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Investigating an overdetermined system of linear equations by using convex functions

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Abstract

The paper studies the application of convex functions in order to prove the existence of optimal solutions of an overdetermined system of linear equations. The study approaches the problem by using even convex functions instead of projections. The research also relies on some special properties of unbounded convex sets, and the lower level sets of continuous functions.

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1. Introduction

We consider a system of m linear equations with n unknowns over the field of real numbers given by

$$\begin{array}{ccccc} a_{11}x_1 + & \dots & + a_{1n}x_n & = & b_1 \\ \vdots & \ddots & \vdots & & \\ a_{m1}x_1 + & \dots & + a_{mn}x_n & = & b_m \end{array}. \tag{1.1}$$

Including the matrices

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \tag{1.2}$$

the given system gets the matrix form

$$Ax = b. \tag{1.3}$$

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Identifying the matrix A with a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$, the column matrix x with a vector of $\mathbb{R}^n$, and the column matrices Ax and b with vectors of $\mathbb{R}^m$, the given system takes the operator form.

If the vector b is not contained in the image $R(A)$ of the operator A, then the equation in formula (1.3) has no solution. Respectively, the system in formula (1.1) is said to be inconsistent. A system of linear equations is said to be overdetermined if there are more equations than unknowns. The overdetermined system with linearly independent equations is evidently inconsistent.

To summarize, employing a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ where $n < m$, and a vector $b \in \mathbb{R}^m$, we receive the equation $Ax = b$ with unknown vectors $x \in \mathbb{R}^n$. If $b \in R(A)$, the set of solutions is the preimage of b as

$$A^{(-1)}(b) = \{x \in \mathbb{R}^n : Ax = b\}. \tag{1.4}$$

If $b \notin R(A)$, the equation has no solution. To include both cases, we use a norm on the space $\mathbb{R}^m$, and the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \|Ax - b\|. \tag{1.5}$$

Then we are looking for the global minimum points x of the function f. Expressed as an equation, it is $f(x) = \min_{y\in\mathbb{R}^n} f(y)$ or

$$\|Ax - b\| = \min_{y\in\mathbb{R}^n} \|Ay - b\|, \tag{1.6}$$

and its solutions are called optimal solutions of the equation $Ax = b$ respecting the given norm. The above equation is reduced to $Ax = b$ if $b \in R(A)$.

The existence of the solution of the minimization problem in formula (1.6) is provided in the framework of the theory of projections. In order to remake it without using projections, we will utilize even convex functions.

Methodical introduction to overdetermined systems of linear equations can be seen in [12]. The general insight into the problem of least absolute deviations can be found in [1]. Linear optimization problems were presented in [2].

2. Affinity and convexity, rays, lower level sets

Let $\mathbb{X}$ be a real vector space. Let $x_1, x_2 \in \mathbb{X}$ be vectors, and let $t_1, t_2 \in \mathbb{R}$ be coefficients. Then the linear combination $t_1x_1 + t_2x_2$ is said to be affine (convex) if $t_2 = 1 - t_1$ ($0 \le t_2 = 1 - t_1 \le 1$). A set $S \subseteq \mathbb{X}$ is said to be affine (convex) if it contains all binomial affine (convex) combinations of its vectors.

The set $A^{(-1)}(b) \subseteq \mathbb{R}^n$ in formula (1.4) is affine. In general, if a set $B \subseteq R(A)$ is affine (convex), then its preimage $A^{(-1)}(B) \subseteq \mathbb{R}^n$ is affine (convex) too.

Let S be an affine (convex) set. A function $f : S \to \mathbb{R}$ is said to be affine (convex) if the equality (inequality)

$$f\big(t_1x_1 + t_2x_2\big) = (\le)\ t_1f(x_1) + t_2f(x_2) \tag{2.1}$$

holds for all affine (convex) combinations $t_1x_1 + t_2x_2$ of points $x_1, x_2 \in S$.

If $x_1, x_2 \in \mathbb{X}$ are vectors, then the ray or half-line R from x_1 passing through x_2 is the subset of $\mathbb{X}$ defined as

$$R = \big\{(1 - t)x_1 + tx_2 : t \ge 0\big\}.$$

If $f : S \to \mathbb{R}$ is a function, then the lower level set L of the function f with a height $l \in \mathbb{R}$ is the subset of S defined as

$$L = \{x \in S : f(x) \le l\}.$$

The lower level sets of a convex function are convex.

The following lemma can be found in [3, Theorem 1, page 23]. We offer the proof by using convex combinations.

2.1. Lemma. *Let $S \subseteq \mathbb{R}^n$ be an unbounded closed convex set, and let $s \in S$ be a point. Then there is a ray from s belonging to S.*

Proof. Without loss of generality, using the translation in the space $\mathbb{R}^n$ by the vector $-s$, we can assume that the set S contains the origin o. So, we are looking for a ray from o contained in S.

Since S is unbounded, we can pick out a sequence $(y_k)_k$ of points $y_k \in S$ such that $\|y_k\| = k$ for every $k \in \mathbb{N}$.

Since S is convex, each line segment between o and y_k is contained in S. Then it follows that the convex combination

$$x_k = \frac{k-1}{k}o + \frac{1}{k}y_k = \frac{1}{k}y_k$$

belongs to S for every $k \in \mathbb{N}$. Each x_k satisfies $\|x_k\| = 1$. We have the sequence $(x_k)_k$ of points x_k belonging to the intersection S_0 of the set S and the unit sphere in $\mathbb{R}^n$. The set S_0 is compact because it is closed and bounded, and therefore the sequence $(x_k)_k$ has a subsequence $(x_{r_k})_k$ converging to some point $x_0 \in S_0$.

Let R be the ray from o passing through x_0. We will verify that R is contained in S. Since $R = \{tx_0 : t \geq 0\}$, we have to show that each ray point tx_0 is in S. Let $t \geq 0$ be a nonnegative coefficient. Since S is convex, the points

$$tx_{r_k} = \frac{t}{r_k}y_{r_k} = \frac{r_k - t}{r_k}o + \frac{t}{r_k}y_{r_k}$$

are in S for $r_k \geq t$. The above points approach tx_0 if k approaches infinity. Since S is closed, the limit tx_0 is in S. □

The proof of Lemma 2.1 is illustrated in Figure 1, and the same is still applicable to an unbounded convex set of $\mathbb{R}^n$ and points of its interior.

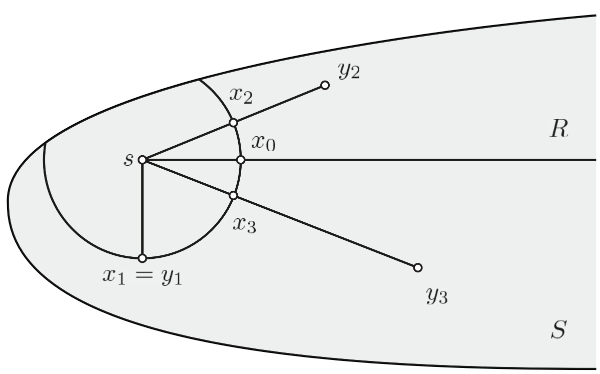

Figure 1. A ray in the unbounded convex set of $\mathbb{R}^2$

Statements similar to the next lemma and corollary are exposed in [11, Theorem 8.4, Theorem 8.5 and Theorem 8.6, pages 30-31]. Bounded sets play a decisive role for the global minimum existence. Considering lower level sets and the global minimum of a continuous function, one can find the following.

2.2. Lemma. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function whose lower level sets are bounded. Then f has the global minimum.*

Proof. First we take a point $x_0 \in \mathbb{R}^n$, and a number $l \geq f(x_0)$. Then we specify the lower level set L of the function f with the height l, which is nonempty. The set L is bounded by the assumption, and closed by the continuity of f. Thus L is compact, and there is a point $x^* \in L$ such that $f(x^*) = \min_{x \in L} f(x)$. The value $f(x^*)$ is the global minimum, which is easy to demonstrate as follows.

Let $x \in \mathbb{R}^n$ be a point. If $x \in L$, then applies $f(x^*) \leq f(x) \leq l$. If $x \notin L$, then we have $f(x^*) \leq l < f(x)$. □

A convex function on the space $\mathbb{R}^n$ approaching infinity on all rays from the origin being examined.

2.3. Corollary. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that*

$$\lim_{t \to \infty} f(tx) = \infty \tag{2.2}$$

for every $x \in \mathbb{R}^n \setminus \{o\}$. Then f has the global minimum.

Proof. Since f is convex on $\mathbb{R}^n$, it is also continuous (a convex function on an open domain is continuous). Respecting the continuity of f in the context of Lemma 2.2, it is sufficient to prove that the lower level sets of f are bounded.

Let L be the lower level set of f with a height $l \geq f(o)$. The set L contains the origin o, and it is closed and convex. If L is not bounded, then it contains some ray R from the origin o by Lemma 2.1. Thus, a point $x_0 \in L \setminus \{o\}$ exists such that the ray $R = \{tx_0 : t \geq 0\}$. Since $R \subseteq L$, for each $t \geq 0$ we have that

$$f(tx_0) \leq l.$$

It follows that the condition in formula (2.2) does not apply to $x_0 \in \mathbb{R}^n \setminus \{o\}$. Hence the lower level set L must be bounded. Consequently, all lower level sets of the function f are bounded. □

Very well written and motivated book in [9] can be recommended as an introductory course to the analysis of convex functions. One chapter of this book refers to the optimization.

The usage of affine and convex combinations is important in the field of mathematical inequalities. A refinement of the Jensen inequality was obtained in [7] by using affine combinations, and improvements of the Hermite-Hadamard inequality were obtained in [8] by using convex combinations.

The book on level set methods and fast marching methods in [10] is intended for mathematicians and applied scientists. This book includes applications from computational geometry, fluid mechanics, computer vision and materials science.

3. Main results

In this section, we assume that the space $\mathbb{R}^n$ is equipped with some norm, and utilize the property of compactness of bounded closed sets in $\mathbb{R}^n$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is an even convex function, then its global minimum is $f(o)$. It is easy to verify by combining the convexity and equation $f(-x) = f(x)$, from which it follows that the inequality

$$f(o) = f\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x) = f(x)$$

holds for every $x \in \mathbb{R}^n$.

In the next lemma and corollary, we will use a convex function $f:\mathbb{R}\to\mathbb{R}$ and the property of nondecreasing slopes of its chords expressed by the double inequality

$$\frac{f(t_2)-f(t_1)}{t_2-t_1}\le\frac{f(t_3)-f(t_1)}{t_3-t_1}\le\frac{f(t_3)-f(t_2)}{t_3-t_2}, \tag{3.1}$$

which holds for every triple of points $t_1,t_2,t_3\in\mathbb{R}$ in the order of $t_1<t_2<t_3$.

The inequality of the first and third term in formula (3.1) can be derived by using the inequality in formula (2.1). To verify the whole double inequality in formula (3.1), we represent its middle term as the convex combination of the first and third term in the form of

$$\frac{f(t_3)-f(t_1)}{t_3-t_1}=\frac{t_2-t_1}{t_3-t_1}\frac{f(t_2)-f(t_1)}{t_2-t_1}+\frac{t_3-t_2}{t_3-t_1}\frac{f(t_3)-f(t_2)}{t_3-t_2}. \tag{3.2}$$

3.1. Lemma. *Let $f:\mathbb{R}\to\mathbb{R}$ be an even convex function. Then f is either constant or meets the limits $\lim_{t\to\pm\infty}f(t)=\infty$.*

Proof. The function f may be a constant because the collection of even convex functions includes all constants.

Suppose that f is not constant. Then a point $t_1>0$ exists so that $f(t_1)>f(0)$. Let $t_2>t_1$. Applying the left-hand side of the inequality in formula (3.1) to the triple $0<t_1<t_2$, we get

$$\frac{f(t_1)-f(0)}{t_1-0}\le\frac{f(t_2)-f(0)}{t_2-0}.$$

Then applies $f(t_2)-f(0)>0$, otherwise we get $f(t_1)\le f(0)$, and therefore

$$f(t_1)-f(0)\le\frac{t_1}{t_2}\big(f(t_2)-f(0)\big)<f(t_2)-f(0), \tag{3.3}$$

which yields $f(t_1)<f(t_2)$. Hence the function f is strictly increasing on the interval $[t_1,\infty)$. Since f is even, there must be $\lim_{t\to\pm\infty}f(t)=\infty$. □

3.2. Corollary. *Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function, and let $g:\mathbb{R}\to\mathbb{R}$ be an even convex function.*

If there exists a number $c\ge 0$ such that the inequality

$$g(t)-c\le f(t)\le g(t)+c \tag{3.4}$$

holds for every $t\in\mathbb{R}$, then the function f has the global minimum.

Proof. According to Lemma 3.1, the function g is constant or meets the limits $\lim_{t\to\pm\infty}g(t)=\infty$.

If g is constant, then using the assumption in formula (3.4) with $c_1=g(0)-c$ and $c_2=g(0)+c$, we obtain that $c_1\le f(t)\le c_2$ for every $t\in\mathbb{R}$. We will show that f is constant. Let $t_1<t_2$. Applying the inequality of the first and third term in formula (3.1) to the triple $t<t_1<t_2$, we get

$$\frac{f(t_1)-f(t)}{t_1-t}\le\frac{f(t_2)-f(t_1)}{t_2-t_1}.$$

Sending t to $-\infty$ and respecting the boundedness of the function f, we obtain that $0\le f(t_2)-f(t_1)$, and so $f(t_1)\le f(t_2)$. Similarly we get the reverse inequality. The conclusion is that $f(t_1)=f(t_2)$.

If $\lim_{t\to\pm\infty}g(t)=\infty$, then the reflection moment applied to formula (3.4) yields $\lim_{t\to\pm\infty}f(t)=\infty$. The function f has the global minimum by Corollary 2.3. □

The above corollary can be generalized to higher dimensions.

3.3. Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $g : \mathbb{R}^n \to \mathbb{R}$ be an even convex function.*

If there exists a number $c \geq 0$ such that the inequality

$$g(x) - c \leq f(x) \leq g(x) + c \tag{3.5}$$

holds for every $x \in \mathbb{R}^n$, then the function f has the global minimum.

Proof. The restriction of the function g to any line in $\mathbb{R}^n$ passing through the origin is an even convex function. Then according to Lemma 3.1, for $x \in \mathbb{R}^n \setminus \{o\}$ we have either $\lim_{t\to\pm\infty} g(tx) = \infty$ or $g(tx) = g(o)$ for every $t \in \mathbb{R}$.

Suppose that g meets the limits $\lim_{t\to\pm\infty} g(tx) = \infty$ for every $x \in \mathbb{R}^n \setminus \{o\}$. Applying the reflection moment to the inequality

$$g(tx) - c \leq f(tx) \leq g(tx) + c$$

by sending t to $\pm\infty$, it follows that $\lim_{t\to\pm\infty} f(tx) = \infty$ for every $x \in \mathbb{R}^n \setminus \{o\}$. The function f has the global minimum by Corollary 2.3.

Suppose that g is constant on the line $X = \{tx_0 : t \in \mathbb{R}\}$ for some point $x_0 \in \mathbb{R}^n \setminus \{o\}$. Then f is constant on the line X by the proof of Corollary 3.2. Without loss of generality, using the rotation in the space $\mathbb{R}^{n+1}$ around the function axis x_{n+1} which the line X turns into axis x_n, we can assume that f and g are constant on the axis x_n. To prove that f has the global minimum, we will apply the mathematical induction on the dimension n. The theorem holds for $n = 1$ by Corollary 3.2. We assume that the theorem holds for $n - 1$, where $n \geq 2$.

Then we define the convex functions f_0 and g_0 on the space $\mathbb{R}^{n-1}$ by

$$f_0(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0)$$

and

$$g_0(x_1, \ldots, x_{n-1}) = g(x_1, \ldots, x_{n-1}, 0),$$

which satisfy the assumptions of the theorem. Including the function f_0 into the induction premise, we can consider that it has the global minimum. The global minimum of f_0 is also the global minimum of f. $\square$

Theorem 3.3 can be applied to overdetermined systems as follows.

3.4. Corollary. *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator where $n < m$, and let $b \in \mathbb{R}^m$ be a point. Let $\|\,\|$ be a norm on $\mathbb{R}^m$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function defined by $f(x) = \|Ax - b\|$. Then f has the global minimum.*

Proof. The convexity of the function f follows from the norm triangle inequality.

Using the inequality

$$\|Ax\| - \|b\| \leq \|Ax - b\| \leq \|Ax\| + \|b\|$$

in the context of formula (3.5) with $f(x) = \|Ax - b\|$, $g(x) = \|Ax\|$ and $c = \|b\|$, we can conclude that f has the global minimum. $\square$

3.5. Corollary. *Let the assumptions of Corollary 3.4 be fulfilled. Let $S \subseteq \mathbb{R}^n$ be the set of the global minimum points of f. Then S is nonempty, closed respecting any norm on $\mathbb{R}^n$, and convex.*

Proof. The set S is nonempty by Corollary 3.4.

The set S is closed respecting any norm on $\mathbb{R}^n$ because the operator A is continuous respecting any pair of norms on $\mathbb{R}^n$ and $\mathbb{R}^m$.

To demonstrate the convexity of S, let us take a convex combination $t_1x_1^* + t_2x_2^*$ of points $x_1^*, x_2^* \in S$. Then the convexity of f yields

$$\begin{aligned} f(t_1x_1^* + t_2x_2^*) &\le t_1f(x_1^*) + t_2f(x_2^*) \\ &= f(x_1^*) = f(x_2^*), \end{aligned}$$

and therefore $f(t_1x_1^* + t_2x_2^*) = f(x_1^*) = f(x_2^*)$ because the value $f(x_1^*) = f(x_2^*)$ is minimal. Thus the combination $t_1x_1^* + t_2x_2^*$ belongs to the set S, providing its convexity. □

4. Applications to overdetermined systems by using p-norms

In numerous studies related to an estimation, we need norms that depend on real numbers. Let $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ be a point, where $m \ge 2$. The p-norms on the space $\mathbb{R}^m$ are defined for numbers $p \ge 1$ by

$$\|y\|_p = \left(\sum_{i=1}^{m} |y_i|^p\right)^{\frac{1}{p}}, \tag{4.1}$$

and their limit case the *max*-norm is expressed by

$$\|y\|_\infty = \max_{1\le i\le m} |y_i|. \tag{4.2}$$

Let A, x and b be as in formula (1.2). Applying the above norms to the point $y = Ax$, and thus to the coordinates $y_i = \sum_{j=1}^{n} a_{ij}x_j - b_i$, the function f in formula (1.5) takes the forms

$$f_p(x) = \|Ax - b\|_p = \left(\sum_{i=1}^{m}\left|\sum_{j=1}^{n} a_{ij}x_j - b_i\right|^p\right)^{\frac{1}{p}} \tag{4.3}$$

and

$$f_\infty(x) = \|Ax - b\|_\infty = \max_{1\le i\le m}\left|\sum_{j=1}^{n} a_{ij}x_j - b_i\right|. \tag{4.4}$$

The investigation of overdetermined systems by using the p-homogeneous metric for $0 < p < 1$ was done in [6]. General analysis of convex functions including their extreme values can be found in [4]. Optimization problems concerning convex functions were discussed in [5].

In the next two examples, we will illustrate the application of functions in formula (4.3) and formula (4.4) in finding the optimal solutions of overdetermined systems. As usual, the functions f_1, f_2 and f_∞ will mostly be used.

To make it easier to present the set of optimal solutions, we will use a notion of the convex (affine) hull. Given a set S in a real vector space, the convex hull conv$S$ (affine hull aff$S$) represents the smallest convex (affine) set in the respective vector space which contains the set S.

4.1. Example. Using the functions f_1, f_2 and f_∞, find the optimal solutions of the overdetermined system

$$\begin{array}{rcrcl} x_1 & & & = & 1 \\ x_1 & - & x_2 & = & 0 \\ & & x_2 & = & 2 \end{array}. \tag{4.5}$$

Using the function

$$f_1(x_1, x_2) = |x_1 - 1| + |x_1 - x_2| + |x_2 - 2|,$$

and eliminating the signs of absolute values, we obtain that $f_1(x_1, x_2) = 1$ on the triangle with vertices $T_1(1,1)$, $T_2(1,2)$ and $T_3(2,2)$. Combining vertices T_1, T_2, T_3 with coefficients $t_1, t_2, t_3 \in [0,1]$ of the sum $t_1+t_2+t_3 = 1$, we can conclude that each convex combination

$$x^* = t_1T_1 + t_2T_2 + t_3T_3 = (t_1 + t_2 + 2t_3, t_1 + 2t_2 + 2t_3)$$

is the minimum point, and the global minimum is

$$f_1(t_1 + t_2 + 2t_3, t_1 + 2t_2 + 2t_3) = 1.$$

So, the set S_1 of optimal solutions respecting the 1-norm is the triangle with vertices T_1, T_2 and T_3. Using the convex hull, we can write

$$S_1 = \operatorname{conv}\{T_1, T_2, T_3\}.$$

Using the function

$$f_2(x_1, x_2) = \sqrt{(x_1 - 1)^2 + (x_1 - x_2)^2 + (x_2 - 2)^2},$$

and applying the differential calculus, we find that $x^* = (4/3, 5/3)$ is the unique minimum point, and so the global minimum is

$$f_2\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\sqrt{3}}{3}.$$

The set of optimal solutions respecting the 2-norm is the singleton

$$S_2 = \{(4/3, 5/3)\}.$$

Using the function

$$f_\infty(x_1, x_2) = \max\big\{|x_1 - 1|,\ |x_1 - x_2|,\ |x_2 - 2|\big\},$$

and removing the signs of absolute values, we obtain that $x^* = (4/3, 5/3)$ is the unique minimum point, and thus the global minimum is

$$f_\infty\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{1}{3}.$$

The set of optimal solutions respecting the max-norm is

$$S_\infty = \{(4/3, 5/3)\}.$$

Knowing the global minimum points, we can determine the projections of the vector b onto the space $R(A)$ as vectors in $R(A)$ of the minimal distance to b.

4.2. Remark. As regards the projections of b onto $R(A)$ relating to the system in formula (4.5), we have to point out the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The space $R(A)$ is the plane in $\mathbb{R}^3$ spanned by the vectors a_1 and a_2. The projections of the vector b onto the plane $R(A)$ respecting the 1-norm are the vectors

$$\begin{aligned} b_1^* &= (t_1 + t_2 + 2t_3)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (t_1 + 2t_2 + 2t_3)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ &= t_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t_3\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}. \end{aligned}$$

The above convex combinations indicate that the set of projections is the triangle with vertices $V_1(1,0,1)$, $V_2(1,-1,2)$ and $V_3(2,0,2)$.

The projection of b onto $R(A)$ respecting the 2-norm and max-norm is the vector

$$b_2^* = b_\infty^* = \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \end{bmatrix}.$$

Let $c \in \mathbb{R}$ be a number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = |x - c|^p$ where $p > 1$. Then the function f is differentiable at each point x with the derivative

$$f'(x) = \begin{cases} -p|x-c|^{p-1} & \text{for} \quad x \le c \\ p|x-c|^{p-1} & \text{for} \quad x \ge c \end{cases}.$$

Another note regarding Example 4.1. Taking $p > 1$, and using the functional

$$f_p(x_1, x_2) = \sqrt[p]{|x_1 - 1|^p + |x_1 - x_2|^p + |x_2 - 2|^p},$$

we find that $x^* = (4/3, 5/3)$ is the unique minimum point with the global minimum

$$f_p\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\sqrt[p]{3}}{3}.$$

4.3. Example. Using the functions f_p for $p \ge 1$ and $p = \infty$, find the optimal solutions of the overdetermined system

$$\begin{array}{rcrcr} x_1 & - & x_2 & = & 0 \\ x_1 & - & x_2 & = & 1 \\ x_1 & - & x_2 & = & -1 \end{array}. \tag{4.6}$$

Using the functions f_p for $p \ge 1$ and $p = \infty$, we find that each point x^* of the line $x_1 = x_2$ is the minimum point, and the global minimum is

$$f_p(x_1, x_1) = \sqrt[p]{2}.$$

The set S_p of optimal solutions respecting any p-norm ($p \ge 1$ and $p = \infty$) is the line $x_1 = x_2$. Using the affine hull of the points $T_1(0,0)$ and $T_2(1,1)$, we can write

$$S_p = \operatorname{aff}\{T_1, T_2\}.$$

4.4. Remark. As regards the projections of b onto $R(A)$ relating to the system in formula (4.6), we have to point out the vectors

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The space $R(A)$ is the line in $\mathbb{R}^3$ spanned by the vector a_1. The projection of the vector b onto the line $R(A)$ respecting any p-norm ($p \ge 1$ and $p = \infty$) is the origin as a result of

$$b_p^* = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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