psi_0 u_n^{(6)}, \\ -\tau_0 u_n - h^2\vartheta_1 u_n'' - h^4\varrho_0 u_n^{(4)} + h^6\sigma_0 u_n^{(6)}, \end{pmatrix}, \tag{4.9}$$

$$\overline{R}^{(1)} = \begin{pmatrix} -\eta_0 u_0 - h\mu_1 u_0' - h^2\lambda_1 u_0^{(2)} + h^6\nu_0 u_0^{(6)}, \\ -\kappa_0 u_0 - h\mu_2 u_0' - h^2\lambda_2 u_0^{(2)} + h^6\omega_0 u_0^{(6)}, \\ -u_0 + \alpha h^6 u_0^{(6)}, \\ 0 \\ \vdots \\ 0 \\ -u_0 + \alpha h^6 u_n^{(6)}, \\ -\kappa_0 u_n + h\mu_2 u_n' - h^2\lambda_2 u_n^{(2)} + h^6\omega_0 u_n^{(6)}, \\ -\eta_0 u_n + h\mu_1 u_n' - h^2\lambda_1 u_n^{(2)} + h^6\nu_0 u_n^{(6)}, \end{pmatrix}, \tag{4.10}$$

where $u_0^{(6)} = f(x_0, u_0), u_n^{(6)} = f(x_n, u_n)$. We suppose that

$$A_0\overline{U}^{(1)} + h^6 B\mathbf{f}^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)}, \tag{4.11}$$

where the vector $\overline{U}^{(1)} = u(x_l), l = 1, 2, ..., n-1$, is the exact solution and $t^{(1)} = [t_1^{(1)}, t_2^{(1)}, ..., t_{n-1}^{(1)}]^T$, is the vector of order $n-1$ of local truncation errors. Also in the same way we can prove the convergence analysis for $\overline{B}$, and $\overline{R}^{(1)}$, of the method. From (4.1) and (4.11) we have:

$$[A]E^{(1)} = [A_0 + h^6 BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \tag{4.12}$$

where

$$\begin{gathered} E^{(1)} = \overline{U}^{(1)} - U^{(1)} = [e_1^{(1)}, e_2^{(1)}, ..., e_{n-1}^{(1)}]^T, \\ \mathbf{f}^{(1)}(\overline{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)}, \end{gathered} \tag{4.13}$$

and $F_k(U^{(1)}) = \mathrm{diag}\{\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\}, l = 1, 2, ..., n-1$, is a diagonal matrix of order $n-1$.

Lemma 4.1 If M is a square matrix of order N and $\|\mathrm{M}\| < 1$, then $(I + \mathrm{M})^{-1}$ exist and $\|(I + \mathrm{M})^{-1}\| \leq \frac{1}{(1 - \|\mathrm{M}\|)}$.

Proof: [26]

Lemma 4.2 The matrix $[A_0 + h^6 BF_k(U^{(1)})]$ in (4.12) is nonsingular, provided $Y < \frac{2554675200}{5506027(b-a)^6}$, where $Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = 1, 2, ..., n-1$. (The norm referred to is the L_∞ norm).

Proof:

We know that $[A_0 + h^6 BF_k(U^{(1)})] = A_0[I + h^6 A_0^{-1} BF_k(U^{(1)})]$, we need to show that inverse of $[I + h^6 A_0^{-1} BF_k(U^{(1)})]$ exist. By using lemma 4.1, we have

$$h^6\|A_0^{-1} BF_k(U^{(1)})\| \leq h^6\|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\| < 1, \tag{4.14}$$

by using (4.5) we obtain $\|B\| \leq \frac{5506027}{4989600}$ and also we have $\|F_k(U^{(1)})\| \leq Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = 1, 2, ..., n-1$, and then by using (4.7) and (4.14) we obtain

$$Y < \frac{2554675200}{5506027(b-a)^6}.$$

As a consequence of Lemmas 4.2 and 4.1 the non-linear system (4.1) has a unique solution if $Y < \frac{2554675200}{5506027(b-a)^6}$.

Theorem 4.3 Let $u(x_l)$ be the exact solution of the boundary value problem (1.1) with boundary conditions (1.2) and we assume u_l, $l = 1, 2, ..., n-1$, be the numerical solution obtained by solving the non-linear system (4.1). Then we have:

$$\|E^{(1)}\| \equiv O(h^8), (\text{provided} Y < \frac{2554675200}{5506027(b-a)^6}, \text{for eight-order method.})$$

Proof: We can write the error equation (4.12) in the following form

$$E^{(1)} = (A_0 + h^6 BF_k(U^{(1)}))^{-1} t^{(1)} = (I + h^6 A_0^{-1} BF_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)},$$

$$\|E^{(1)}\| \leq \|(I + h^6 A_0^{-1} BF_k(U^{(1)}))^{-1}\| \|A_0^{-1}\| \|t^{(1)}\| \|,$$

It follows that

$$\text{(4.15)} \quad \|E^{(1)}\| \leq \frac{\|A_0^{-1}\| \|t^{(1)}\|}{1 - h^6 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\|},$$

provided that $h^6 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1$. Also we have

$$\text{(4.16)} \quad \|t^{(1)}\| \leq \frac{108013637}{54486432000} h^{14} M_{14},$$

$$\alpha = \frac{1}{30240}, \beta = \frac{41}{5040}, \gamma = \frac{2189}{10080}, \delta = \frac{4153}{7560},$$

where $M_{14} = \max|u^{(14)}(\xi)|, a \leq \xi \leq b$.

Substituting $\|A_0^{-1}\|, \|F_k(U^{(1)})\|$, $\|B\|$ and $\|t^{(1)}\|$ from above relations in (4.15) and simplifying we obtain

$$\text{(4.17)} \quad \|E^{(1)}\| \leq \frac{108013637(b-a)^6 h^8 M_{14}}{10920(2554675200 - 5506027(b-a)^6 Y)} \equiv O(h^8),$$

It is a eight-order convergent method provided

$$\text{(4.18)} \quad Y < \frac{2554675200}{5506027(b-a)^6}.$$

5. Numerical illustration

In this section for sake of briefness the eight-order presented method are applied to the following test problems. If we choose $\alpha = \frac{1}{30240}, \beta = \frac{41}{5040}, \gamma = \frac{2189}{10080}$ and $\delta = \frac{4153}{7560}$ we obtained the eight-order method. Examples 1-9 have been solved and also compared the obtained solution with the exact solution. The maximum absolute errors in solutions of eight-order method are tabulated in Tables 1-10. The maximum absolute errors in solutions of examples 1-9 are compared with methods in [12, 13, 25, 11, 15, 18, 19, 20] moreover, in figures 1-3 we plot the graphs of exact and numerical solution for the non-linear examples 5, 8 and 9, with different values of step size h.

Example 1. Consider the following linear problem [12, 19, 20]

$$u^{(6)}(x) = u(x) - 6e^x, \quad 0 \leq x \leq 1,$$

$$u(0) = 1, u^{'}(0) = 0, u^{''}(0) = -1, u(1) = 0, u^{'}(1) = -e, u^{''}(1) = -2e.$$

The exact solution for this problem is $u(x) = (1-x)e^x$.

Example 2. Consider the following linear problem [12, 19]

$$u^{(6)}(x) = -u(x) + 6\cos(x), \quad 0 \leq x \leq 1,$$

$$u(0) = 0, u'(0) = -1, u''(0) = 2, u(1) = 0, u'(1) = \sin(1), u''(1) = 2\cos(1).$$

The exact solution for this problem is $u(x) = (x-1)\sin(x)$.

Example 3. Consider the following linear problem [20]

$$u^{(6)}(x) = -(5x+1)u(x) + (185x - 25x^2 + 10x^4)\cos(x) + (270 - 36x^2)\sin(x),$$

$$-1 \leq x \leq 1,$$

$$u(-1) = 4\cos(1), u'(-1) = \cos(1) + 4\sin(1), u''(-1) = -16\cos(1) + 2\sin(1),$$

$$u(1) = -2\cos(1), u'(1) = \cos(1) + 2\sin(1), u''(1) = 14\cos(1) - 2\sin(1).$$

The exact solution for this problem is $u(x) = (2x^3 - 5x + 1)\cos(x)$.

Example 4. Consider the following linear problem [20]

$$u^{(6)}(x) = -u(x) + 6(2x\cos(x) + 5\sin(x)), \quad -1 \leq x \leq 1,$$

$$u(-1) = 0, u'(-1) = 2\sin(1), u''(-1) = -4\cos(1) - 2\sin(1),$$

$$u(1) = 0, u'(1) = 2\sin(1), u''(1) = 4\cos(1) + 2\sin(1).$$

The exact solution for this problem is $u(x) = (x^2 - 1)\sin(x)$.

Example 5. Consider the following non-linear problem [11, 15, 25]

$$u^{(6)}(x) = e^x u^2(x), \quad 0 \leq x \leq 1,$$

$$u(0) = -u'(0) = u''(0) = 1, u(1) = -u'(1) = u''(1) = e^{-1}.$$

The exact solution for this problem is $u(x) = e^{-x}$.

Example 6. Consider the following linear problem [15]

$$u^{(6)}(x) = u(x) - 6e^x, \quad 0 \leq x \leq 1,$$

$$u(0) = 1, u''(0) = -1, u^{(4)}(0) = -3, u(1) = 0, u''(1) = -2e, u^{(4)}(1) = -4e.$$

The exact solution for this problem is $u(x) = (1-x)e^x$.

We solved this example with different order of methods and the computed results are tabulated in tables 6-7.

Example 7. Consider the following linear problem [18]

$$u^{(6)}(x) = -u(x) + 6(2x\cos(x) + 5\sin(x)), \quad 0 \leq x \leq 1,$$

$$u(0) = u''(0) = u^{(4)}(0) = u(1) = 0,$$

$$u''(1) = 2\sin(1) + 4\cos(1), u^{(4)}(1) = -12\sin(1) - 8\cos(1).$$

The exact solution for this problem is $u(x) = (x^2 - 1)\sin(x)$.

Example 8. Consider the following non-linear problem [11, 13, 15, 25]

$$u^{(6)}(x) = e^{-x}u^2(x), \quad 0 \leq x \leq 1,$$

$$u(0) = u''(0) = u^{(4)}(0) = 1, u(1) = u''(1) = u^{(4)}(1) = e.$$

The exact solution for this problem is $u(x) = e^x$.

Table 1: Maximum absolute errors of Example 1

n	Our method	Method in [19]	Method in [20]	Method in [12]
8	6.4170×10^{-16}	3.6463×10^{-6}	1.37×10^{-6}	6.64×10^{-9}
16	1.6230×10^{-18}	3.0209×10^{-7}	1.08×10^{-7}	1.04×10^{-9}
32	7.6442×10^{-21}	2.1369×10^{-8}	2.25×10^{-8}	7.66×10^{-11}
64	3.0225×10^{-23}	1.2289×10^{-9}	7.04×10^{-9}	9.39×10^{-11}
128	1.1822×10^{-25}	1.4821×10^{-9}	7.46×10^{-9}	-
256	4.6183×10^{-28}	-	-	-
512	1.8040×10^{-30}	-	-	-

Table 2: Maximum absolute errors of Example 2

n	Our method	Method in [19]	Method in [12]
8	3.5357×10^{-16}	1.8429×10^{-6}	2.95×10^{-9}
16	9.0187×10^{-19}	1.3951×10^{-7}	4.50×10^{-10}
32	4.1814×10^{-21}	9.4848×10^{-9}	3.65×10^{-11}
64	1.6501×10^{-23}	5.6293×10^{-10}	5.92×10^{-11}
128	6.4497×10^{-26}	6.4848×10^{-10}	-
256	2.5198×10^{-28}	-	-
512	9.8430×10^{-31}	-	-

Table 3: Maximum absolute errors of Example 3

n	Our method	Method in [20]
8	1.2569×10^{-10}	1.17×10^{-4}
16	6.8147×10^{-14}	1.62×10^{-5}
32	9.2677×10^{-16}	3.80×10^{-6}
64	3.8581×10^{-18}	9.52×10^{-7}
128	1.5136×10^{-20}	8.68×10^{-7}
256	5.9141×10^{-23}	-
512	2.3102×10^{-25}	-

Table 4: Maximum absolute errors of Example 4

n	Our method	Method in [20]
8	4.8027×10^{-12}	8.25×10^{-6}
16	2.5527×10^{-15}	1.13×10^{-6}
32	3.5500×10^{-17}	2.64×10^{-7}
64	1.4796×10^{-19}	6.96×10^{-8}
128	5.8057×10^{-22}	7.17×10^{-8}
256	2.2684×10^{-24}	-
512	8.8614×10^{-27}	-

Example 9. Consider the following non-linear problem [11, 13, 18]

$$u^{(6)}(x) = -u^2(x) + (x^2-1)^2\sin^2(x) + (31-x^2)\sin(x) + 12x\cos(x),$$

$$0 \le x \le 1,$$

$$u(0) = u''(0) = u^{(4)}(0) = u(1) = 0,$$

$$u''(1) = 2\sin(1) + 4\cos(1), u^{(4)}(1) = -12\sin(1) - 8\cos(1).$$

The exact solution for this problem is $u(x) = (x^2-1)\sin(x)$.

Table 5: Maximum absolute errors of Example 5

x	Our method	Method in [11]	Method in [25]	Method in [15]
0.1	6.459×10^{-20}	1.56×10^{-9}	3.1×10^{-14}	-2.347×10^{-7}
0.2	2.664×10^{-19}	3.02×10^{-9}	1.9×10^{-13}	-1.389×10^{-6}
0.3	2.725×10^{-19}	3.84×10^{-9}	4.8×10^{-13}	-3.307×10^{-6}
0.4	1.473×10^{-19}	3.67×10^{-9}	8.0×10^{-13}	-5.203×10^{-6}
0.5	5.148×10^{-20}	2.34×10^{-9}	1.0×10^{-12}	-6.198×10^{-6}
0.6	6.907×10^{-20}	1.58×10^{-11}	1.0×10^{-12}	-5.780×10^{-6}
0.7	1.514×10^{-19}	2.58×10^{-9}	8.1×10^{-13}	-4.082×10^{-6}
0.8	1.663×10^{-19}	5.10×10^{-9}	4.3×10^{-13}	-1.903×10^{-6}
0.9	4.233×10^{-20}	5.02×10^{-9}	9.2×10^{-13}	-3.570×10^{-7}

Table 6: Maximum absolute errors of Example 6

x	$\alpha=0,\beta=0,$ $\gamma=\frac{1}{4},\delta=\frac{1}{2}$	$\alpha=0,\beta=\frac{1}{120},$ $\gamma=\frac{13}{60},\delta=\frac{11}{20}$
0.1	4.7183×10^{-9}	1.9793×10^{-12}
0.2	8.9841×10^{-9}	3.7550×10^{-12}
0.3	1.2383×10^{-8}	5.1624×10^{-12}
0.4	1.4582×10^{-8}	6.0777×10^{-12}
0.5	1.5361×10^{-8}	6.4198×10^{-12}
0.6	1.4636×10^{-8}	6.1523×10^{-12}
0.7	1.2470×10^{-8}	5.2864×10^{-12}
0.8	9.0702×10^{-9}	3.8842×10^{-12}
0.9	4.7706×10^{-9}	2.0630×10^{-12}

Table 7: Maximum absolute errors of Example 6
(for $\alpha=\frac{1}{30240},\beta=\frac{41}{5040},\gamma=\frac{2189}{10080}$ and $\delta=\frac{4153}{7560}$)

x	Our method	Method in [15]
0.1	1.9865×10^{-14}	4.0933×10^{-4}
0.2	3.7664×10^{-14}	7.7820×10^{-4}
0.3	5.1722×10^{-14}	1.07048×10^{-3}
0.4	6.0790×10^{-14}	1.25787×10^{-3}
0.5	6.4068×10^{-14}	1.32238×10^{-3}
0.6	6.1232×10^{-14}	1.25787×10^{-3}
0.7	5.2455×10^{-14}	1.07048×10^{-3}
0.8	3.8425×10^{-14}	7.7820×10^{-4}
0.9	2.0357×10^{-14}	4.0933×10^{-4}

Table 8: Maximum absolute errors of Example 7

n	Our method	Method in [18]
8	2.5215×10^{-12}	1.652489367×10^{-8}
16	2.7189×10^{-15}	$2.497231310\times10^{-10}$
32	4.0730×10^{-18}	$2.125805087\times10^{-11}$
64	9.8572×10^{-21}	-
128	3.2795×10^{-23}	-
256	1.2263×10^{-25}	-
512	4.7374×10^{-28}	-

Table 9: Maximum absolute errors of Example 8

x	Our method	Method in [11]	Method in [25]	Method in [15]
0.1	1.466×10^{-15}	8.34×10^{-10}	-4.8×10^{-11}	-1.233×10^{-4}
0.2	2.780×10^{-15}	8.26×10^{-10}	-9.1×10^{-11}	-2.354×10^{-4}
0.3	3.817×10^{-15}	4.15×10^{-10}	-1.3×10^{-10}	-3.257×10^{-4}
0.4	4.486×10^{-15}	7.90×10^{-11}	-1.5×10^{-10}	-3.855×10^{-4}
0.5	4.726×10^{-15}	4.49×10^{-10}	-1.6×10^{-10}	-4.086×10^{-4}
0.6	4.516×10^{-15}	5.91×10^{-10}	-1.6×10^{-10}	-3.919×10^{-4}
0.7	3.868×10^{-15}	4.98×10^{-10}	-1.4×10^{-10}	-3.361×10^{-4}
0.8	2.833×10^{-15}	2.51×10^{-10}	-9.9×10^{-11}	-2.459×10^{-4}
0.9	1.500×10^{-15}	8.87×10^{-12}	-5.2×10^{-11}	-1.299×10^{-4}

n	Our method	Method in [13]
8	4.2504×10^{-14}	3.79×10^{-10}
16	4.9352×10^{-17}	2.51×10^{-11}
32	7.5809×10^{-20}	5.10×10^{-10}
64	1.7794×10^{-22}	-
128	5.7714×10^{-25}	-
256	2.1379×10^{-27}	-
512	8.2366×10^{-30}	-

Table 10: Maximum absolute errors of Example 9

n	Our method	Method in [11]	Method in [18]	Method in [13]
8	2.5172×10^{-12}	1.78×10^{-7}	1.6497×10^{-8}	9.69×10^{-9}
16	2.7143×10^{-15}	1.43×10^{-8}	2.5983×10^{-10}	2.04×10^{-10}
32	4.0661×10^{-18}	1.01×10^{-9}	6.8631×10^{-11}	5.43×10^{-11}
64	9.8406×10^{-21}	6.75×10^{-11}	-	-
128	3.2740×10^{-23}	-	-	-
256	1.2242×10^{-25}	-	-	-
512	4.7294×10^{-28}	-	-	-

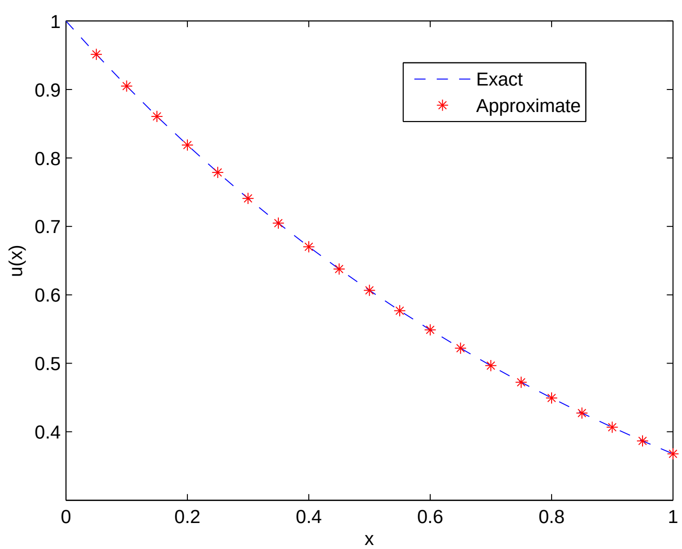

Figure 1: The graph of exact and approximation solutions with $0 \leq x \leq 1$ and $h = \frac{1}{20}$ for Example 5.

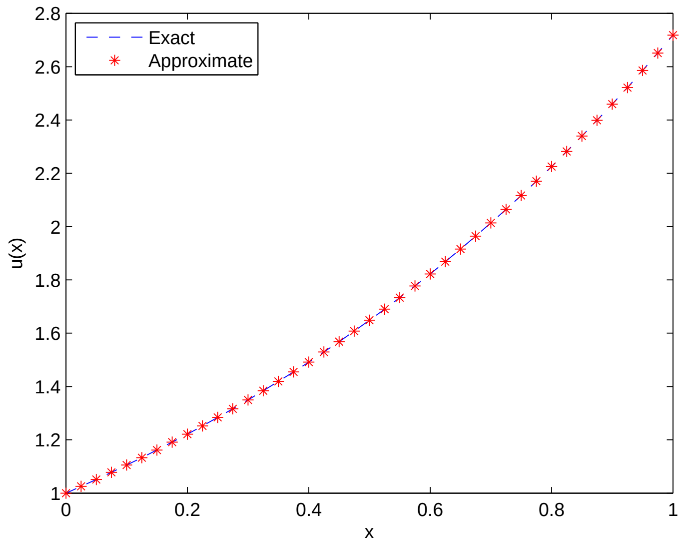

Figure 2: The graph of exact and approximation solutions with $0 \leq x \leq 1$ and $h = \frac{1}{40}$ for Example 8.

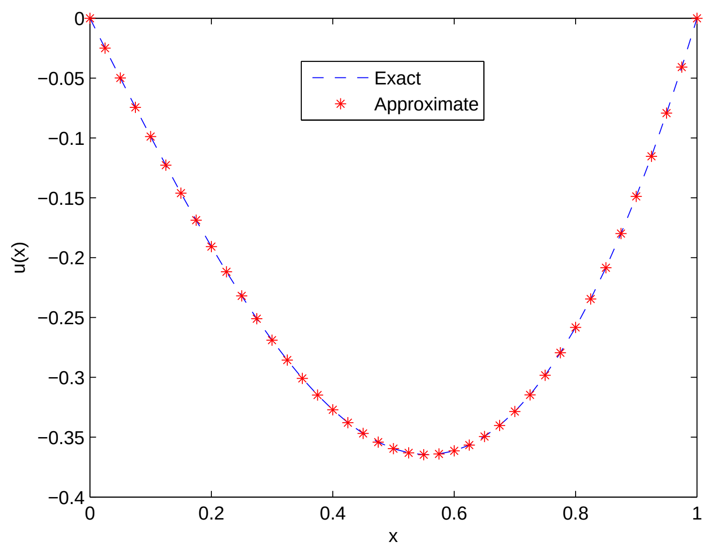

Figure 3: The graph of exact and approximation solutions with $0 \leq x \leq 1$ and $h = \frac{1}{40}$ for Example 9.

Conclusion

We approximate solution of the sixth-order linear and non-linear boundary value problems by using non-polynomial spline, we developed the class of various order of $4, 6$ and 8 methods. The new approach enable us to approximate the solution at every point of the range of integration. Tables 1-10 show that our approach produced better in the sense that max $| e_i |$ is minimum in comparison with the methods developed in [12, 13, 25, 11, 15, 18, 19, 20]. The results obtain by our methods are observed to be better than that obtained results by Arshad Khan [12, 13], Ullah et al. [25], Navnit Jha et al. [11], Noor et al. [15], Ramadan et al. [18] and Siddiqi et al. [19, 20] as discussed in examples 1-9.

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