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Bayesian inference for the Pareto lifetime model in the presence of outliers under progressive censoring with binomial removals

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Abstract

Here we have used Type II progressive censoring with random removal for the Pareto lifetime model in the presence of outliers. The number of units removed at each failure time follows a Binomial distribution. The analysis is based on Bayesian approach. In the last, we have given examples with real data.

Keywords: Pareto distribution; Bayesian estimation; Prior; Progressive censoring; Type II censoring; Linex loss function; Outliers.

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1. Introduction

Amin [2] developed Bayesian procedures in the context of parameter estimation and prediction of future observations from the classical Pareto distribution. Bayes estimators as well as Bayesian credible regions are derived for the parameters of the density function, as well as the survival probability and hazard rate. Also she has illustrated derivation of the predictive distribution of individual future observations. Inferences are based on the progressive Type II censored data with random removals where the number of units removed at each failure time follow a Binomial distribution. Analysis is carried out using the natural conjugate prior. For more details see Arnold and Press [3] and [4], Dunsmore and Amin [15] and [16] and Nigm and Hamdy [21].
Pareto distribution has found widespread use as a model for various socioeconomic phenomena. The Pareto has also been used in reliability and lifetime modeling (see for example Berger and Mandelbrot [6], Davis and Feldstein [8], Freiling [17] and Harris

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[18]).
We assume that the random variables $(X_1, X_2, ..., X_n)$ are such that any m of them are distributed with probability density function

$$(1.1)\qquad f_2(x;\alpha,\beta,\theta) = \frac{\alpha(\beta\theta)^\alpha}{x^{\alpha+1}}, \qquad 0 < \beta\theta \le x,\ \ \alpha > 0,\ \beta > 1,\ \theta > 0,$$

and remaining $(n-m)$ random variables are distributed as

$$(1.2)\qquad f_1(x;\alpha,\theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad 0 < \theta \le x,\ \ \alpha > 0.$$

In this paper, we have derived the Bayesian estimators of parameters of the Pareto distribution in the presence of outliers under progressive Type II censoring with random removals where the number of units removed at each failure time follow a Binomial distribution. At the end, we have given the examples of real data.

2. Model

The joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of m outliers is given by

$$f(x_1, x_2, .., x_n; \alpha, \beta, \theta)$$

$$(2.1)\qquad = \frac{\alpha^n\theta^{n\alpha}\beta^{m\alpha}}{C(n,m)}(\prod_{i=1}^{n} x_i)^{-(\alpha+1)} \sum_{A_1=1}^{n-m+1}\ \sum_{A_2=A_1+1}^{n-m+2} \cdots \sum_{A_m=A_{m-1}+1}^{n}\ \prod_{j=1}^{m} \mathbf{I}(x_{A_j} - \beta\theta),$$

where $C(n,m) = \frac{n!}{m!(n-m)!}$ and $\mathbf{I}$ is the indicator function defined as

$$\mathbf{I}(y) = \begin{cases} 1 & y > 0, \\ 0 & otherwise. \end{cases}$$

Note that from (1) and (2), marginal distribution of X_i is

$$(2.2)\qquad f(x_i;\alpha,\beta,\theta) = b\frac{\alpha(\beta\theta)^\alpha}{x_i^{\alpha+1}}\mathbf{I}(x_i - \beta\theta) + \bar{b}\frac{\alpha\theta^\alpha}{x_i^{\alpha+1}}\mathbf{I}(x_i - \theta), \quad \alpha > 0,\ \beta > 1,\ \theta > 0,$$

where $b = \frac{m}{n}$, $\bar{b} = 1 - b$ and $(X_1, X_2, ..., X_n)$ are not independent (For more details see Dixit [9], Dixit et al. [10], Dixit et al. [13], Dixit and Nasiri [14] and Dixit and Jabbari Nooghabi [11, 12]).
Also, the survival functions respect to (1) and (2) are

$$(2.3)\qquad S_2(x;\alpha,\beta,\theta) = \left(\frac{\beta\theta}{x}\right)^\alpha \mathbf{I}(x - \beta\theta), \quad \alpha > 0,\ \beta > 1,\ \theta > 0,$$

and

$$(2.4)\qquad S_1(x;\alpha,\theta) = \left(\frac{\theta}{x}\right)^\alpha \mathbf{I}(x - \theta), \quad \alpha > 0,\ \theta > 0.$$

A natural joint conjugate prior for (α,θ) was first suggested by Lwin [20] and later generalized by Arnold and Press [3]. The prior, called the Power Gamma prior (or modified Lwin prior), denoted by $\text{PG}(\nu, \lambda, \mu, \epsilon)$ is described as follows.

$$g(\alpha,\theta) = \frac{\lambda}{\Gamma(\nu)}(\ln(\mu) - \lambda\ln(\epsilon))^\nu \theta^{\lambda\alpha-1}\alpha^\nu\mu^{-\alpha}, \quad \alpha > 0,\ 0 < \theta < \epsilon,$$

$$(2.5)\qquad \nu, \lambda, \mu, \epsilon > 0,\ 0 < \epsilon^\lambda < \mu.$$

Then

$$(2.6)\qquad g(\alpha) = \frac{(\ln(\mu) - \lambda\ln(\epsilon))^\nu}{\Gamma(\nu)}\alpha^{\nu-1}e^{-\alpha(\ln(\mu)-\lambda\ln(\epsilon))}, \quad \alpha > 0,$$

and

$$g(\theta|\alpha) = \lambda\alpha\theta^{\lambda\alpha-1}\epsilon^{-\lambda\alpha}, \quad 0 < \theta < \epsilon. \tag{2.7}$$

Also we assume the following prior density function for parameter β.

$$g(\beta) = \frac{1}{\beta\ln(d)}, \quad 1 < \beta < d,\ d > 1. \tag{2.8}$$

Therefore

$$g(\alpha,\beta,\theta) = \frac{\lambda}{\Gamma(\nu)\ln(d)}(\ln(\mu) - \lambda\ln(\epsilon))^{\nu}\theta^{\lambda\alpha-1}\alpha^{\nu}\mu^{-\alpha}\beta^{-1}, \quad \alpha > 0,\ 0 < \theta < \epsilon,\ 1 < \beta < d,$$
$$\nu, \lambda, \mu, \epsilon > 0,\ 0 < \epsilon^{\lambda} < \mu,\ d > 1. \tag{2.9}$$

Under progressive Type II censoring, a group of n individuals are observed from time 0 and the test is terminated at the time of the rth failure. When the ith item fails (i= 1, 2,..., $r-1$), k_i of the surviving items are removed from the experiment (k_1=0, 1,..., $n-r$ and k_i=0, 1,..., $n-r-\sum_{j=1}^{i-1}k_j$). When the rth failure is observed, the remaining $k_r = n-r-\sum_{j=1}^{r-1}k_j$ surviving units are all removed. Here, we assume that when the ith item fails (i= 1, 2,..., $r-1$), t_i and u_i of the surviving items are removed from the 'no-outliers' and outliers observations, respectively. Also, when the rth failure is observed, the remaining $t_r = n-m-(r-s)-\sum_{j=1}^{r-s-1}t_j$ and $u_r = m-s-\sum_{j=1}^{s-1}u_j$ surviving units are all removed from the 'no-outliers' and outliers observations, respectively. So $k_i = u_i + t_i$ for i=1, 2,..., r. For progressive Type II censoring with predetermined k_i's, the extension version of the likelihood in the presence of outliers can be defined as

$$L(\boldsymbol{x}|\boldsymbol{K}{=}\boldsymbol{k})$$
$$= \frac{C_1}{C(r,s)}\prod_{i=1}^{r} f_1(x_{(i)})[S_1(x_{(i)})]^{t_i}\sum_{A_1=1}^{r-s+1}\ldots\sum_{A_s=A_{s-1}+1}^{r}\prod_{j=1}^{s}\frac{f_2(x_{(A_j)})[S_2(x_{(A_j)})]^{u_{A_j}}}{f_1(x_{(A_j)})[S_1(x_{(A_j)})]^{t_{A_j}}}, \tag{2.10}$$

where the realized values are denoted by $\boldsymbol{X} = (X_{(1)}, X_{(2)}, ..., X_{(r)})$, $\boldsymbol{K} = (\boldsymbol{T}, \boldsymbol{U}) = ((T_1,U_1),(T_2,U_2),...,(T_{r-1},U_{r-1}))$, s is the number of outliers observation out of r, $C(r,s) = \frac{r!}{s!(s-r)!}$ and the constant C_1 is

$$C_1 = n(n-k_1-1)(n-k_1-k_2-2)\ldots\left(n-\sum_{i=1}^{r-1}k_i - r + 1\right).$$

One should note that if we put $m = 0$ and $s = 0$, then the likelihood is reduced to homogeneous case as in Amin [2] and Cohen [7].

Expression (12) is derived from conditioning on k_i , however, in some practical situations these numbers of k_i may occur at random as a result of the unexpected dropout of experimental units. Under random removals, at the failure of an item, each of the remaining live items will either be dropped out of the test or will continue. Each unit acting independently of the others with a probability for each to be dropped out equal to p. Thus, following Tse et al. [22], we assume that K_i ($i = 1, 2, ..., r-1$), the number of items dropped out at time $X_{(i)}$, assumes the following distributions:

The random variable T_1 follows the binomial distribution with parameters $n-m-(r-s)$ and p (denoted as $Bin(n-m-(r-s),p)$), whereas the variables $T_i|t_1,t_2,...,t_{i-1}$ follow the $Bin(n-m-(r-s)-\sum_{j=1}^{i-1}t_j, p)$ distributions for $i = 2, 3, ..., r-1$, respectively.

Also, The random variable U_1 follows the $Bin(m-s,p)$, whereas the variables $U_i|u_1,u_2,...,u_{i-1}$ follow the $Bin(m-s-\sum_{j=1}^{i-1}u_j, p)$ distributions for $i = 2, 3, ..., r-1$, respectively.

Furthermore, we assume that K_i is independent of X_i. The likelihood function of X and $K = (T, U)$ can be found as

$$L(\boldsymbol{x}, (\boldsymbol{t}, \boldsymbol{u})) = L(\boldsymbol{x}|\boldsymbol{K}=\boldsymbol{k})A_0, \tag{2.11}$$

where

$$\begin{aligned} A_0 &= P(T_1 = t_1)\prod_{i=2}^{r-1} P(T_i = t_i|T_1 = t_1, T_2 = t_2, ..., T_{i-1} = t_{i-1}) \\ &\times P(U_1 = u_1)\prod_{i=2}^{r-1} P(U_i = u_i|U_1 = u_1, U_2 = u_2, ..., U_{i-1} = u_{i-1}). \end{aligned}$$

Therefore after substituting the values in (12) and (13) and using some algebra, we get

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{k}|\alpha, \beta, \theta) &= A_0\frac{C_1}{C(r,s)}\alpha^r\beta^{\alpha s}\theta^{\alpha(r+\sum_{i=1}^r t_i)}\prod_{i=1}^r x_{(i)}^{-\alpha(t_i+1)-1} \\ &\times \sum_{A_1,\ldots,A_s}^{*} \beta^{\alpha\sum_{j=1}^s u_{A_j}}\theta^{\alpha\sum_{j=1}^s (u_{A_j}-t_{A_j})}\prod_{j=1}^s x_{(A_j)}^{-\alpha(u_{A_j}-t_{A_j})}, \end{aligned} \tag{2.12}$$

where

$$\sum_{A_1,\ldots,A_s}^{*} = \sum_{A_1=1}^{r-s+1}\cdots\sum_{A_s=A_{s-1}+1}^{r}.$$

3. Posterior distributions

In the previous section, we found the likelihood under progressive type II censoring with binomial removals as in (14). Now, we obtain the posterior density of (α, β, θ).

3.1. Theorem.

Posterior densities of α, β and θ are

$$\begin{aligned} &h(\alpha, \beta, \theta|\boldsymbol{x}, \boldsymbol{k}) \\ &= \frac{\alpha^{r+\nu}\mu^{-\alpha}}{B_0\Gamma(r+\nu)}\left[\prod_{i=1}^r x_{(i)}^{-\alpha(t_i+1)}\right]\sum_{A_1,\ldots,A_s}^{*}\theta^{\alpha B_3-1}\beta^{\alpha B_6-1}\prod_{j=1}^s x_{(A_j)}^{-\alpha(u_{A_j}-t_{A_j})}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &h(\alpha|\boldsymbol{x}, \boldsymbol{k}) \\ &= \frac{\alpha^{r+\nu-2}\mu^{-\alpha}}{B_0\Gamma(r+\nu)}\left[\prod_{i=1}^r x_{(i)}^{-\alpha(t_i+1)}\right]\sum_{A_1,\ldots,A_s}^{*}\frac{\left[\prod_{j=1}^s x_{(A_j)}^{-\alpha(u_{A_j}-t_{A_j})}\right]\omega^{\alpha B_3}\left(d^{\alpha B_6}-1\right)}{B_3B_6}, \\ &\qquad \alpha > 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &h(\beta|\boldsymbol{x}, \boldsymbol{k}) \\ &= \frac{1}{B_0\beta}\sum_{A_1,\ldots,A_s}^{*}\frac{\left[-B_3\ln(\omega)+\ln\left(\mu\beta^{-B_6}\ [\prod_{i=1}^r x_{(i)}^{t_i+1}][\prod_{j=1}^s x_{(A_j)}^{u_{A_j}-t_{A_j}}]\right)\right]^{-r-\nu}}{B_3}, \\ &\qquad 1 < \beta < d, \end{aligned} \tag{3.3}$$

and

$$h(\theta|\boldsymbol{x},\boldsymbol{k})$$

$$(3.4)\quad = \frac{1}{B_0\theta}\sum_{A_1,\dots,A_s}^{*}\frac{1}{B_6}\left\{[-B_3\ln(\theta)+B_1]^{-r-\nu}-[-B_3\ln(\theta)+B_2]^{-r-\nu}\right\},\ \ 0<\theta<\omega,$$

where

$$(3.5)\quad B_0=\sum_{A_1,\dots,A_s}^{*}\frac{1}{B_6B_3(r+\nu-1)}\left\{[-B_3\ln(\omega)+B_1]^{-r-\nu+1}-[-B_3\ln(\omega)+B_2]^{-r-\nu+1}\right\},$$

$$(3.6)\quad B_1=\ln\left(\mu d^{-B_6}\left[\prod_{i=1}^{r}x_{(i)}^{t_i+1}\right]\left[\prod_{j=1}^{s}x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right),$$

$$(3.7)\quad B_2=\ln\left(\mu\left[\prod_{i=1}^{r}x_{(i)}^{t_i+1}\right]\left[\prod_{j=1}^{s}x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right),$$

$$(3.8)\quad B_3=r+\lambda+\sum_{i=1}^{r}t_i+\sum_{j=1}^{s}(u_{A_j}-t_{A_j}),$$

$$(3.9)\quad B_6=s+\sum_{j=1}^{s}u_{A_j},$$

and $\omega=min(x_{(1)},\epsilon)$.

Proof. Applying the joint prior density of the parameters (α,β,θ) in (11) and using (14), we have

$$g(\boldsymbol{x},\boldsymbol{k})=\int_0^{\omega}\int_0^{\infty}\int_1^{d}L(\boldsymbol{x},\boldsymbol{k}|\alpha,\beta,\theta)g(\alpha,\beta,\theta)d\beta d\alpha d\theta,$$

So

$$\begin{aligned}
g(\boldsymbol{x},\boldsymbol{k},\alpha,\theta) &= \int_1^{d}L(\boldsymbol{x},\boldsymbol{k}|\alpha,\beta,\theta)g(\alpha,\beta,\theta)d\beta\\
&= \frac{A_0C_1\lambda\theta^{-1}}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^{\nu}\left(\prod_{i=1}^{r}x_{(i)}^{-1}\right)\\
&\times \sum_{A_1,\dots,A_s}^{*}\alpha^{r+\nu}\left[\mu\theta^{-B_3}\left(\prod_{i=1}^{r}x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^{s}x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha}\int_1^{d}\beta^{\ \alpha B_6-1}d\beta\\
&= \frac{A_0C_1\lambda\theta^{-1}}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^{\nu}\left(\prod_{i=1}^{r}x_{(i)}^{-1}\right)\\
&\times \sum_{A_1,\dots,A_s}^{*}\frac{\alpha^{r+\nu-1}}{B_6}\left[\mu\theta^{-B_3}\left(\prod_{i=1}^{r}x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^{s}x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha}[d^{\ \alpha B_6}-1].
\end{aligned}$$

Then integrating respect to α, we get

$$\begin{aligned}
g(\boldsymbol{x},\boldsymbol{k},\theta) &= \int_0^\infty g(\boldsymbol{x},\boldsymbol{k},\alpha,\theta)d\alpha \\
&= \frac{A_0C_1\lambda\theta^{-1}}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^\nu\left(\prod_{i=1}^r x_{(i)}^{-1}\right) \\
&\times \sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\{\int_0^\infty \alpha^{r+\nu-1}\left[\mu\theta^{-B_3}d^{-B_6}\left(\prod_{i=1}^r x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^s x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha}d\alpha \\
&- \int_0^\infty \alpha^{r+\nu-1}\left[\mu\theta^{-B_3}\left(\prod_{i=1}^r x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^s x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha}d\alpha\} \\
&= \frac{A_0C_1\lambda\Gamma(r+\nu)}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^\nu\left(\prod_{i=1}^r x_{(i)}^{-1}\right) \\
&\times \sum_{A_1,\ldots,A_s}^{*}\frac{\theta^{-1}}{B_6}\left\{[-B_3\ln(\theta)+B_1]^{-r-\nu}-[-B_3\ln(\theta)+B_2]^{-r-\nu}\right\}.
\end{aligned}$$

Now integrating respect to θ imply that

$$\begin{aligned}
g(\boldsymbol{x},\boldsymbol{k}) &= \int_0^\omega g(\boldsymbol{x},\boldsymbol{k},\theta)d\theta \\
&= \frac{A_0C_1\lambda\Gamma(r+\nu)}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^\nu\left(\prod_{i=1}^r x_{(i)}^{-1}\right) \\
&\times \sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\left\{\int_0^\omega \theta^{-1}\left[-B_3\ln(\theta)+B_1\right]^{-r-\nu}d\theta-\int_0^\omega \theta^{-1}\left[-B_3\ln(\theta)+B_2\right]^{-r-\nu}d\theta\right\} \\
&= \frac{A_0C_1\lambda\Gamma(r+\nu)}{C(r,s)\Gamma(\nu)\ln(d)}(\ln(\mu)-\lambda\ln(\epsilon))^\nu\left(\prod_{i=1}^r x_{(i)}^{-1}\right) \\
&\times \sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3B_6(r+\nu-1)}\left\{[-B_3\ln(\omega)+B_1]^{-r-\nu+1}-[-B_3\ln(\omega)+B_2]^{-r-\nu+1}\right\}.
\end{aligned}$$

We know that

$$h(\alpha,\beta,\theta|\boldsymbol{x},\boldsymbol{k})=\frac{L(\boldsymbol{x},\boldsymbol{k}|\alpha,\beta,\theta)g(\alpha,\beta,\theta)}{g(\boldsymbol{x},\boldsymbol{k})}.$$

Thus using some elementary algebra, joint posterior density of (α,β,θ) is obtained. Also we have

$$\begin{aligned}
h(\alpha,\theta|\boldsymbol{x},\boldsymbol{k}) &= \int_1^d h(\alpha,\beta,\theta|\boldsymbol{x},\boldsymbol{k})d\beta \\
&= \frac{\alpha^{r+\nu}\mu^{-\alpha}}{B_0\Gamma(r+\nu)}\left[\prod_{i=1}^r x_{(i)}^{-\alpha(t_i+1)}\right]\sum_{A_1,\ldots,A_s}^{*}\theta^{\alpha B_3-1}\left[\prod_{j=1}^s x_{(A_j)}^{-\alpha(u_{A_j}-t_{A_j})}\right]\int_1^d \beta^{\alpha B_6-1}d\beta \\
&= \frac{1}{B_0\Gamma(r+\nu)\theta}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\{\alpha^{r+\nu-1}\left[\mu d^{-B_6}\theta^{-B_3}\left(\prod_{i=1}^r x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^s x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha} \\
&- \alpha^{r+\nu-1}\left[\mu\theta^{-B_3}\left(\prod_{i=1}^r x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^s x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]^{-\alpha}\}.
\end{aligned}$$

Then

$$\begin{aligned} h(\alpha|\boldsymbol{x},\boldsymbol{k}) &= \int_0^\omega h(\alpha,\theta|\boldsymbol{x},\boldsymbol{k})d\theta \\ &= \frac{\alpha^{r+\nu-1}\mu^{-\alpha}}{B_0\Gamma(r+\nu)}\left[\prod_{i=1}^{r} x_{(i)}^{-\alpha(t_i+1)}\right]\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\left[\prod_{j=1}^{s} x_{(A_j)}^{-\alpha(u_{A_j}-t_{A_j})}\right]\left(d^{\alpha B_6}-1\right) \\ &\times \int_0^\omega \theta^{\alpha B_3-1}d\theta. \end{aligned}$$

So we get the marginal posterior density of α as in (16).
Further

$$\begin{aligned} h(\theta|\boldsymbol{x},\boldsymbol{k}) &= \int_0^\infty h(\alpha,\theta|\boldsymbol{x},\boldsymbol{k})d\alpha \\ &= \frac{1}{B_0\Gamma(r+\nu)\theta}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\{\int_0^\infty \alpha^{r+\nu-1}e^{-\alpha(-B_3\ln(\theta)+B_1)}d\alpha \\ &- \int_0^\infty \alpha^{r+\nu-1}e^{-\alpha(-B_3\ln(\theta)+B_2)}d\alpha\}. \end{aligned}$$

So by evaluating the above integrals, we get the marginal posterior density of θ as in (18).
Finally for posterior density of β, we have

$$\begin{aligned} h(\beta,\theta|\boldsymbol{x},\boldsymbol{k}) &= \int_0^\infty h(\alpha,\beta,\theta|\boldsymbol{x},\boldsymbol{k})d\alpha \\ &= \frac{1}{B_0\beta\theta\Gamma(r+\nu)}\sum_{A_1,\ldots,A_s}^{*} \\ &\times \int_0^\infty \alpha^{r+\nu}\exp\left(-\alpha\ln\left[\mu\theta^{-B_3}\beta^{-B_6}\left(\prod_{i=1}^{r} x_{(i)}^{t_i+1}\right)\left(\prod_{j=1}^{s} x_{(A_j)}^{u_{A_j}-t_{A_j}}\right)\right]\right)d\alpha \\ &= \frac{r+\nu}{B_0\beta\theta}\sum_{A_1,\ldots,A_s}^{*}\left[-B_3\ln(\theta)+\ln\left(\mu\beta^{-B_6}\left[\prod_{i=1}^{r} x_{(i)}^{t_i+1}\right]\left[\prod_{j=1}^{s} x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right)\right]^{-r-\nu-1}. \end{aligned}$$

So

$$\begin{aligned} h(\beta|\boldsymbol{x},\boldsymbol{k}) &= \int_0^\omega h(\beta,\theta|\boldsymbol{x},\boldsymbol{k})d\theta = \frac{r+\nu}{B_0\beta}\sum_{A_1,\ldots,A_s}^{*} \\ &\times \int_0^\omega \theta^{-1}\left[-B_3\ln(\theta)+\ln\left(\mu\beta^{-B_6}\left[\prod_{i=1}^{r} x_{(i)}^{t_i+1}\right]\left[\prod_{j=1}^{s} x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right)\right]^{-r-\nu-1}d\theta. \end{aligned}$$

Therefore, we can obtain the posterior density of β as in (17).

4. Bayes estimators

If our loss function is squared error loss, then the posterior means of α, β and θ using (16), (17) and (18) represent the appropriate Bayes estimators. Results will be derived under Progressive Type II censoring with Binomial removals. Bayes estimators and credible region for homogenous case of Pareto distribution (ie. $m = 0$ and $\beta = 1$) under progressive Type II censoring and squared error loss and absolute error loss are given in Amin [2]. Some of this material for homogenous case of Pareto distribution was derived earlier for the particular case of Type II censoring, that is when $k_i = 0$ for $i = 1, 2, ..., r-1$ and $k_r = n - r$ in Arnold and Press [3], [4], Dunsmore and Amin [15], [16] and Nigm and Hamdy [21].

4.1. Under squared error loss function. The squared loss function for parameter α and decision rule δ is defined by

$$L(\alpha,\delta) = (\delta-\alpha)^2. \tag{4.1}$$

4.1. Theorem.

A) Bayes estimator of α is

$$\hat{\alpha} = \frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{{B_5}^{-r-\nu} - {B_4}^{-r-\nu}}{B_3 B_6}, \tag{4.2}$$

B) Bayes estimator of β is

$$\hat{\beta} = \frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{{B_6}^{-r-\nu} e^{\frac{B_4}{B_6}}}{B_3} \left[\Gamma\left(-r-\nu+1, -\ln(d) + \frac{B_4}{B_6}\right) - \Gamma\left(-r-\nu+1, \frac{B_4}{B_6}\right)\right], \tag{4.3}$$

C) Bayes estimator of θ is

$$\hat{\theta} = \frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{{B_3}^{-r-\nu}}{B_6} \tag{4.4}$$

$$\left[e^{\frac{B_1}{B_3}} \Gamma\left(-r-\nu+1, -\ln(\omega) + \frac{B_1}{B_3}\right) - e^{\frac{B_2}{B_3}} \Gamma\left(-r-\nu+1, -\ln(\omega) + \frac{B_2}{B_3}\right)\right],$$

where B_0, B_1, B_2, B_3 and B_6 are defined as in (19), (20), (21), (22) and (23), respectively,

$$B_4 = \ln\left(\mu\omega^{-B_3} \left[\prod_{i=1}^{r} x_{(i)}^{t_i+1}\right] \left[\prod_{j=1}^{s} x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right), \tag{4.5}$$

$$B_5 = \ln\left(\mu\omega^{-B_3} d^{-B_6} \left[\prod_{i=1}^{r} x_{(i)}^{t_i+1}\right] \left[\prod_{j=1}^{s} x_{(A_j)}^{u_{A_j}-t_{A_j}}\right]\right), \tag{4.6}$$

and

$$\Gamma(a,y) = \int_y^\infty t^{a-1} e^{-t} dt,$$

is the incomplete Gamma function (for more details see Abramowitz and Stegun [1]).
Proof.
The proof is given in the appendix.

4.2. Under Linex loss function. In this subsection, we will obtain the Bayes estimator of the parameters α, β and θ under Linex loss function. We know that the Linex loss function for parameter α and decision rule δ is

$$L(\alpha,\delta) = e^{c(\delta-\alpha)} - c(\delta-\alpha) - 1, \quad -\infty < c < \infty. \tag{4.7}$$

For $c > 0$, the loss function $L(\alpha,\delta)$ is quite asymmetric about 0 with overestimation being more costly than under-estimation. As $|\delta-\alpha| \to \infty$, the loss $L(\alpha,\delta)$ increases almost exponentially when $\delta-\alpha > 0$ and almost linearly when $\delta-\alpha < 0$. For $c < 0$, the linearity-exponentiality phenomenon is reversed. Also, when $|\delta-\alpha|$ is very small, $L(\alpha,\delta)$ is near $\frac{c(\delta-\alpha)^2}{2}$.

4.2. Theorem.

A) Bayes estimator of α is

$$\tilde{\alpha} = -\frac{1}{c}\ln\left(\frac{1}{B_0(r+\nu-1)}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3B_6}\left[(c+B_5)^{-r-\nu+1}-(c+B_4)^{-r-\nu+1}\right]\right), \tag{4.8}$$

B) Bayes estimator of β is

$$\tilde{\beta} = -\frac{1}{c}\ln\left(\frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{B_9}{B_3}\right), \tag{4.9}$$

C) Bayes estimator of θ is

$$\tilde{\theta} = -\frac{1}{c}\ln\left(\frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{B_7-B_8}{B_6}\right), \tag{4.10}$$

where

$$B_7 = B_3{}^{-r-\nu}\sum_{j=0}^{\infty}\frac{(-c)^j}{j!}j^{r+\nu-1}e^{\frac{jB_1}{B_3}}\,\Gamma\left(-r-\nu+1,-j\ln(\omega)+\frac{jB_1}{B_3}\right), \tag{4.11}$$

$$B_8 = B_3{}^{-r-\nu}\sum_{j=0}^{\infty}\frac{(-c)^j}{j!}j^{r+\nu-1}e^{\frac{jB_2}{B_3}}\,\Gamma\left(-r-\nu+1,-j\ln(\omega)+\frac{jB_2}{B_3}\right), \tag{4.12}$$

$$B_9 = B_6{}^{-r-\nu}\sum_{j=0}^{\infty}\frac{(-c)^j}{j!}j^{r+\nu-1}e^{\frac{jB_4}{B_6}} \left[\Gamma\left(-r-\nu+1,-j\ln(d)+\frac{jB_4}{B_6}\right)-\Gamma\left(-r-\nu+1,\frac{jB_4}{B_6}\right)\right], \tag{4.13}$$

and B_0, B_1, B_2, B_3, B_4, B_5 and B_6 are defined as in (19), (20), (21), (22), (28), (29) and (23), respectively.

Proof.

For proof refer to the appendix.

Note: One should note that Amin [2] had not found the Bayes estimator of α and θ for the homogenous case of the Pareto distribution under progressive censoring with binomial removals and Linex loss function. So we can obtain them as follows.

4.3. Theorem.

A) Bayes estimator of α for homogenous case of the Pareto distribution is

$$\tilde{\alpha}_1 = -\frac{r+\nu}{c}\ln\left(\frac{B_{10}}{B_{10}+c}\right), \tag{4.14}$$

B) Bayes estimator of θ for homogenous case of the Pareto distribution is

$$\tilde{\theta}_1 = -\frac{1}{c}\ln\left(\frac{(n+\lambda)(r+\nu)B_{11}}{B_{10}}\right), \tag{4.15}$$

where

$$B_{10} = \ln(\mu) + \sum_{i=1}^{r}(k_i+1)\ln(x_{(i)}) - (n+\lambda)\ln(\omega),$$

and

$$B_{11} = \sum_{j=0}^{\infty}\left(\frac{n+\lambda}{B_{10}}\right)^j C(r+\nu+j,j)\sum_{i=0}^{\infty}\frac{(-c\,\omega)^i}{i!}\left[\frac{(-1)^j\Gamma(j+1)}{i^{j+1}}\right].$$

Proof.

The proof is given in the appendix.

4.3. Credible regions. In this subsection, we will obtain $100(1-\gamma)\%$ symmetric credible region for the parameters α, β and θ.

4.4. Theorem.

A) By using Newton-Raphson method, the lower (α_L) and upper (α_U) bounds of $100(1-\gamma)\%$ symmetric credible region for α are obtained as follows.

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \{B_5^{-r-\nu+2} - B_4^{-r-\nu+2} + \sum_{l=0}^{r+\nu-2} \frac{1}{l!} \left[\frac{(\alpha_L B_4)^l e^{-\alpha_L B_4}}{B_4^{r+\nu-2}} - \frac{(\alpha_L B_5)^l e^{-\alpha_L B_5}}{B_5^{r+\nu-2}} \right] \} = \frac{\gamma}{2}, \tag{4.16}$$

and

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \sum_{l=0}^{r+\nu-2} \frac{1}{l!} \left[\frac{(\alpha_U B_5)^l e^{-\alpha_U B_5}}{B_5^{r+\nu-2}} - \frac{(\alpha_U B_4)^l e^{-\alpha_U B_4}}{B_4^{r+\nu-2}} \right] = \frac{\gamma}{2}, \tag{4.17}$$

B) By using Newton-Raphson method, the lower (β_L) and upper (β_U) bounds of $100(1-\gamma)\%$ symmetric credible region for β are found as follows.

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \left[[B_4 - B_6 \ln(\beta_L)]^{-r-\nu+1} - B_4^{-r-\nu+1} \right] = \frac{\gamma}{2}, \tag{4.18}$$

and

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \left[[B_4 - B_6 \ln(d)]^{-r-\nu+1} - [B_4 - B_6 \ln(\beta_U)]^{-r-\nu+1} \right] = \frac{\gamma}{2}, \tag{4.19}$$

C) By using Newton-Raphson method, the lower (θ_L) and upper (θ_U) bounds of $100(1-\gamma)\%$ symmetric credible region for θ are obtained as follows.

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \left[[B_1 - B_3 \ln(\theta_L)]^{-r-\nu+1} - [B_2 - B_3 \ln(\theta_L)]^{-r-\nu+1} \right] = \frac{\gamma}{2}, \tag{4.20}$$

and

$$\frac{1}{B_0(r+\nu-1)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3 B_6} \{ [B_1 - B_3 \ln(\omega)]^{-r-\nu+1} - [B_1 - B_3 \ln(\theta_U)]^{-r-\nu+1} + [B_2 - B_3 \ln(\omega)]^{-r-\nu+1} - [B_2 - B_3 \ln(\theta_U)]^{-r-\nu+1} \} = \frac{\gamma}{2}, \tag{4.21}$$

where B_0, B_1, B_2, B_3, B_4, B_5 and B_6 are defined as in (19), (20), (21), (22), (28), (29) and (23), respectively.

Proof.

To proof see the appendix.

5. Numerical study

5.1. Simulation data. Assume that a lifetimes of n parts of electronic instruments followed the Pareto distribution in the presence of outliers as in (1) and (2). They are put on the test simultaneously. We observed that the first rth items are failed and the times of failure (in hours) including the number of surviving items removed from the process at the failure of each item named as t_i (for 'no-outlier' data) and u_i (for outliers data). At first, we have simulated the values of α, β and θ from the (8), (9) and (10), respectively, by using the following fixed values as: $\nu = 7$, $\mu = 100$, $\epsilon = 50$, $\lambda = 0.2$ and $d = 3$. The simulated parameters from (8), (9) and (10) by using 1000 replications are $\alpha = 1.836382$, $\beta = 1.825401$ and $\theta = 12.781246$. Then the data were generated from the Pareto distribution in the presence of outliers with parameters $\alpha = 1.836382$, $\beta = 1.825401$ and $\theta = 12.781246$ for different values of n, m, s and $r = 6$. Also data for dropouts t_i and u_i were generated from Binomial distribution as: $T_1 \sim Bin(n-m-(r-s), p = 0.05)$, $T_i|t_1, t_2, ..., t_{i-1} \sim Bin(n-m-(r-s)-\sum_{j=1}^{i-1} t_j, 0.05)$, $U_1 \sim Bin(m-s, 0.05)$ and $U_i|u_1, u_2, ..., u_{i-1} \sim Bin(m-s-\sum_{j=1}^{i-1} u_j, 0.05)$ for $i = 2, 3, ..., r-1$. Therefore, repeating 1000 times, the Bayesian estimators and determinant of the covariance matrix of estimate of the parameters are derived and shown in Table 1. The determinant is calculated from the following formula.

$$Generalized\ \ variance = \begin{vmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta}) & Cov(\hat{\alpha},\hat{\theta}) \\ Cov(\hat{\alpha},\hat{\beta}) & Var(\hat{\beta}) & Cov(\hat{\beta},\hat{\theta}) \\ Cov(\hat{\alpha},\hat{\theta}) & Cov(\hat{\beta},\hat{\theta}) & Var(\hat{\theta}) \end{vmatrix}.$$

Further, a 95% symmetric two-sided Bayes probability interval of the parameters α, β and θ are shown in Table 2.

Table 1. Bayes estimators and the determinant

(n,m,s)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	determinant	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\theta}$	determinant
(10,1,1)	2.798	2.270	1.417e+01	7.503e-07	5.692	2.394	1.437e+01	2.814e-06
(20,1,1)	3.278	2.328	1.337e+01	4.833e-08	6.319	2.442	1.346e+01	2.091e-07
(30,1,1)	3.433	2.346	1.313e+01	8.051e-09	6.529	2.456	1.318e+01	3.664e-08
(40,1,1)	3.482	2.351	1.301e+01	1.678e-09	6.594	2.460	1.305e+01	7.937e-09
(10,2,1)	3.301	2.507	1.553e+01	2.125e-03	6.433	2.586	1.571e+01	3.431e-03
(20,2,1)	4.236	2.604	1.417e+01	7.388e-04	7.774	2.660	1.423e+01	1.210e-03
(30,2,1)	4.503	2.621	1.366e+01	2.442e-04	8.198	2.673	1.370e+01	4.254e-04
(40,2,1)	4.673	2.629	1.344e+01	8.966e-05	8.474	2.679	1.346e+01	1.643e-04
(10,2,2)	2.948	2.531	1.580e+01	4.230e-04	5.919	2.602	1.608e+01	8.373e-04
(20,2,2)	3.200	2.562	1.435e+01	1.138e-04	6.277	2.626	1.451e+01	2.322e-04
(30,2,2)	3.321	2.575	1.384e+01	4.507e-05	6.455	2.637	1.395e+01	9.192e-05
(40,2,2)	3.421	2.585	1.357e+01	3.150e-05	6.608	2.644	1.364e+01	6.642e-05
(10,3,2)	3.163	2.571	1.703e+01	4.597e-03	6.270	2.634	1.733e+01	1.119e-02
(20,3,2)	3.486	2.599	1.502e+01	2.511e-03	6.801	2.657	1.518e+01	8.164e-03
(30,3,2)	3.621	2.616	1.432e+01	1.182e-03	7.008	2.669	1.442e+01	3.974e-03
(40,3,2)	3.762	2.627	1.393e+01	8.696e-04	7.283	2.679	1.400e+01	4.967e-04
(10,3,3)	4.441	2.752	1.735e+01	5.183e-03	8.485	2.779	1.756e+01	3.468e-02
(20,3,3)	5.017	2.773	1.497e+01	1.508e-03	9.757	2.797	1.509e+01	8.896e-03
(30,3,3)	5.583	2.800	1.434e+01	5.281e-04	11.470	2.818	1.441e+01	2.800e-03
(40,3,3)	6.277	2.821	1.402e+01	4.769e-04	13.491	2.836	1.406e+01	2.495e-03

'hat' is to estimate under square error loss and 'tilde' for under Linex loss.

Again, assuming that the prior parameters for the joint prior density are $\nu = 10$, $\mu = 75$, $\epsilon = 100$, $\lambda = 0.5$ and $d = 3$. The simulated parameters from the joint prior density are $\alpha = 5.018414$, $\beta = 2.491183$ and $\theta = 69.665391$. Same as the previous procedure, we have obtained the Bayesian estimates and the determinant. The results for different values of n, m, s and $r = 6$ are inserted in Table 3. Also, the 95% symmetric two-sided Bayes probability interval for the parameters are shown in Table 4 for different values of n, m and s.

Table 2. 95% symmetric two-sided Bayes probability interval for the parameters

(n,m,s)	α_L	α_U	β_L	β_U	θ_L	θ_U
(10,1,1)	0.979	4.114	1.137	2.956	10.862	14.051
(20,1,1)	0.948	4.181	1.105	2.969	10.995	13.712
(30,1,1)	0.957	4.218	1.106	2.969	11.321	13.642
(40,1,1)	0.939	4.149	1.104	2.968	11.521	14.125
(10,2,1)	1.041	4.335	1.372	2.967	11.230	14.027
(20,2,1)	1.005	4.475	1.313	2.981	11.285	13.714
(30,2,1)	1.009	4.490	1.311	2.981	11.526	13.574
(40,2,1)	1.004	4.472	1.309	2.981	11.721	14.153
(10,2,2)	1.255	5.469	1.497	2.986	13.183	16.493
(20,2,2)	1.346	5.897	1.550	2.987	12.510	14.948
(30,2,2)	1.436	6.275	1.600	2.988	12.850	14.316
(40,2,2)	1.483	6.480	1.627	2.989	14.167	13.979
(10,3,2)	1.245	5.390	1.536	2.987	14.200	17.624
(20,3,2)	1.140	5.130	1.487	2.986	13.040	16.027
(30,3,2)	1.061	4.866	1.444	2.985	12.782	15.362
(40,3,2)	0.974	4.571	1.400	2.984	12.588	15.002
(10,3,3)	1.840	6.608	2.034	2.993	14.903	17.638
(20,3,3)	2.223	6.307	2.103	2.995	13.728	15.990
(30,3,3)	2.299	5.849	2.126	2.991	13.568	15.366
(40,3,3)	2.212	5.437	2.146	2.973	13.638	15.089

Table 3. Bayes estimators and the determinant

(n,m,s)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	determinant	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\theta}$	determinant
(10,1,1)	10.201	2.711	8.590e+01	4.175e-04	5.105	2.637	3.118e+01	4.498e-05
(20,1,1)	11.570	2.745	7.899e+01	1.500e-05	5.816	2.686	3.463e+01	1.247e-05
(30,1,1)	12.073	2.755	7.620e+01	1.194e-06	6.062	2.702	3.700e+01	3.727e-06
(40,1,1)	12.167	2.757	7.465e+01	1.014e-06	6.111	2.704	3.869e+01	2.532e-06
(10,2,1)	11.640	1.896	9.514e+01	1.387e-02	5.732	1.886	3.265e+01	4.157e-02
(20,2,1)	13.767	1.907	8.552e+01	3.656e-03	6.745	1.898	3.677e+01	2.457e-03
(30,2,1)	14.259	1.907	8.133e+01	1.380e-03	6.954	1.898	3.904e+01	1.828e-03
(40,2,1)	14.660	1.910	7.895e+01	7.044e-04	7.125	1.902	4.094e+01	1.139e-03
(10,2,2)	10.051	1.893	9.532e+01	1.195e-03	5.052	1.883	3.019e+01	1.802e-04
(20,2,2)	10.571	1.899	8.601e+01	6.466e-04	5.276	1.890	3.181e+01	5.057e-05
(30,2,2)	11.043	1.904	8.184e+01	2.646e-04	5.485	1.895	3.352e+01	4.666e-05
(40,2,2)	11.311	1.906	7.932e+01	1.039e-04	5.606	1.899	3.527e+01	3.914e-05
(10,3,2)	15.022	1.919	9.124e+01	2.668e-01	6.681	1.912	3.393e+01	4.055e-03
(20,3,2)	14.437	1.916	9.131e+01	2.105e-01	6.497	1.909	3.364e+01	3.832e-03
(30,3,2)	14.661	1.920	8.609e+01	9.803e-02	6.653	1.914	3.530e+01	3.726e-03
(40,3,2)	14.348	1.921	8.298e+01	1.444e-02	6.619	1.915	3.673e+01	1.952e-03
(10,3,3)	12.501	1.744	9.854e+01	3.207e-02	6.041	1.741	3.131e+01	9.068e-04
(20,3,3)	17.324	1.759	9.141e+01	1.140e-02	7.820	1.757	3.466e+01	6.981e-04
(30,3,3)	18.730	1.762	8.604e+01	8.963e-03	8.285	1.761	3.620e+01	4.294e-04
(40,3,3)	19.364	1.764	8.301e+01	1.128e-03	8.528	1.763	3.761e+01	1.845e-04

'hat' is to estimate under square error loss and 'tilde' for under Linex loss.

Further, to investigate how the value of p (the removal probability) can affect on the variability of the model parameter estimate, we have used three points of p as 0.15, 0.50 and 0.80. Then, the simulation study is used to estimate the parameters respect to p. Estimate of the parameters and the determinant are derived for the prior parameters $\alpha = 1.836382$, $\beta = 1.825401$ and $\theta = 12.781246$ and different values of n, m, s and $r = 6$.

The results are shown in Table 5.
From the Tables 1 and 3, it has been seen that the determinant of covariance matrix of the Bayesian estimators of the parameters α, β and θ are decreased as n increased.
Also, according to Table 5, when n is fixed, in some of the cases the generalized variance is increasing when removal probability p, increases; but when n increases the generalized variance is always decreasing.

Table 4. 95% symmetric two-sided Bayes probability interval for the parameters

(n,m,s)	α_L	α_U	β_L	β_U	θ_L	θ_U
(10,1,1)	4.282	10.899	3.353	4.225	78.671	88.076
(20,1,1)	3.606	9.952	2.599	3.625	75.857	85.175
(30,1,1)	3.068	8.798	2.201	3.253	73.985	86.655
(40,1,1)	2.736	7.986	2.004	3.104	72.978	87.221
(10,2,1)	5.743	7.767	3.914	4.067	93.489	97.060
(20,2,1)	5.597	10.403	4.337	4.696	78.463	90.247
(30,2,1)	4.596	9.816	4.536	4.971	78.052	88.568
(40,2,1)	4.016	9.125	4.590	5.099	76.757	88.261
(10,2,2)	6.861	9.416	3.539	3.685	93.867	97.142
(20,2,2)	6.523	8.772	3.817	3.929	90.899	92.360
(30,2,2)	6.255	9.025	3.857	3.979	90.500	91.402
(40,2,2)	5.859	8.714	3.774	3.936	89.955	90.935
(10,3,2)	6.518	8.911	3.487	3.620	95.849	99.231
(20,3,2)	6.598	8.708	3.733	3.800	92.209	93.657
(30,3,2)	6.255	8.235	3.801	3.878	91.745	92.594
(40,3,2)	5.798	8.738	3.790	3.884	91.105	91.837
(10,3,3)	6.044	7.882	3.191	3.193	94.936	95.357
(20,3,3)	5.532	8.309	3.179	3.174	89.987	89.974
(30,3,3)	5.247	6.997	3.365	3.372	87.987	88.016
(40,3,3)	4.980	7.842	3.491	3.500	87.338	87.381

Table 5. Bayes estimators and the determinant

(n,m,s,p)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	determinant	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\theta}$	determinant
(20,1,1,0.15)	2.912	2.284	1.350e+01	1.804e-07	5.843	2.406	1.356e+01	6.634e-07
(40,1,1,0.15)	3.068	2.303	1.312e+01	2.162e-08	6.046	2.421	1.314e+01	8.386e-08
(20,1,1,0.50)	2.364	2.212	1.347e+01	3.126e-07	5.125	2.346	1.353e+01	1.049e-06
(40,1,1,0.50)	2.411	2.219	1.312e+01	5.728e-08	5.187	2.351	1.314e+01	1.873e-07
(20,1,1,0.80)	2.263	2.198	1.338e+01	3.908e-07	4.994	2.334	1.349e+01	1.291e-06
(40,1,1,0.80)	2.278	2.200	1.302e+01	6.485e-08	5.014	2.336	1.307e+01	2.015e-07
(20,2,1,0.15)	3.382	2.465	1.420e+01	6.045e-04	6.535	2.551	1.425e+01	8.741e-04
(40,2,1,0.15)	3.670	2.491	1.348e+01	1.335e-04	6.938	2.571	1.349e+01	2.082e-04
(20,2,1,0.50)	2.682	2.383	1.415e+01	9.150e-05	5.596	2.489	1.422e+01	1.544e-04
(40,2,1,0.50)	2.679	2.378	1.347e+01	2.140e-05	5.586	2.485	1.349e+01	3.581e-05
(20,2,1,0.80)	2.700	2.426	1.403e+01	7.289e-05	5.621	2.523	1.414e+01	1.315e-04
(40,2,1,0.80)	2.717	2.438	1.333e+01	1.611e-05	5.630	2.532	1.338e+01	2.428e-05
(20,2,2,0.15)	3.303	2.573	1.426e+01	2.906e-05	6.432	2.635	1.435e+01	5.818e-05
(40,2,2,0.15)	3.623	2.605	1.353e+01	5.066e-06	6.908	2.660	1.355e+01	1.079e-05
(20,2,2,0.50)	3.078	2.549	1.415e+01	3.254e-05	6.102	2.616	1.422e+01	4.287e-05
(40,2,2,0.50)	3.275	2.570	1.347e+01	3.934e-06	6.389	2.633	1.349e+01	7.628e-06
(20,2,2,0.80)	2.868	2.522	1.399e+01	3.266e-05	5.805	2.594	1.412e+01	4.810e-05
(40,2,2,0.80)	2.950	2.533	1.327e+01	5.386e-06	5.917	2.604	1.336e+01	9.274e-06
(20,3,2,0.15)	4.228	2.685	1.493e+01	2.815e-03	8.233	2.726	1.500e+01	2.411e-02
(40,3,2,0.15)	4.579	2.705	1.389e+01	4.850e-04	8.836	2.742	1.391e+01	2.863e-03
(20,3,2,0.50)	4.295	2.725	1.484e+01	7.479e-04	8.267	2.759	1.489e+01	4.048e-03
(40,3,2,0.50)	4.457	2.736	1.382e+01	1.109e-04	8.533	2.767	1.384e+01	4.748e-04
(20,3,2,0.80)	3.889	2.710	1.470e+01	4.985e-04	7.490	2.746	1.481e+01	3.025e-03
(40,3,2,0.80)	4.145	2.727	1.364e+01	1.231e-04	7.934	2.759	1.371e+01	6.751e-04
(20,3,3,0.15)	3.493	1.811	1.480e+01	8.369e-06	6.713	1.825	1.492e+01	2.551e-05
(40,3,3,0.15)	3.858	1.827	1.382e+01	1.198e-06	7.272	1.839	1.386e+01	4.356e-06
(20,3,3,0.50)	3.197	1.796	1.468e+01	4.435e-06	6.281	1.812	1.479e+01	1.171e-05
(40,3,3,0.50)	3.453	1.808	1.374e+01	1.198e-06	6.664	1.823	1.378e+01	3.017e-06
(20,3,3,0.80)	2.862	1.777	1.453e+01	4.464e-06	5.791	1.796	1.471e+01	1.008e-05
(40,3,3,0.80)	2.920	1.781	1.349e+01	1.373e-06	5.880	1.799	1.362e+01	3.006e-06

'hat' is to estimate under square error loss and 'tilde' for under Linex loss.

All the programs and the simulation codes are written by using R software.

5.2. Illustrative examples. Here we consider an example from Amin [2].

Example 1: HIV+data

Hosmer and Lemeshow [19] produced data describing the survival experience of a group of 100 HIV+ members of a large health maintenance organization. Subjects were enrolled in the study from 1 January 1989 to 31 December 1991. The study was completed on 31 December 1995. After a confirmed diagnosis of HIV+ members were followed until death because of acquired immunodeficiency syndrome (AIDS) or AIDS-related complications, until the end of the study or until the subject was lost to follow-up. According to Amin, the data follows the Pareto distribution. We know that the HIV+ members dead by affecting AIDS or AIDS-related problems, then it is clear that data follows the Pareto distribution in the presence of outliers. Here the AIDS-related factors are shown the outlier's data. Survival times (in months) are given below. Quantities indicated with asterisk denoted censored observations (For more details see Amin [2]).

$$1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1^*,1^*,2,2,2,2,2,2,2^*,2^*,2^*,2^*,2^*,3,3,3,3,3,3,3,3,3,3,3^*,3^*,4,4,4,4,4^*,5,5,5,5,5,5,5,6,6,6^*,7,7,7,7,7,7,7,7^*,8,8,8,8,9,9,9,10,10,10,10^*,11,11,11,12,12,12^*,12^*,13,14,15,15,19^*,22,24^*,30,31,32,34,35,36,43,53,54,56^*,57,58,60^*,60^*.$$

For $n = 100$, $m = 1$, $r = 80$, $s = 1$, $u_i = 0$, $(i = 1, 2, ..., 12)$ and $\boldsymbol{t} = (2, 5, 2, 1, 1, 1, 1, 2, 1, 1, 2, 1)$, Bayes estimates and 95% symmetric two-sided Bayes probability interval of the parameters are derived under the prior specification for $\nu = 7$, $\mu = 8$, $\epsilon = 5$, $\lambda = 1$, $d = 3$ and $c = 4$ are given in Table 6 (upper value). Also, under the non-informative prior density (or $\nu = -1$, $\mu = 1$, $\epsilon \to \infty$, $\lambda = 0$) for α and θ and specified prior for β, $d = 3$ and $c = 4$, the corresponding values are given in Table 6 (lower value).

Table 6. Bayes estimator and 95% symmetric two-sided Bayes probability interval for the parameters $(m = 1, s = 1)$

$\hat{\alpha}$	$\tilde{\alpha}$	α_L	α_U	$\hat{\beta}$	$\tilde{\beta}$	β_L	β_U	$\hat{\theta}$	$\tilde{\theta}$	θ_L	θ_U
0.826	0.765	0.610	0.872	1.317	1.899	1.000	2.657	0.926	0.971	0.812	1.025
0.721	0.661	0.541	0.869	1.177	1.913	1.000	2.431	0.899	0.935	0.801	0.995

Upper value in each cell refers to the specified prior and lower value to the non-informative prior, 'hat' notation is indicated the estimation under square error loss and 'tilde' for estimation under Linex loss.

Further, for $n = 100$, $m = 2$, $r = 80$, $s = 1$, $u_i = 0$, $(i = 1, 2, ..., 11)$, $u_{12} = 1$ and $\boldsymbol{t} = (2, 5, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1)$ Bayes estimates and 95% symmetric two-sided credible regions of the parameters are derived under the prior specification for $\nu = 7$, $\mu = 8$, $\epsilon = 5$, $\lambda = 1$, $d = 3$ and $c = 4$ are shown in Table 7 (upper value). Also, under the non-informative prior density for α and θ and specified prior for β, $d = 3$ and $c = 4$ the corresponding values are given in Table 7 (lower value).

Table 7. Bayes estimator and 95% symmetric two-sided Bayes probability interval for the parameters $(m = 2, s = 1)$

$\hat{\alpha}$	$\tilde{\alpha}$	α_L	α_U	$\hat{\beta}$	$\tilde{\beta}$	β_L	β_U	$\hat{\theta}$	$\tilde{\theta}$	θ_L	θ_U
0.956	0.816	0.611	0.986	1.568	1.873	1.000	2.203	0.935	0.981	0.901	1.103
0.905	0.954	0.712	0.996	1.669	1.604	1.000	2.511	0.595	0.644	0.315	1.002

Upper value in each cell refers to the specified prior and lower value to the non-informative prior, 'hat' notation is indicated the estimation under square error loss and 'tilde' for estimation under Linex loss.

Example 2:
A life test for a new insulating material used 25 specimens. The specimens were tested simultaneously at 30 KV (considerably higher than the rated voltage of 20 KV). Further, it is also observed that there is some noise in the voltage rate. So the data is in the presence of outliers. The test was run until 15 of the specimens failed (under Type II progressive censoring). In other hand, when any specimen (from first to 15th) failed, according to the binomial distribution of dropout random variables, the corresponding number of surviving items are removed from the observations (same as the procedure which is described in section 2, pages 3 and 4). The failure times were recorded as 1.08,12.20,17.80,19.10,26.00,27.90,28.20, 32.20,35.90,43.50,44.00,45.20,45.70,46.30 and 47.80 hours.
Here for $n = 25$, $m = 1$, $r = 15$, $s = 1$, $u_i = 0$, $(i = 1, 2, ..., 15)$ and $\boldsymbol{t} = (0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1)$, we can obtain the Bayes estimates under squared and Linex loss function and 95% symmetric two-sided Bayes probability interval. The results under specified prior density for $\nu = 7$, $\mu = 8$, $\epsilon = 5$, $\lambda = 1$, $d = 3$ and $c = 4$ are given in Table 8 (upper value). The corresponding results under the non-informative prior density for α and θ and specified prior for β, $d = 3$ and $c = 4$ are shown in Table 8 (lower value).

Table 8. Bayes estimator and 95% symmetric two-sided Bayes probability interval for the parameters $(m = 1, s = 1)$

$\hat{\alpha}$	$\tilde{\alpha}$	α_L	α_U	$\hat{\beta}$	$\tilde{\beta}$	β_L	β_U	$\hat{\theta}$	$\tilde{\theta}$	θ_L	θ_U
4.365	4.253	0.124	5.419	2.052	2.172	1.038	2.939	3.687	1.399	0.677	3.757
1.780	1.734	0.045	3.568	1.642	1.718	1.030	2.925	3.386	0.805	0.256	3.461

Upper value in each cell refers to the specified prior and lower value to the non-informative prior, 'hat' notation is indicated the estimation under square error loss and 'tilde' for estimation under Linex loss.

Also, for $n = 25$, $m = 2$, $r = 15$, $s = 1$, $u_i = 0$, $(i = 1, 2, ..., 14)$, $u_{15} = 1$ and $\boldsymbol{t} = (0, 0, 2, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1)$, the Bayes estimates under squared and Linex loss function and the 95% symmetric two-sided Bayes probability interval under specified prior density for $\nu = 7$, $\mu = 8$, $\epsilon = 5$, $\lambda = 1$, $d = 3$ and $c = 4$ are shown in Table 9 (upper value). The corresponding values under the non-informative prior density for α and θ and specified prior for β, $d = 3$ and $c = 4$ are inserted in Table 9 (lower value).

Table 9. Bayes estimator and 95% symmetric two-sided Bayes probability interval for the parameters $(m = 2, s = 1)$

$\hat{\alpha}$	$\tilde{\alpha}$	α_L	α_U	$\hat{\beta}$	$\tilde{\beta}$	β_L	β_U	$\hat{\theta}$	$\tilde{\theta}$	θ_L	θ_U
3.129	3.047	0.124	5.419	1.357	1.523	1.038	2.939	2.570	2.180	0.339	3.777
2.035	1.981	0.063	4.032	1.313	1.511	1.034	2.933	2.841	2.270	0.210	3.716

Upper value in each cell refers to the specified prior and lower value to the non-informative prior, 'hat' notation is indicated the estimation under square error loss and 'tilde' for estimation under Linex loss.

Further, for $n = 25$, $m = 2$, $r = 15$, $s = 2$, $u_i = 0$, $(i = 1, 2, ..., 15)$ and $\boldsymbol{t} = (2, 0, 2, 0, 1, 0, 0, 1, 0, 0,$
$1, 1, 0, 1, 1)$, we can obtain the Bayes estimates under squared and Linex loss function and the 95% symmetric two-sided Bayes probability interval. The values under specified prior density for $\nu = 7$, $\mu = 8$, $\epsilon = 5$, $\lambda = 1$, $d = 3$ and $c = 4$ are shown in Table 10 (upper value). Also, the corresponding results under the non-informative prior density for α and θ and specified prior for β, $d = 3$ and $c = 4$ are given in Table 10 (lower value).

Table 10. Bayes estimator and 95% symmetric two-sided Bayes probability interval for the parameters $(m = 2, s = 2)$

$\hat{\alpha}$	$\tilde{\alpha}$	α_L	α_U	$\hat{\beta}$	$\tilde{\beta}$	β_L	β_U	$\hat{\theta}$	$\tilde{\theta}$	θ_L	θ_U
3.218	3.131	0.130	5.675	1.666	1.561	1.056	2.954	2.565	2.804	0.340	3.774
2.093	2.036	0.066	4.218	1.638	1.535	1.043	2.945	2.853	2.256	0.211	3.771

Upper value in each cell refers to the specified prior and lower value to the non-informative prior, 'hat' notation is indicated the estimation under square error loss and 'tilde' for estimation under Linex loss.

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7. Appendix

Proof of Theorem 4.1:
Case A: According to definition of squared error loss function we should find the mean of the estimator. So

$$\begin{aligned}\hat{\alpha} &= \mathrm{E}(\alpha|\boldsymbol{x},\boldsymbol{k})\\ &= \frac{1}{B_0\Gamma(r+\nu)}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3B_6}\left[\int_0^\infty \alpha^{r+\nu-1}e^{-\alpha B_5}d\alpha - \int_0^\infty \alpha^{r+\nu-1}e^{-\alpha B_4}d\alpha\right].\end{aligned}$$

Hence by evaluating the integrals, we get the Bayes estimator of α.
Cases B:

$$\begin{aligned}\hat{\beta} &= \mathrm{E}(\beta|\boldsymbol{x},\boldsymbol{k})\\ &= \frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3}\int_1^d [-B_6\ln(\beta)+B_4]^{-r-\nu}\,d\beta.\end{aligned}$$

By using the following transformation

$$y = \frac{-B_6\ln(\beta)+B_4}{B_6},$$

we have

$$\hat{\beta} = \frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{B_6{}^{-r-\nu}\exp\left(\frac{B_4}{B_6}\right)}{B_3}\int_{-\ln(d)+\frac{B_4}{B_6}}^{\frac{B_4}{B_6}} y^{-r-\nu}e^{-y}dy,$$

and the Bayes estimator of β is obtained as in (26).
Case C:

$$\begin{aligned}\hat{\theta} &= \mathrm{E}(\theta|\boldsymbol{x},\boldsymbol{k})\\ &= \frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}\left[\int_0^\omega [-B_3\ln(\theta)+B_1]^{-r-\nu}\,d\theta - \int_0^\omega [-B_3\ln(\theta)+B_2]^{-r-\nu}\,d\theta\right]\\ &= \frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{B_3{}^{-r-\nu}}{B_6}\left[e^{\frac{B_1}{B_3}}\int_{-\ln(\omega)+\frac{B_1}{B_3}}^{\infty} y^{-r-\nu}e^{-y}dy - e^{\frac{B_2}{B_3}}\int_{-\ln(\omega)+\frac{B_2}{B_3}}^{\infty} z^{-r-\nu}e^{-z}dz\right],\end{aligned}$$

using the following transformations.

$$y = \frac{-B_3\ln(\theta)+B_1}{B_3},$$

and

$$z = \frac{-B_3\ln(\theta)+B_2}{B_3}.$$

Therefore Bayes estimator of θ can be easily obtained and the proof is complete.
Proof of Theorem 4.2:
Case A: According to definition of Linex loss function we have

$$\begin{aligned}\phi(\delta) &= \mathrm{E}(L(\alpha,\delta)) = \int_0^\infty L(\alpha,\delta)h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha\\ &= e^{c\delta}\int_0^\infty e^{-c\alpha}h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha - c\delta + c\int_0^\infty \alpha h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha - 1.\end{aligned}$$

Differentiating $\phi(\delta)$ respect to δ, we get

$$\frac{\partial\phi(\delta)}{\partial\delta} = ce^{c\delta}\int_0^\infty e^{-c\alpha}h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha - c = 0.$$

Hence by solving the above equation respect to δ, the Byes estimator of α is given by

$$\begin{aligned}\tilde{\alpha} &= -\frac{1}{c}\ln\left(\int_0^\infty e^{-c\alpha}h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha\right)\\ &= -\frac{1}{c}\ln\left(\frac{1}{B_0\Gamma(r+\nu)}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3B_6}\left[\int_0^\infty \alpha^{r+\nu-2}e^{-\alpha(c+B_5)}d\alpha-\int_0^\infty \alpha^{r+\nu-2}e^{-\alpha(c+B_4)}d\alpha\right]\right).\end{aligned}$$

By evaluating the above integrals, the Bayes estimator of α is given in (31).
Case B:

$$\begin{aligned}\tilde{\beta} &= -\frac{1}{c}\ln\left(\int_1^d e^{-c\beta}h(\beta|\boldsymbol{x},\boldsymbol{k})d\beta\right)\\ &= -\frac{1}{c}\ln\left(\frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_3}\int_1^d \beta^{-1}[-B_6\ln(\beta)+B_4]^{-r-\nu}e^{-c\beta}d\beta\right).\end{aligned}$$

By evaluating the integral, we can get the Bayes estimator of β in (32).
Case C:

$$\begin{aligned}\tilde{\theta} &= -\frac{1}{c}\ln\left(\int_0^\omega e^{-c\theta}h(\theta|\boldsymbol{x},\boldsymbol{k})d\theta\right)\\ &= -\frac{1}{c}\ln(\frac{1}{B_0}\sum_{A_1,\ldots,A_s}^{*}\frac{1}{B_6}[\int_0^\omega \theta^{-1}[-B_3\ln(\theta)+B_1]^{-r-\nu}e^{-c\theta}d\theta\\ &- \int_0^\omega \theta^{-1}[-B_3\ln(\theta)+B_2]^{-r-\nu}e^{-c\theta}d\theta]).\end{aligned}$$

Similarly, we get the Bayes estimator of θ in (33) and the proof is finished.
Proof of Theorem 4.3:
Case A:

$$\begin{aligned}\tilde{\alpha}_1 &= -\frac{1}{c}\ln\left(\int_0^\infty e^{-c\alpha}h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha\right)\\ &= -\frac{1}{c}\ln\left(\frac{B_{10}^{r+\nu}}{\Gamma(r+\nu)}\int_0^\infty \alpha^{r+\nu-1}e^{-\alpha(c+B_{10})}d\alpha\right).\end{aligned}$$

After evaluating, we get the Bayes estimator of α in (37).
Case B:

$$\begin{aligned}\tilde{\theta}_1 &= -\frac{1}{c}\ln\left(\int_0^\omega e^{-c\theta}h(\theta|\boldsymbol{x},\boldsymbol{k})d\theta\right)\\ &= -\frac{1}{c}\ln\left((n+\lambda)(r+\nu)B_{10}^{r+\nu}\int_0^\omega \theta^{-1}\left[B_{10}-\ln\left(\frac{\theta}{\omega}\right)\right]^{-r-\nu-1}e^{-c\theta}d\theta\right).\end{aligned}$$

Since

$$\left[B_{10}-(n+\lambda)\ln\left(\frac{\theta}{\omega}\right)\right]^{-r-\nu-1} = B_{10}^{-r-\nu-1}\sum_{j=0}^{\infty}C(r+\nu+j,j)\left[\frac{n+\lambda}{B_{10}}\ln\left(\frac{\theta}{\omega}\right)\right]^j,$$

$$\tilde{\theta}_1 = -\frac{1}{c}\ln\left((n+\lambda)(r+\nu)B_{10}^{-1}\sum_{j=0}^{\infty}\left(\frac{n+\lambda}{B_{10}}\right)^j C(r+\nu+j,j)\int_0^\omega \theta^{-1}\left[\ln\left(\frac{\theta}{\omega}\right)\right]^j e^{-c\theta}d\theta\right).$$

Let $z=\ln(\frac{\theta}{\omega})$, we get

$$\tilde{\theta}_1 = -\frac{1}{c}\ln\left((n+\lambda)(r+\nu)B_{10}^{-1}\sum_{j=0}^{\infty}\left(\frac{n+\lambda}{B_{10}}\right)^j C(r+\nu+j,j)\int_{-\infty}^0 z^j e^{-c\ \omega\exp(z)}dz\right).$$

Set

$$e^{-c\omega\exp(z)} = \sum_{i=0}^{\infty} \frac{(-c\omega)^i e^{iz}}{i!}.$$

Then the Bayes estimator of θ is given in (38) and the proof is complete.

Proof of Theorem 4.4:

Case A: The symmetric $100(1-\gamma)\%$ two-sided Bayes probability interval for α could be easily derived from the following integrals.

$$\int_0^{\alpha_L} h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha = \frac{\gamma}{2} \quad \textit{and} \quad \int_{\alpha_U}^{\infty} h(\alpha|\boldsymbol{x},\boldsymbol{k})d\alpha = \frac{\gamma}{2}.$$

Hence

$$\frac{1}{B_0\Gamma(r+\nu)} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3B_6} \{\int_0^{\alpha_L} \alpha^{r+\nu-2}e^{-\alpha B_5}d\alpha - \int_0^{\alpha_L} \alpha^{r+\nu-2}e^{-\alpha B_4}d\alpha\} = \frac{\gamma}{2}.$$

We know that

$$\int_0^{\alpha_L} \alpha^{r+\nu-2}e^{-\alpha B_5}d\alpha = \frac{\Gamma(r+\nu-1)}{B_5^{r+\nu-2}} \left[1 - \sum_{l=0}^{r+\nu-2} \frac{(\alpha_L B_5)^l e^{-\alpha_L B_5}}{l!}\right].$$

So by using simple algebra, we can get (39). Also, we can find (40) by using the following relation

$$\int_{\alpha_U}^{\infty} \alpha^{r+\nu-2}e^{-\alpha B_5}d\alpha = \frac{\Gamma(r+\nu-1)}{B_5^{r+\nu-2}} \sum_{l=0}^{r+\nu-2} \frac{(\alpha_U B_5)^l e^{-\alpha_U B_5}}{l!}.$$

Case B:

$$\frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3} \int_1^{\beta_L} \beta^{-1}\left[B_4 - B_6\ln(\beta)\right]^{-r-\nu} d\beta = \frac{\gamma}{2}.$$

Let $z = B_4 - B_6\ln(\beta)$. Then we can get (41). Also

$$\frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_3} \int_{\beta_U}^{d} \beta^{-1}\left[B_4 - B_6\ln(\beta)\right]^{-r-\nu} d\beta = \frac{\gamma}{2},$$

then similarly we can get (42).

Case C:

$$\frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_6} \left[\int_0^{\theta_L} \theta^{-1}[B_1 - B_3\ln(\theta)]^{-r-\nu}d\theta - \int_0^{\theta_L} \theta^{-1}[B_2 - B_3\ln(\theta)]^{-r-\nu}d\theta\right] = \frac{\gamma}{2}.$$

Let $z_1 = B_1 - B_3\ln(\theta)$ and $z_2 = B_2 - B_3\ln(\theta)$. So

$$\frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{-B_6B_3} \left[\int_0^{B_1 - B_3\ln(\theta_L)} z_1^{-r-\nu}dz_1 - \int_0^{B_2 - B_3\ln(\theta_L)} z_2^{-r-\nu}dz_2\right] = \frac{\gamma}{2}.$$

Then, we can get (43). Also

$$\frac{1}{B_0} \sum_{A_1,\ldots,A_s}^{*} \frac{1}{B_6} \left[\int_{\theta_U}^{\omega} \theta^{-1}[B_1 - B_3\ln(\theta)]^{-r-\nu}d\theta - \int_{\theta_U}^{\omega} \theta^{-1}[B_2 - B_3\ln(\theta)]^{-r-\nu}d\theta\right] = \frac{\gamma}{2}.$$

With the same transformation in (43), we can find (44) and the proof is finished.