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On Π- coherence of rings

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Abstract

Let n be a fixed positive integer. A ring R is called left n-Π- coherent if every n-generated torsionless left R-module is finitely presented, some characterizations and applications of n-Π-coherent rings are obtained.

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1. Introduction

Recall that a left R-module M is called torsionless if M can be embedded into some direct product of $_RR$, or equivalently, if the natural map $i : M \to M^{**}$ is monic, where M^* denotes $\mathrm{Hom}_R(M, R)$. A ring R is called *left* Π*-coherent* [2] if every finitely generated torsionless left R-module is finitely presented. Clearly, a left left Π- coherent ring is left coherent, so, in [14], Π-coherent rings are also called strongly coherent rings. Π- coherent rings have been studied by a series of authors (see, for example, [2, 5, 9, 10, 14, 19]).

In this article, we extend the concept of left Π- coherent rings to *left* n-Π- *coherent* rings, we call a ring R left n-Π- coherent if every n-generated torsionless left R-module is finitely presented.

In section 2, we give a series of characterizations of left n-Π-coherent rings. As corollaries, some characterizations of left Π- coherent rings are obtained. To characterize left n-Π- coherent rings, we shall study n-projective modules, this concept was introduced in [22]. Moreover, by using the concept of 1-Π-coherent rings, we give a new characterization of Quasi-Frobenius rings.

In Section 3, we call a ring R right *n-GF* if every n-generated right R-module embeds in a free module. n-GF rings are characterized by n-projective modules, conditions under which left n-Π-coherent rings are right n-GF rings are given. As corollaries, conditions under which left Π-coherent (resp., left 1-Π-coherent) rings are right FGF (resp., right CF) rings are given.

A ring R is called *right n-semihereditary* [23] if every n-generated right ideal of R is projective. By [23, Theorem 1], a ring R is right n-semihereditary if and only if every n-generated submodule of a projective right R-module is projective. In Section 4 , we call a ring R *right strongly n-semihereditary* if every n-generated torsionless right R-module is projective. n-semihereditary

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rings are characterized by n-projective modules, conditions under which right IF rings are right strongly n-semihereditary rings are given, and conditions under which left n-Π- coherent rings are right n-semihereditary rings are given too. As corollaries, some new characterizations of PP rings and semihereditary rings are given, conditions under which left Π-coherent (resp., left 1-Π- coherent) rings are right semihereditary (resp., right PP) rings are given. Furthermore , by using n-Π-coherent rings and n-projective modules, we give a series characterization of commutative strongly n-semihereditary rings.

Throughout this paper, n is a positive integer, R is an associative ring with identity, and all modules considered are unitary. In general, for a set S, we write S^n for the set of all formal $1 \times n$ matrices whose entries are elements of S, and S_n for the set of all formal $n \times 1$ matrices whose entries are elements of S. Let N be a left R-module, $X \subseteq N_n$ and $A \subseteq R^n$. Then we denote $\mathbf{r}_{N_n}(A) = \{u \in N_n : au = 0, \forall a \in A\}$ and $\mathbf{l}_{R^n}(X) = \{a \in R^n : ax = 0, \forall x \in X\}$.

2. n-Π- coherent rings

Recall that a right R-module M is called *finitely projective (resp., singly projective)* [1] if for every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C finitely generated (resp., cyclic) right R-module, there exists $h : C \to N$ such that $g = fh$. In [22], Zhu extended the two concepts to *n-projective modules*. Following [22], a right R-module M is called *n-projective* if for any epimorphism $f : N \to M$ and for every n-generated submodule M_0 of M, there exists a homomorphism $g : M_0 \to N$ such that fg is the identity map of M_0.

The following Theorem will be used frequently in the sequel.

2.1. Theorem. *The following are equivalent for a right R-module M:*

(1) M is n-projective.

(2) For every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C an n-generated right R-module, there exists $h : C \to N$ such that $g = fh$.

(3) For any n-generated right R-module N and any homomorphism $f : N \to M, f$ factors through a finitely generated free right R-module F, that is, there exist $g : N \to F$ and $h : F \to M$ such that $f = hg$.

(4) For any n-generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a finitely generated free right R-module F.

(5) For any n-generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a free right R-module F.

(6) For any n-generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a finitely projective right R-module F.

(7) For any n-generated submodule N of M, the inclusion map $\iota : N \to M$ factors through an n-projective right R-module P.

Proof. (1) $\Leftrightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Let F_1 be a free module and $\pi : F_1 \to M$ be an epimorphism. Since M is n-projective, there exists a homomorphism $g : N \to F_1$ such that $f = \pi g$. Note that N is n-generated, $\mathrm{Im}(g)$ is finitely generated , so there is a finitely generated free module F such that $\mathrm{Im}(g) \subseteq F \subseteq F_1$. Let $\iota : F \to F_1$ be the inclusion map and $h = \pi\iota$. Then h is a homomorphism from F to M and $f = hg$.

(3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7). It is obvious.

(4) $\Rightarrow$ (2). Let $f : N \to M$ be an epimorphism and $g : C \to M$ be any homomorphism, where C is an n-generated right R-module. Then $\mathrm{Im}(g)$ is n-generated. By (4), the inclusion $\iota : \mathrm{Im}(g) \to M$ factors through a finitely generated free right R-module F, i.e., there exist $\varphi : \mathrm{Im}(g) \to F$ and $\psi : F \to M$ such that $\iota = \psi\varphi$. Since F is projective, there exists a homomorphism $\theta : F \to N$ such that $\psi = f\theta$. Now write $h = \theta\varphi g$, then h is a homomorphism from C to N, and $g = \iota g = \psi\varphi g = (f\theta)\varphi g = fh$. Thus (2) holds.

(7) ⇒ (5). Let N be an n-generated submodule of M, and $\iota : N \to M$ be the inclusion map. By (7), there exist an n-projective right R-module P, a homomorphism $\alpha : N \to P$ and a homomorphism $\beta : P \to M$ such that $\iota = \beta\alpha$. Let $\pi : F \to P$ be an epimorphism, here F is a free module. Since P is n-projective, there exists a homomorphism $g : N \to F$ such that $\alpha = \pi g$. Now write $h = \beta\pi$. Then $h \in \mathrm{Hom}_R(F, M)$ and $\iota = hg$. □

2.2. Corollary. *Every n-generated n-projective module is projective.*

2.3. Corollary. *The following are equivalent for a right R-module M:*

(1) M is finitely projective.

(2) For any epimorphism $f : N \to M$ and for every finitely generated submodule M_0 of M, there exists a homomorphism $g : M_0 \to N$ such that fg is the identity map of M_0.

(3) For any finitely generated right R-module N and any homomorphism $f : N \to M$, f factors through a finitely generated free right R-module F, that is, there exist $g : N \to F$ and $h : F \to M$ such that $f = hg$.

(4) For any finitely generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a finitely generated free right R-module F.

(5) For any finitely generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a free right R-module F.

(6) For any finitely generated submodule N of M, the inclusion map $\iota : N \to M$ factors through a finitely projective right R-module P.

2.4. Proposition. *Every pure submodule of an n-projective module is n-projective.*

Proof. Let M be an n-projective right R-module and M' a pure submodule of M. Let C be an n-generated right R-module and f be a homomorphism from C to M'. Write $\iota : M' \to M$ be the inclusion map. Since M is n-projective, by Theorem 2.1, ιf factor through a finitely generated free right R-module F, that is, there exist $g : C \to F$ and $\varphi : F \to M$ such that $\iota f = \varphi g$. So we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & C & \xrightarrow{g} & F & \xrightarrow{\pi_1} & F/\mathrm{Im}(g) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & M' & \xrightarrow{\iota} & M & \xrightarrow{\pi_2} & M/M' & & \end{array}$$

, where π_1 and π_2 are the natural epimorphisms, and $\psi(x + \mathrm{Im}(g)) = \pi_2\varphi(x)$. Since M' is pure in M and $F/\mathrm{Im}(g)$ is finitely presented, there exists a homomorphism $\alpha : F/\mathrm{Im}(g) \to M$ such that $\psi = \pi_2\alpha$. Thus, by Diagram Lemma (see [18], page 53), there exists a homomorphism h from F to M' such that $f = hg$. Therefore, M' is n-projective. □

2.5. Corollary. [1, Proposition 14] *Every pure submodule of a finitely (or singly) projective module is finitely (or singly) projective.*

Let $\mathcal{F}$ be a class of right R-modules and M a right R-module. Following [11], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. It is easy to see that an epic $\mathcal{F}$-preenvelope is an $\mathcal{F}$-envelope.

2.6. Theorem. *Let $\mathcal{F}$ be a class of right R-modules closed under pure submodules and isomorphisms. Then the following statements are equivalent:*

(1) $\mathcal{F}$ is closed under direct product.

(2) Every right R-module has an $\mathcal{F}$-preenvelope.

Proof. (1) ⇒ (2). Let N be any right R-module. By [11, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ dependent on Card(N) and Card(R) such that for any homomorphism $f : N \to F$ with $F \in \mathcal{F}$, there is a pure submodule S of F such that $f(N) \subseteq S$ and Card $S \leq \aleph_\alpha$. Thus f has a factorization $N \to S \to F$ with $S \in \mathcal{F}$ since $\mathcal{F}$ is closed under pure submodules. Now let $\{\varphi_\beta\}_{\beta\in B}$ be the family of all such homomorphisms $\varphi_\beta : N \to S_\beta$ with Card $S_\beta \leq \aleph_\alpha$ and $S_\beta \in \mathcal{F}$. Then any homomorphism $N \to F$ with $F \in \mathcal{F}$ has a factorization $N \to S_i \to F$ for some $i \in B$. Thus the homomorphism $N \to \prod_{\beta\in B} S_\beta$ induced by all φ_β is an $\mathcal{F}$-preenvelope since $\prod_{\beta\in B} S_\beta \in \mathcal{F}$ by (1).

(2) ⇒ (1). For any family $\{F_i\}_{i\in I}$ of right R-modules in $\mathcal{F}$, by hypothesis, $\prod_{i\in I} F_i$ has an $\mathcal{F}$-preenvelope $\varphi : \prod_{i\in I} F_i \to F$. Let $p_i : \prod_{i\in I} F_i \to F_i$ be the projective. Then there exists $f_i : F \to F_i$ such that $p_i = f_i\varphi$. Define $\psi : F \to \prod_{i\in I} F_i$ by $\psi(x) = (f_i(x))$ for each $x \in F$, then it is easy to check that $\psi\varphi = 1$. Hence $\prod_{i\in I} F_i$ is isomorphic to a direct summand of F, and so $\prod_{i\in I} F_i \in \mathcal{F}$. □

2.7. Proposition. *If M is an n-generated right R-module, then every projective preenvelope of M is an n-projective preenvelope of M .*

Proof. Let $f : M \to P$ be a projective preenvelope of M. Then P is clearly n-projective. And for any n-projective right R-module P' and any homomorphism $g : M \to P'$, by Theorem 2.1, g factors through a finitely generated free right R-module F, that is, there exist $\alpha : M \to F$ and $\beta : F \to P'$ such that $g = \beta\alpha$. Since $f : M \to P$ is a projective preenvelope of M, there exists a homomorphism $\gamma : P \to F$ such that $\alpha = \gamma f$. Now let $h = \beta\gamma$. Then $g = hf$. So f is an n-projective preenvelope of M. □

2.8. Corollary. *(1) If M is a finitely generated right R-module, then every projective preenvelope of M is a finitely projective preenvelope of M .*

(2) If M is a cyclic right R-module, then every projective preenvelope of M is a singly projective preenvelope of M .

Proof. (1). By a similar way to the proof of Proposition 2.7.

(2). It follows immediately from Proposition 2.7. □

Inspired by the concept of Π- coherent rings, we have the following definition.

2.9. Definition. A ring R is called left n-Π- coherent if every n-generated torsionless left R-module is finitely presented.

Similarly, we have the concept of right n-Π-coherent rings. Clearly, a ring R is left Π-coherent if and only if it is left n-Π-coherent for every positive integer n. A left $(n + 1)$-Π-coherent ring is left n-Π-coherent, but the converse does not hold in general.

2.10. Lemma. *Let $X = \{\alpha_i : i \in I\}$ be a subset of R_n. Then $\mathbf{l}_{R^n}(X) \cong P^*$, where $P = R_n/\sum_{i\in I}\alpha_i R$.*

Proof. Define $\sigma : \mathbf{l}_{R^n}(X) \to P^*$ by $\sigma(\beta) = f_\beta$, where $f_\beta(\overline{\gamma}) = \beta\gamma$. Then it is easy to check that σ is a left R- isomorphism. □

Now we characterize left n-Π-coherent rings as follows.

2.11. Theorem. *The following statements are equivalent for a ring R:*

(1) R is left n-Π-coherent.

(2) If $0 \to K \xrightarrow{f} M \xrightarrow{g} T$ is an exact sequence of left R-modules, where M is n-generated and T is torsionless, then K is finitely generated.

(3) $\mathbf{l}_{R^n}(X)$ is a finitely generated submodule of ${}_RR^n$ for any subset X of R_n.

(4 For any n-generated right R-module M, the dual module M^* is a finitely generated left R-module.

Proof. (1) ⇒ (2). Since R is left n-Π-coherent and $\mathrm{Im}(g)$ is an n-generated torsionless left R-module, $\mathrm{Im}(g)$ is finitely presented. Note that the sequence $0 \to \mathrm{Ker}(g) \to M \to \mathrm{Im}(g) \to 0$ is exact, we have that $\mathrm{Ker}(g)$ is finitely generated. Thus $K \cong \mathrm{Im}(f) = \mathrm{Ker}(g)$ is finitely generated.

(2) ⇒ (3). Let $X = \{\alpha_i : i \in I\}$. Then we have an exact sequence of left R-modules $0 \to \mathbf{l}_{R^n}(X) \to R^n \xrightarrow{g} R^I$, where $g(\beta) = (\beta\alpha_i)_{i\in I}$. By (2), $\mathbf{l}_{R^n}(X)$ is a finitely generated left R-module.

(3) ⇒ (1). Let $T = Rt_1 + \cdots + Rt_n$ be an n-generated submodule of R^I, where $t_j = (a_{ij})_{i\in I}$. Write $\alpha_i = (a_{i1}, \cdots, a_{in})', i \in I$, $X = \{\alpha_i \mid i \in I\}$. Then we have an exact sequence of left R-modules $0 \to \mathbf{l}_{R^n}(X) \to R^n \to T \to 0$. By (3), $\mathbf{l}_{R^n}(X)$ is finitely generated, so T is finitely presented.

(3) ⇔ (4) follows from Lemma 2.10. □

2.12. Corollary. *The following statements are equivalent for a ring R:*

(1) R is left Π*-coherent.*

(2) If $0 \to K \xrightarrow{f} M \xrightarrow{g} T$ *is an exact sequence of left R-modules, where M is finitely generated, T is torsionless, then K is finitely generated.*

(3) $\mathbf{l}_{R^n}(X)$ *is a finitely generated submodule of* ${}_RR^n$ *for any positive integer n, any subset X of* R_n.

(4) For any finitely generated right R-module M, the dual module M^* *is a finitely generated left R-module.*

We note that the equivalence of (1), (3) and (4) in Corollary 2.12 was shown in [2, Theorem 1], but the method we use in the proof of our Theorem 2.11 is different from that of [2, Theorem 1]. Following [10], a ring R is said to be a *right ⋆-ring* provided that every finitely generated right R-module has finitely generated dual, so, by Corollary 2.12, right ⋆-rings are identified with left Π-coherent rings.

2.13. Corollary. *Let R be a right coherent left n-*Π*-coherent ring. Then it is a right n-*Π*-coherent ring.*

Proof. Let M be an n-generated torsionless right R-module. Since R is left n-Π-coherent, by Theorem 2.11, M^* is a finitely generated left R-module, and so there exists a finitely generated free left R-module F such that $F \to M^* \to 0$ is exact, which induces an exact sequence $0 \to M^{**} \to F^*$. But M is torsionless, the natural map $i : M \to M^{**}$ is monic, and so the sequence $0 \to M \to F^*$ is exact, it shows that M is a finitely generated submodule of a free right R-module. Note that R is right coherent, we have that M is finitely presented by [4, Theorem 2.1]. Therefore, R is a right n-Π-coherent ring. □

2.14. Corollary. [10, Corollary 2.5B] *A right coherent left* Π*-coherent ring is right* Π*-coherent.*

Following [20], a ring R is called *left (m,n)-coherent* if every n-generated submodule of the left R-module R^m is finitely presented.

2.15. Example. *Let K be a field , and* $x, y_1, y_2, ...$ *be commuting indeterminates,* $S = K[x, y_1, y_2, ...]$ *and* $R = K[x^2, x^3, y_1, y_2, ..., xy_1, xy_2, ...]$*. Then R is a subring of the domain S, so R is 1-*Π*-coherent. But by* [20, Example 5.8]*, R is not (1,2)-coherent and so it is not 2-*Π*-coherent.*

2.16. Theorem. *The following are equivalent for a ring R:*

(1) R is a left n-Π*-coherent ring.*

(2) Every n-generated right R-module has a projective preenvelope.

(3) Every n-generated right R-module has an n-projective preenvelope.

(4) Every direct product of n-projective right R-modules is n-projective.

(5) Any direct product of copies of R_R *is n-projective.*

(6) Every right R-module has an n-projective preenvelope.

Proof. (1) ⇒ (2). Let M be an n-generated right R-module. Since M^* is finitely generated, there exists a generating set $\{f_j \in M^* : j = 1, 2, ..., m\}$. Define $f : M \to R^m$; $x \mapsto (f_1(x), f_2(x), \cdots, f_m(x)), x \in$

M. We shall show that f is a projective preenvelope. It is enough to show that for any positive integer k and any homomorphism $g : M \to R^k$, there exists a homomorphism $h : R^m \to R^k$ such that $g = hf$. Let $\pi_i : R^k \to R$ be the ith projection, $i = 1, 2, ..., k$. Then there exist $r_{ij} \in R$, $j = 1, 2, ..., m$ such that $\pi_i g = \sum_{j=1}^{m} r_{ij} f_j$. Define $h : R^m \to R^k; (a_1, a_2, \cdots, a_m) \mapsto (\sum_{j=1}^{m} r_{1j}a_j, \sum_{j=1}^{m} r_{2j}a_j, \cdots, \sum_{j=1}^{m} r_{kj}a_j)$. Then $g = hf$.

(2) $\Rightarrow$ (3). By Proposition 2.7.

(3) $\Rightarrow$ (4). Let $\{M_i\}_{i\in I}$ be a family of n-projective right R-modules and N any n-generated submodule of $\prod_{i\in I} M_i$. Let $\iota : N \to \prod_{i\in I} M_i$ be the inclusion map and $\pi_i : \prod_{i\in I} M_i \to M_i$ be the ith projection. Let $i \in I$. Since M_i is n-projective, there exist a finitely generated free right R-module F_i, homomorphisms $g_i : N \to F_i$ and $h_i : F_i \to M_i$ such that $\pi_i\iota = h_i g_i$ by Theorem 2.1(3). Note that N has an n-projective preenvelope $f : N \to P$ by (3), and so there is $\varphi_i : P \to F_i$ such that $g_i = \varphi_i f$. Define $g : P \to \prod_{i\in I} M_i$ by $g(x) = ((h_i\varphi_i)(x))$. Then $\iota = gf$. Thus $\prod_{i\in I} M_i$ is n-projective by Theorem 2.1(7).

(4) $\Leftrightarrow$ (6). By Proposition 2.4 and Theorem 2.6.

(4) $\Rightarrow$ (5). It is clear.

(5) $\Rightarrow$ (1). Let M be an n-generated right R-module. For every index set I, there is a canonical homomorphism $\tau : R^I \otimes M^* \to (M^*)^I$ defined by $\tau((r_i) \otimes \alpha) = (r_i\alpha)$. We shall show that τ is epic. Indeed, let $(f_i) \in (M^*)^I$. Define $f : M \to (R_R)^I$ by $f(x) = (f_i(x))$. Then f is a right R-homomorphism. By (5), $(R_R)^I$ is n-projective. So by Theorem 2.1, there exist a finitely generated free right R-module R^m, a homomorphism $g : M \to R^m$ and a homomorphism $h : R^m \to R^I$ such that $f = hg$. Let $\pi_i : R^I \to R$ be the ith projection, $p_j : R^m \to R$ be the jth projection and $\iota_j : R \to R^m$ the jth injection, $j = 1, 2, \cdots, m$. Put $a_j = h\iota_j(1)$ and $g_j = p_j g$. Then for any $i \in I$ and any $x \in M$, we have $f_i(x) = \pi_i f(x) = \pi_i hg(x) = \pi_i h(\sum_{j=1}^{m} \iota_j p_j)g(x) = \pi_i \sum_{j=1}^{m} h\iota_j p_j g(x) = \pi_i \sum_{j=1}^{m} h\iota_j(g_j(x)) = \pi_i \sum_{j=1}^{m} h\iota_j(1)(g_j(x)) = \pi_i \sum_{j=1}^{m} a_j(g_j(x))$, so $f_i = \sum_{j=1}^{m} \pi_i(a_j)g_j$, and thus $(f_i) = \tau(\sum_{j=1}^{m} a_j \otimes g_j)$. This shows that τ is an epimorphism, and so M^* is a finitely generated left R-module by [11, Lemma 3.2.21]. Therefore, by Theorem 2.11, R is a left n-Π-coherent ring. □

2.17. Corollary. *The following are equivalent for a ring R:*

(1) R is a left Π*-coherent ring.*

(2) Every finitely generated right R-module has a projective preenvelope.

(3) Every finitely generated right R-module has a finitely projective preenvelope.

(4) Any direct product of finitely projective right R-modules is finitely projective.

(5) Any direct product of copies of R_R *is finitely projective.*

(6) Every right R-module has a finitely projective preenvelope.

Proof. By a similar way to the proof of Theorem 2.16. □

We note that the equivalence of (1), (2), (3), (4) and (6) in Corollary 2.17 was shown in [9, Corollary 3.6 , Corollary 3.12], Corollary 3.6 and Corollary 3.12 in [9] follow from [9, Proposition 3.4] and [9, Theorem 3.10] respectively, but the way we use in the proof of our Theorem 2.16 is different from that of [9, Proposition 3.4, Theorem 3.10].

Recall that a ring R is called *Quasi-Frobenius* if it is right or left self-injective and right or left artinian, or equivalently, if it is right or left self-injective and right or left noetherian; a ring R is called *left Kasch* if every simple left R-module embeds in R. At the end of this section, we give a new characterization of Quasi-Frobenius rings by 1-Π-coherent rings.

2.18. Theorem. *The following are equivalent for a ring R:*

(1) R is a Quasi-Frobenius ring.

(2) R is a left self-injective, left 1-Π*-coherent left Kasch ring.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1). Since R is left self-injective and left Kasch, by [17, Proposition 1.44], $R = E(_RR)$ is a cogenerator, and so R/T is torsionless for every left ideal T of R. Note that R is left 1-Π-coherent, R/T is finitely presented, and hence T is finitely generated. Thus, R is a left noetherian left self-injective ring, and so it is a Quasi-Frobenius ring. □

3. n-GF-rings

Recall that a ring R is called *right CF* [17] if every cyclic right R-module embeds in a free module, a ring R is called *right FGF* [10] if every finitely generated right R-module embeds in a free module, a ring R is called *right 2-GF* [7, 15] if every 2-generated right R-module embeds in a free module. These rings have important role in the studies of Quasi-Frobenius rings. Now we extend these concepts as follows.

3.1. Definition. A ring R is called a *right n-GF* ring if every n-generated right R-module embeds in a free module.

3.2. Proposition. *The following are equivalent for a ring R:*

(1) R is a right n-GF ring.

(2) Every injective right R-module is n-projective.

(3) The injective envelope of any n-generated right R-module is n-projective.

Proof. (1) ⇒ (2) . Let E be an injective right R-module. Then for every epimorphism $f : N \to E$ and any homomorphism $g : C \to E$ with C being an n-generated right R-module. By (1), there exists a free module F and a monomorphism $\iota : C \to F$. So there exists a $h : F \to E$ such that $g = h\iota$, and hence there exists a $\varphi : F \to N$ such that $h = f\varphi$. Thus, $\varphi\iota$ is a homomorphism from C to N and $g = f(\varphi\iota)$. By Theorem 2.1, E is n-projective.

(2) ⇒ (3) . It is clear.

(3) ⇒ (1) . Let N be an n-generated right R-module. Let $\iota : N \to E(N)$ be the inclusion map and $\pi : F \to E(N)$ be an epimorphism, where F is a free module. Since $E(N)$ is n-projective, there exists a homomorphism $f : N \to F$ such that $\iota = \pi f$. It is easy to see that f is monic, and so (1) follows. □

3.3. Corollary. *The following are equivalent for a ring R:*

(1) R is a right CF ring.

(2) Every injective right R-module is singly projective.

(3) The injective envelope of any cyclic right R-module is singly projective.

3.4. Corollary. [14, Theorem 2.10] *The following are equivalent for a ring R:*

(1) R is a right FGF ring.

(2) Every injective right R-module is finitely projective.

(3) The injective envelope of any finitely generated right R-module is finitely projective.

Recall that a ring R is called a *right dual ring* if $\mathbf{r}_R\mathbf{l}_R(T) = T$ for every right ideal T.

3.5. Definition. A ring R is called a *right n-dual ring* if $\mathbf{r}_{R_n}\mathbf{l}_{R^n}(T) = T$ for every submodule T of the right R-module R_n. A ring R is called a *right strongly dual ring* if it is a right n-dual ring for each positive integer n.

It is easy to see that a ring R is a right n-dual ring if and only if R_n/T is torsionless for every submodule T of the right R-module R_n. The following theorem is partly inspired by [9, Corollary 4.3] .

3.6. Theorem. *The following are equivalent for a left n-Π-coherent ring R:*

(1) R is a right n-GF ring.

(2) Every right R-module has a monic n-projective preenvelope.

(3) Every n-generated right R-module has a monic n-projective preenvelope.
(4) Every n-generated right R-module has a monic projective preenvelope.
(5) R is a right n-dual ring.

Proof. (1) ⇒ (2). Let M be any right R-module. Since R is left n-Π-coherent, by Theorem 2.16, M has an n-projective preenvelope $f : M \to P$. Since R is a right n-GF ring, by Proposition 3.2, $E(M)$ is n-projective. Let $\iota : M \to E(M)$ be the inclusion map. Then there exists a homomorphism $g : P \to E(M)$ such that $\iota = gf$, and hence f is monic.

(2) ⇒ (3) is trivial.

(3) ⇒ (4). Let M be an n-generated right R-module. Then M has a monic n-projective preenvelope $f : M \to P$ by (3). Thus, by Theorem 2.1(4), there exist a finitely generated free right R-module F, a monomorphism $g : M \to F$ and a homomorphism $h : F \to P$ such that $f = hg$. Now let P' be a projective right R-module and φ be a homomorphism from M to P'. Then there exists a homomorphism $\theta : P \to P'$ such that $\varphi = \theta f$. Thus, θh is a homomorphism from F to P' and $\varphi = (\theta h)g$. Therefore, $g : M \to F$ is a monic projective preenvelope of M.

(4) ⇒ (5). Let T be any submodule of the right R-module R_n. Then by (4), R_n/T embeds in a projective module, so it is torsionless, and thus $\mathbf{r}_{R_n}\mathbf{l}_{R^n}(T) = T$.

(5) ⇒ (1). Let M be an n-generated right R-module. Then there is an exact sequence $0 \to M \to R_R^I$ for some index set I by (5). Note that R_R^I is n-projective by Theorem 2.16 since R is left n-Π-coherent, M embeds in a finitely generated free right R-module by Theorem 2.1(4), that is, R is a right n-GF ring. □

3.7. Corollary. *The following are equivalent for a left* 1-Π-*coherent ring R:*
(1) R is a right CF ring.
(2) Every right R-module has a monic singly projective preenvelope.
(3) Every cyclic right R-module has a monic singly projective preenvelope.
(4) Every cyclic right R-module has a monic projective preenvelope.
(5) R is a right dual ring.

3.8. Corollary. *The following are equivalent for a left* Π-*coherent ring R:*
(1) R is a right FGF ring.
(2) Every right R-module has a monic finitely projective preenvelope.
(3)Every finitely generated right R-module has a monic finitely projective preenvelope.
(4) Every finitely generated right R-module has a monic projective preenvelope.
(5) R is a right strongly dual ring.

Proof. By a similar way to the proof of Theorem 3.6. □

4. n-semihereditary rings and strongly n-semihereditary rings

Recall that a ring R is called *right semihereditary* ([3], p.14) if every finitely generated right ideal of R is projective, a ring R is called *right PP* [13] if every principal right ideal of R is projective. Following [21, 23], a ring R is called *right n-semihereditary* if every n-generated right ideal of R is projective. By [23, Theorem 1], a ring R is right n-semihereditary if and only if every n-generated submodule of a projective right R-module is projective, so a ring R is right PP if and only if every cyclic submodule of a projective right R-module is projective . In [22], n-semihereditary rings are also called n-hereditary rings. Here we characterize right n-semihereditary rings in terms of n-projective modules.

4.1. Theorem. *The following are equivalent for a ring R:*
(1) R is a right n-semihereditary ring.
(2) Any submodule of an n-projective right R-module is n-projective.

Proof. (1) ⇒ (2). Suppose that N is a submodule of an n-projective right R-module P, and K be an n-generated submodule of N. Let $\lambda : N \to P$ and $\iota : K \to N$ be the inclusion maps. Since P is n-projective, by Theorem 2.1(4), $\lambda\iota$ factors through a finitely generated free right R-module F, and so K embeds in F, it follows that K is projective since R is right n-semihereditary. Therefore, N is n-projective by Theorem 2.1(6).

(2) ⇒ (1). It follows from the fact that every n-generated n-projective module is projective. □

4.2. Corollary. *The following are equivalent for a ring R:*

(1) R is a right PP ring.

(2) Any submodule of a singly projective right R-module is singly projective.

4.3. Corollary. *The following are equivalent for a ring R:*

(1) R is a right semihereditary ring.

(2) Any submodule of a finitely projective right R-module is finitely projective.

Note that a ring R is right n-semihereditary if and only every n-generated submodule of a projective right R-module is projective, a ring R is right semihereditary if and only every finitely generated submodule of a projective right R-module is projective, and observe that every submodule of a projective right R-module is torsionless, we have naturally the following definition.

4.4. Definition. A ring R is called *right strongly n-semihereditary* if every n-generated torsionless right R-module is projective. A ring R is called *right strongly semihereditary* if every finitely generated torsionless right R-module is projective. A ring R is called *right strongly PP* if every cyclic torsionless right R-module is projective.

It is easy to see that a right strongly n-semihereditary ring is both right n-semihereditary and right n-Π-coherent.

4.5. Proposition. *The following are equivalent for a ring R:*

(1) R is a semisimple Artinian ring.

(2) R is a right strongly PP and right dual ring.

Proof. (1) ⇒ (2). It is obvious.

(2) ⇒ (1). Let I be a right ideal of R. Since R is right dual, by [17, Lemma 1.40], R/I is torsionless. Since R is right strongly PP, R/I is projective, and so I is a direct summand. It follows that R is a semisimple Artinian ring. □

Recall that a ring R is called *left P-injective* [16], if every R-homomorphism from a principal left ideal of R to R extends to a endomorphism of R.

4.6. Proposition. *Let R be a right strongly PP and left P-injective ring. Then it is a von Neumann regular ring.*

Proof. Let $a \in R$. Since R is left P-injective, by [16, Lemma 1.1], we have $\mathbf{r}_R\mathbf{l}_R(aR) = aR$. So, by [17, Lemma 1.40], R/aR is torsionless. But R is right strongly PP, R/aR is projective, and thus aR is a direct summand. It follows that R is a von Neumann regular ring. □

4.7. Remark. In [6, Example 1], Chen and Ding constructed a ring R which is left hereditary and right P-injective, but it is not von Neumann regular. Similarly, we can construct a ring which is right hereditary and left P-injective, but it is not von Neumann regular. So, by Proposition 4.6, we see that right hereditary rings need not be right strongly PP, and hence right n-semihereditary rings need not be right strongly n-semihereditary in general.

Recall that a ring R is said to be a *right IF ring* [12] if every injective right R-module is flat. A ring R is said to be a *right FGTF ring* [10] if every finitely generated torsionless right R-module embeds in a projective, or equivalently, in a free right R-module. We call a ring R a *right n-GTF ring* if every n-generated torsionless right R-module embeds in a free right R-module. It is easy to see that a ring R is a right n-GTF ring if and only if every n-generated torsionless right R-module embeds in a projective right R-module.

4.8. Theorem. *The following are equivalent for a ring R :*
(1) R is a right strongly n-semihereditary ring.
(2) R is right n-semihereditary and right n-GTF.
Furthermore, if R is a right IF ring, then the above conditions are equivalent to:
(3) R is right n-semihereditary and right n-Π-coherent.

Proof. (1) $\Leftrightarrow$ (2); and (1) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Let M be an n-generated torsionless right R-module. Since R is right n-Π-coherent, M is finitely presented. Note that R is a right IF ring, by [8, Theorem 1], M is a submodule of a free module. Since R is right n-semihereditary, every n-generated submodule of a free right R-module is projective, so M is projective. And (1) follows. □

4.9. Corollary. *The following are equivalent for a ring R :*
(1) R is a right strongly PP ring.
(2) R is right PP and right 1-GTF.
Furthermore, if R is a right IF ring, then the above conditions are equivalent to:
(3) R is right PP and right 1-Π-coherent.

4.10. Corollary. *The following are equivalent for a ring R :*
(1) R is a right strongly semihereditary ring.
(2) R is right semihereditary and right FGTF.
Furthermore, if R is a right IF ring, then the above conditions are equivalent to:
(3) R is right semihereditary and right Π-coherent.

4.11. Theorem. *The following are equivalent for a left n-Π-coherent ring R:*
(1) R is a right strongly n-semihereditary ring.
(2) R is a right n-semihereditary ring.
(3) Any submodule of an n-projective right R-module is n-projective.
(4) Every right R-module has an epic n-projective preenvelope.
(5) Every torsionless right R-module is n-projective.
(6) Every n-generated right R-module has an epic projective preenvelope.
(7) Every n-generated right R-module has an epic n-projective preenvelope.

Proof. (1) $\Rightarrow$ (2) is obvious.

(5) $\Rightarrow$ (1) . By Corollary 2.2.

(2) $\Rightarrow$ (3). By Theorem 4.1.

(3) $\Rightarrow$ (4). Let M be any right R-module. Since R is left n-Π-coherent, by Theorem 2.16, M has an n-projective preenvelope $f : M \to P$. Note that $im(f)$ is n-projective by (3), so $M \to im(f)$ is an epic n-projective preenvelope. Note that for any class of right R-modules $\mathcal{F}$, each epic $\mathcal{F}$-preenvelope is an envelope, we have (4).

(4) $\Rightarrow$ (5). Let M be a torsionless right R-module. Then there is a monomorphism $i : M \to R_R^I$ for some index set I. Since R is left n-Π-coherent, by Theorem 2.16, R_R^I is n-projective. Let $f : M \to P$ be an epic n-projective envelope. Then there exists a homomorphism $g : P \to R_R^I$ such that $i = gf$. Thus f is an isomorphism, and so M is n-projective.

(3) $\Rightarrow$ (6). Let M be any n-generated right R-module. Since R is left n-Π-coherent, by Theorem 2.16, M has an n-projective preenvelope $f : M \to P$. Note that $im(f)$ is n-projective by (3), and n-generated n-projective module is projective, so $M \to im(f)$ is an epic projective preenvelope, and therefore it is an epic projective envelope.

(6) $\Rightarrow$ (7). By Proposition 2.7.

(7) $\Rightarrow$ (1). Let M be an n-generated torsionless right R-module. Then there is a monomorphism $i : M \to R_R^I$ for some index set I. Since R is left n-Π-coherent, by Theorem 2.16, R_R^I is n-projective.

By (7), there is an epic n-projective envelope $f : M \to P$. So there exists a homomorphism $g : P \to R_R^I$ such that $i = gf$. Thus f is an isomorphism, and hence M is n-projective. Noting that an n-generated n-projective module is projective, we have that M is projective. □

4.12. Corollary. *The following are equivalent for a left* Π*-coherent ring R:*

(1) R is a right strongly semihereditary ring.
(2) R is a right semihereditary ring.
(3) Any submodule of a finitely projective right R-module is finitely projective.
(4) Every right R-module has an epic finitely projective envelope.
(5) Every torsionless right R-module is finitely projective.
(6) Every finitely generated right R-module has an epic projective envelope.
(7) Every finitely generated right R-module has an epic finitely projective envelope.

Proof. By a similar way to the proof of Theorem 4.11. □

4.13. Corollary. *The following are equivalent for a left 1-*Π*-coherent ring R:*

(1) R is a right strongly PP ring.
(2) R is a right PP ring.
(3) Any submodule of a singly projective right R-module is singly projective.
(4) Every right R-module has an epic singly projective envelope.
(5) Every torsionless right R-module is singly projective.
(6) Every cyclic right R-module has an epic projective envelope.
(7) Every cyclic right R-module has an epic singly projective envelope.

4.14. Theorem. *The following are equivalent for a commutative ring R:*

(1) R is a strongly n-semihereditary ring.
(2) R is n-Π*-coherent and n-semihereditary.*
(3) R is n-Π*-coherent and every submodule of an n-projective R-module is n-projective.*
(4) R is n-Π*-coherent and every ideal is n-projective.*
(5) R is n-Π*-coherent and every finitely generated ideal is n-projective.*
(6) Every R-module has an epic n-projective envelope.
(7) Every n-generated R-module has an epic finitely generated projective envelope.
(8) Every n-generated R-module has an epic projective envelope.
(9) Every n-generated R-module has an epic n-projective envelope.
(10) Every torsionless R-module is n-projective.

Proof. (1) ⇒ (2); (3) ⇒ (4) ⇒ (5); and (7) ⇒ (8) are obvious.

(2) ⇒ (1). By Theorem 4.11.

(2) ⇒ (3). By Theorem 4.1.

(5) ⇒ (2). By Corollary 2.2.

(3) ⇒ (6). By Theorem 4.11.

(6)⇒(7). Let M be an n-generated right R-module. Then by (6), M has an epic n-projective envelope $f : M \to P$. By Theorem 2.1, f factors through a finitely generated free right R-module F, that is, there exist $g : M \to F$ and $h : F \to P$ such that $f = hg$. Since F is n-projective, there exists $\varphi : P \to F$ such that $g = \varphi f$. So $f = (h\varphi)f$, and hence $h\varphi = 1_P$ since f is epic. Hence, P is isomorphic to a direct summand of F, and thus P is finitely generated projective.

(8) ⇒ (9). By Proposition 2.7.

(9)⇒(1). Assume (9). Then it is clear that R is n-Π-coherent by Theorem 2.16 (3). So, by Theorem 4.11(7), R is a strongly n-semihereditary ring.

(2), (6) ⇒ (10). By Theorem 4.11.

(10) ⇒ (1). By Corollary 2.2. □

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