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# An alternative item sum technique for improved estimators of population mean in sensitive surveys

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**Abstract**

The item sum technique (IST) was developed for the measurement of quantitative sensitive variables. This method is closely related to the unmatched count technique (UCT), which was developed to measure the proportion of dichotomous sensitive items in a human population surveys. In this article, firstly, we proposed an improved IST which has a fruitful advantage that it does not require two subsamples as in usual IST and there is also no need of finding optimum subsample sizes. We derived the mean and variance of the proposed estimator and compare it with the usual IST both theoretically and numerically. Secondly, we suggest some alternative family of estimators of the population mean of sensitive variable and compare them with estimator, based on the proposed one sample version of IST. Thirdly, we utilize auxiliary information in estimation of population mean, say $\mu_s$ of sensitive variable. It is established that the estimator based on the proposed IST is always more efficient than its usual counterpart. The estimator using second raw moment of the auxiliary variable is observed to be more efficient than the other auxiliary information based estimators, namely, the ratio, product and regression estimators. The usual and proposed ISTs are applied to estimate the average number of classes missed by the student during the last semester at the Quaid-i-Azam University. Estimated average of number of missed classes and 95% confidence intervals are reported showing that the proposed IST yields precise estimates compared to the usual IST.

**Keywords:** auxiliary information, item sum technique, sensitive variable, unmatched count technique.

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## 1. Introduction

It is well understood that survey is a major instrument in most of the applied research. Surveys for obtaining information on sensitive or stigmatizing characteristics are plagued by the problem of distorted responses or refusals by respondents because it is natural tendency of humans, whenever a sensitive question is asked directly from them they hide their information. For example, most of the people prefer to hide their income, savings, tax wages and illegal characters (e.g. cheating or criminal behavior) from the society. These refusals and vague answers may cause a biased estimation of unknown population parameters, as well as increased variances of the estimates. A question arises here as to how can we get truthful answers to a sensitive or stigmatizing characteristics from respondents. To deal with this situation, Warner [39] introduced randomized response technique (RRT). The main purpose of introducing this technique was to estimate the true proportion of sensitive characteristics while protecting the privacy of respondents. This technique reduces the evasive answering bias and increases the response rate. Horvitz et al. [18], Greenberd [16], Kuk [24], Mangat and Singh [26], Mangat [27], [4], Kim and Warde [23], Chang et al. [1], Gjestvang and Singh [15], Chaudhuri [2], Mehta et al. [29], Gupta et al. [17] and many other authors suggested different modifications and theoretical investigations of the properties of the Warner [39] RRT. Although, RRT has been used in many research areas such as physical and social sciences because of its advantages, there are several difficulties and limitations associated with RRTs. Firstly, interviews that are conducted through using RRT take more time and cost to complete, than the other types of interviews. Geurts [13] discussed financial limitations of RRT. He reported that RRT requires larger sample sizes to obtain the confidence intervals comparable with the direct questioning technique. Another main problem of RRTs reported by Hubbard et al. [19] is making a decision about what kind of the randomization device would be the best for obtaining information on sensitive or stigmatizing characteristics. Chaudhuri and Christofides [3] also gave a criticism on the RRTs. According to them RRT is confined with the respondents' skill to understand and handling the device. An ingenious respondent may feel or understand that his/her response can be traced back to his/her real status, if he/she does understand the mathematical logic behind the randomization device. Because of these difficulties and limitations, alternative techniques have been suggested in the literature. Some of these include the Unmatched Count Technique (Smith, Federer and Raghavarao [35]), the Nominative technique (Miller [31]), The Three card method (Droitcour et al. [12]) and Item sum technique (Trappmann et al. [38]). These alternatives were suggested to avoid untruthful responses on sensitive questions particularly concerning personal issues or illegal acts. Now we give a brief introduction of the item count technique and its recent quantified version known as item sum technique.

**1.1. The unmatched count technique.** The Unmatched count technique (UCT) is a survey methodology that is designed to estimate the proportion of people in the population bearing stigmatizing characteristic. This technique was firstly introduced by Smith, Federer and Raghavarao [35], they used the term "Block Total Response Technique", Later Miller [30] further developed and empirically tested this technique. This technique is also known as the list experiment technique and item count technique (Dalton, Wimbush and Daily [7], Dalton et al. [8], Droitcour et al. [11], Coutts and Jann [6]). In this method, the survey respondents are divided randomly into two subsamples. Each member of the first subsample is presented with the list of $g$ non-stigmatizing (or non-key) items only and each one of the participant of the second subsample is presented with a list containing $g+1$ items, i.e., the same $g$ non-stigmatizing items presented in the first subsample and one stigmatizing item. All stigmatizing and non-stigmatizing items have binary outcomes (e.g., 'Yes' or 'No'). Respondents in both subsamples are then asked to report only the number of 'yes' answers to the investigator. The $g$ non-sensitive items may or may not be the same in both subsamples.

An unbiased estimator of the proportion of sensitive item(e.g. tax evasion) in the population can be estimated by the mean difference of 'yes' answers from the two random subsamples. UCT have been used in numerous studies such as racial prejudice (Kuklinski, Cobb and Gilens [25], Gilens, Sniderman and Kuklinski [14]), drug use (Droitcour et al. [11]), hate crime victimization among college students (Rayburn, Earleywine and Davison [33]), shoplifting (Tsuchiya, Hirai and Ono [37]), and attitude about immigration (Janus [22]). The Unmatched count technique has a major advantage over the randomized response technique, that no randomization device is required and the concept of counting items is relatively simple and the procedure is easy to manage in human population surveys. UCT has also disadvantages which are related to the protection of privacy. In case all $g+1$ items are applicable (or none) to a respondent of the second sample, then his/her response reveals his/her status concerning the stigmatizing characteristic and thus the issue of privacy protection arises. Coutts and Jann [6] also reported the statistical low power as its major disadvantage. According to them the estimates obtained from the UCT have typically large standard errors as compared to the estimates obtained from the RRT based on the same sample size. Another limitation of UCT is that it takes more time and cost because, in this technique, two subsamples are required to estimate population proportion of sensitive variable. Hussain et al. [21] focused on this issue and proposed an improved UCT which requires only single sample. Furthermore, it does not involve finding optimum subsample sizes. The said technique is statistically efficient and more attractive in terms of time and cost.

**1.2. The item sum technique.** Extending the usual UCT, Trappmann et al. [38] proposed a quantified version of UCT, named as the Item Sum Technique(IST) to estimate mean of sensitive variable. In this technique, the survey respondents are randomly divided into two independent subsamples. Each member of the first subsample is presented with the list containing $g+1$ items, with $g$ of those related to non-sensitive characteristics ($T_i$) and one related to sensitive characteristic (S). Each one of the participant of the second subsample is presented with a list containing only the $g$ non-sensitive items. All sensitive and non-sensitive items are quantitative in nature. Respondents in both subsamples are then requested to report the total score applicable to them, without reporting the individual scores on each of the items. Although, in theory, there is no restriction on the choice of number of non-key items but Trappmann et al. [38], suggested to use one non-key(non-sensitive) item in order to improve statistical efficiency of procedure. An unbiased estimator of the population mean of sensitive item, say $\mu_s$, from the IST data can be estimated by the mean difference of answers between the two random subsamples.

let $Y_j$ be the reported total score of the $jth$ respondent in the first subsample ($j = 1, 2, ..., n_1$) and $Z_k$ be the reported total score of the $kth$ respondent in the second subsample ($k = 1, 2, ..., n_2$). Then, an unbiased estimator of population mean of sensitive variable is given by:

$$(1.1) \qquad \hat{\mu}_{s_1} = \bar{y} - \bar{z},$$

where $\bar{y}$ and $\bar{z}$ are the sample means of first and second subsamples, respectively. The variance of $\hat{\mu}_{s_1}$ is given by

$$(1.2) \qquad Var(\hat{\mu}_{s_1}) = \frac{\sigma_s^2}{n_1} + \frac{n\sum_{i=1}^{g}\sigma_{t_i}^2}{n_1 n_2}$$

where $\sigma_s^2$ is the population variance of sensitive variable and $\sigma_{t_i}^2$ is the population variance of $ith$ non sensitive variable. In usual IST, two subsamples are required to obtain an unbiased estimator of population mean of stigmatizing variable and finding optimum subsample sizes are also required. We focus on these issues and proposed an alternative IST which is based on one sample and does not require finding optimum subsample sizes. The proposed technique is a quantified version of UCT proposed by Hussain et al. [21].

## 2. Proposed item sum technique

In this section, we propose an alternative IST, avoiding the need of two subsamples. Let $\mu_s$ be the population mean of the sensitive variable of interest. Our purpose is to estimate $\mu_s$. The procedure is described as follows.

Assume that a simple random sample of size $n$ is drawn from the population. Each one of the participants in the sample is provided a list of $g$ items. The $ith$ item contains the addition of queries about a stigmatizing sensitive (S) and non-stigmatizing ($T_i$) variables. The respondents are directed to report only the total score of all items, without reporting the individual scores of each item. Both the non-stigmatizing ($T_i$) and a stigmatizing sensitive (S) variables are quantitative in nature(and preferably measured on the same scale). For example, respondents in the sample may be presented the following items.

(1) Last digit of your cell phone number + Number of times you smoked *shisha*(harmful type of tobacco) last month.
(2) Date on your birth day was + Number of times you smoked *shisha* last month.
(3) Last digit of your CNIC(Computerized National Identity Card) number + Number of times you smoked *shisha* last month.
(4) Number of hours you watched T.V last day + Number of times you smoked *shisha* last month.

We assume that all ($T_i$) and (S) variables are unrelated to each other and the distribution of non-sensitive($T_i$) variables are known to the interviewers. It is important that the number of non-stigmatizing questions should be chosen wisely by the investigators. Although according to theory there is no restriction on the choice of the number of non-stigmatizing questions but, too short and too many questions can create a problem, related to the privacy protection of the respondents and the statistical deficiency of the technique, respectively.

Let $T_1, ..., T_g$ denote the non sensitive variables and S be the sensitive variable. Let $Y_j$ denote the total score of $jth$ respondent in the sample ($j = 1, 2, ..., n$), then mathematically, it can be written as:

$$(2.1) \qquad Y_j = gS + \sum_{i=1}^{g} T_i$$

Taking expectation on (2.1) we have

$$E(Y_j) = gE(S) + \sum_{i=1}^{g} E(T_i),$$

$$= g\mu_s + \sum_{i=1}^{g} \mu_{t_i},$$

(2.2) $$\mu_s = \frac{E(Y_j) - \sum_{i=1}^{g} \mu_{t_i}}{g}.$$

This suggest, defining an unbiased estimator of $\mu_s$ as

(2.3) $$\hat{\mu}_{s_P} = \frac{\bar{y} - \sum_{i=1}^{g} \mu_{t_i}}{g},$$

where $\mu_{t_i}$ denoting the population mean of the $ith$ non-sensitive variable $(i = 1, 2, ..., g)$ and $\bar{y}$ is the sample mean of reported response. The estimator $\hat{\mu}_{s_P}$ in (2.3), obtained through proposed IST, does not require two subsamples (as in usual IST).

The variance of the estimator $\hat{\mu}_{s_P}$ is given by:

$$Var(\hat{\mu}_{s_P}) = \frac{Var(\bar{y})}{g^2} = \frac{Var(y)}{ng^2},$$

(2.4) $$Var(\hat{\mu}_{s_P}) = \frac{\sigma_s^2}{n} + \frac{\sum_{i=1}^{g} \sigma_{t_i}^2}{ng^2},$$

and when sample is drawn without replacement, the variance of $\hat{\mu}_{s_P}$ is given by

(2.5) $$Var(\hat{\mu}_{s_P}) = \frac{1-f}{ng^2}(g^2\sigma_s^2 + \sum_{i=1}^{g} \sigma_{t_i}^2),$$

where $\sigma_s^2$ is the population variance of sensitive variable and $\sigma_{t_i}^2$ is the population variance of $ith$ non sensitive variable and $f = \frac{n}{N}$.

**2.1. Efficiency comparison.** Here, we present efficiency comparison of the proposed estimator $\hat{\mu}_{s_P}$ with the estimator $\hat{\mu}_{s_1}$ based on the usual IST.

Consider

$$Var(\hat{\mu}_{s_1}) - Var(\hat{\mu}_{s_P}) \geq 0$$

$$\frac{\sigma_s^2}{n_1} + \frac{n\sum_{i=1}^{g} \sigma_{t_i}^2}{n_1 n_2} - \frac{\sigma_s^2}{n} - \frac{\sum_{i=1}^{g} \sigma_{t_i}^2}{ng^2} \geq 0$$

$$\frac{n_2\sigma_s^2}{n_1 n} + \frac{(n^2g^2 - n_1n_2)\sum_{i=1}^{g} \sigma_{t_i}^2}{nn_1n_2g^2} \geq 0.$$

Since, all the quantities on right hand size of above inequality are strictly positive, the above inequality always holds true. Hence, we can infer that the proposed estimator $\hat{\mu}_{s_P}$ is always more efficient than the estimator $\hat{\mu}_{s_1}$.

The Percent Relative Efficiency (PRE) of $\hat{\mu}_{s_P}$ with respect to $\hat{\mu}_{s_1}$ is define as

$$PRE(\hat{\mu}_{s_P}, \hat{\mu}_{s_1}) = \frac{Var(\hat{\mu}_{s_1})}{Var(\hat{\mu}_{s_P})} \times 100.$$

Using the above definition, PRE has been worked out for the different fixed values of the parameters involved and is arranged in the Tables 1-3(see appendix). One may easily observe that the proposed estimator, $\hat{\mu}_{s_P}$ is always more efficient than the estimator, $\hat{\mu}_{s_1}$.

The proposed and usual ISTs are illustrated through a practical example of estimating the average number of classes missed by the students during the last semester at Quaid-i-Azam University. We set $g = 4$ and select a random sample of 100 students. For collecting data through usual IST sample was randomly divided into two equal halves. The estimate and 95 % confidence interval through usual IST are 3.52 and $[3.18, 3.85]$, respectively. The estimate and 95 % confidence interval through the proposed IST are 3.28 and $[3.12, 3.43]$, respectively. This clearly shows that estimated variance of the proposed estimator is smaller than the usual IST estimator.

## 3. Some alternative classes of estimators for $\mu_s$

**3.1. Searls' method of estimation.** Motivated by searls [34], we define a family of estimators of the population mean of the sensitive variable $\mu_s$ as

$$\hat{\mu}_{s_\lambda} = \lambda \hat{\mu}_{s_P}, \tag{3.1}$$

where $\lambda$ is constant to be chosen suitably by the interviewer or researcher and $\hat{\mu}_{s_P}$ is the proposed estimator, defined in (2.3). The bias and mean square error of $\hat{\mu}_{s_\lambda}$ are respectively given as:

$$Bias(\hat{\mu}_{s_\lambda}) = (\lambda - 1)\mu_s, \tag{3.2}$$

and

$$MSE(\hat{\mu}_{s_\lambda}) = \lambda^2 Var(\hat{\mu}_{s_P}) + (\lambda - 1)^2 \mu_s^2. \tag{3.3}$$

**Efficiency comparison.** The proposed estimator $\hat{\mu}_{s_\lambda}$ is, relatively more efficient than the estimator $\hat{\mu}_{s_P}$ if

$$MSE(\hat{\mu}_{s_\lambda}) - Var(\hat{\mu}_{s_P}) \leq 0$$

$$(\lambda^2 - 1)Var(\hat{\mu}_{s_P}) + (\lambda - 1)^2 \mu_s^2 \leq 0$$

It can be shown that $MSE(\hat{\mu}_{s_\lambda}) - Var(\hat{\mu}_{s_P}) \leq 0$ if and only if

$$\frac{\mu_s^2 - Var(\hat{\mu}_{s_P})}{\mu_s^2 + Var(\hat{\mu}_{s_P})} < \lambda \leq 1, \tag{3.4}$$

where $Var(\hat{\mu}_{s_P})$ is given by (2.4)

Using (3.4) we have computed the ranges of $\lambda$ in which suggested estimator $\hat{\mu}_{s_\lambda}$ is, relatively more efficient than the proposed estimator $\hat{\mu}_{s_P}$ for different values of $(n, \sigma_s^2)$. R-package is used to find these ranges. The ranges, so obtained, are presented in the Tables 4 and 5 (see appendix).

**3.1.1.** *Optimum estimator amongst the family of estimator $\hat{\mu}_{s_\lambda}$.* By differentiating (3.3) with respect to $\lambda$ and equating to zero, we can find optimum value of $\lambda$ which minimizes the $MSE(\hat{\mu}_{s_\lambda})$, as:

$$\lambda_{opt} = \frac{\mu_s^2}{\mu_s^2 + Var(\hat{\mu}_{s_P})}. \tag{3.5}$$

Thus, the optimum estimator,say $\hat{\mu}_{s_{\lambda_{opt}}}$ is given by

$$\hat{\mu}_{s_{\lambda_{opt}}} = \lambda_{opt} \hat{\mu}_{s_P}. \tag{3.6}$$

Its Bias and MSE is given by

$$Bias(\hat{\mu}_{s_{\lambda_{opt}}}) = (\lambda_{opt} - 1)\mu_s,$$

and

$$(3.7) \qquad MSE(\hat{\mu}_{s_{\lambda_{opt}}}) = \frac{\mu_s^2 Var(\hat{\mu}_{s_P})}{\mu_s^2 + Var(\hat{\mu}_{s_P})}.$$

The relative efficiency of the optimum estimator$\hat{\mu}_{s_{\lambda_{opt}}}$ with respect to proposed estimator $\hat{\mu}_{s_P}$ is given by

$$RE = \frac{Var(\hat{\mu}_{s_P})}{MSE(\hat{\mu}_{s_{\lambda_{opt}}})},$$

$$(3.8) \qquad RE = 1 + \frac{Var(\hat{\mu}_{s_P})}{\mu_s^2}.$$

From (3.8), it is obvious that optimum estimator is always more efficient than the estimator $\hat{\mu}_{s_P}$.

**3.2. Estimation method which utilizes the *priori* information.** It is a well known fact that by using prior knowledge about parameter, estimation becomes more precise and efficient. Bayesian approach of estimation is an example of using the prior information in the form of prior distribution. Some times, in order to have more precise and statistically efficient estimator, we may use prior knowledge together with the sample information. Motivated by Thompson [36] and Mathur snd Singh [28], we define another estimator for $\mu_s$. Let $\mu_{s_0}$ be the prior estimate or guessed value of the population mean of the sensitive variable $\mu_s$. Then, we define a new estimator as

$$(3.9) \qquad \hat{\mu}_{s_k} = K\hat{\mu}_{s_P} + (1-K)\mu_{s_0}, \quad 0 < K \leq 1$$

where $K$ is a constant specified by the investigator according to his/her belief in the prior estimate $\mu_{s_0}$. The value of $K$ closer to 0 specifies strong belief in $\mu_{s_0}$ and closer to 1 indicates strong belief in $\mu_s$.

The bias and mean square error of $\hat{\mu}_{s_k}$ are, respectively, given by:

$$(3.10) \qquad Bias(\hat{\mu}_{s_k}) = (1-K)\mu_s w,$$

$$(3.11) \qquad MSE(\hat{\mu}_{s_k}) = w^2(1-K)^2\mu_s^2 + K^2 Var(\hat{\mu}_{s_P}),$$

where $w = (\frac{\mu_{s_0}}{\mu_s} - 1)$.

**Efficiency comparison.** The proposed estimator $\hat{\mu}_{s_k}$ is relatively more efficient than the estimator $\hat{\mu}_{s_P}$ if

$MSE(\hat{\mu}_{s_k})$ - $Var(\hat{\mu}_{s_P})\leq 0$

$$= w^2(1-K)^2\mu_s^2 + K^2 Var(\hat{\mu}_{s_P}) - Var(\hat{\mu}_{s_P}) \leq 0.$$

It can be shown that $MSE(\hat{\mu}_{s_k}) - Var(\hat{\mu}_{s_P}) \leq 0$, if and only if

$$(3.12) \qquad w^2 < \frac{(1+K)Var(\hat{\mu}_{s_P})}{(1-K)\mu_s^2}, \quad 0 < K \leq 1$$

or

$$(3.13) \qquad \frac{w^2\mu_s^2 - Var(\hat{\mu}_{s_P})}{w^2\mu_s^2 + Var(\hat{\mu}_{s_P})} < K \leq 1,$$

where $Var(\hat{\mu}_{s_P})$ is given by (2.4)

Through R-Package, using (3.13) we have computed ranges of K in which suggested estimator is more efficient than the proposed estimator , for different values of ($n$ and $w$). These ranges are presented in the Tables 6-10 (see appendix).

**3.2.1.** *Optimum estimators of $\hat{\mu}_{s_k}$.* By differentiating (3.11) with respect to $K$ and equating to zero, we can find optimum value of $K$ which minimizes the $MSE(\hat{\mu}_{s_k})$ . The optimum value of $K$ is given by

$$K_{opt} = \frac{w^2\mu_s^2}{w^2\mu_s^2 + Var(\hat{\mu}_{s_P})}. \tag{3.14}$$

Thus, the optimum estimator, say $\hat{\mu}_{s_{k_{opt}}}$ is given by

$$\hat{\mu}_{s_{k_{opt}}} = K_{opt}\hat{\mu}_{s_P} + (1 - K_{opt})\mu_{s_0}, \tag{3.15}$$

and its MSE is given by

$$MSE(\hat{\mu}_{s_{k_{opt}}}) = \frac{w^2\mu_s^2 Var(\hat{\mu}_{s_P})}{w^2\mu_s^2 + Var(\hat{\mu}_{s_P})}. \tag{3.16}$$

The relative efficiency of the optimum estimator $\hat{\mu}_{s_{k_{opt}}}$ with respect to our proposed estimator $\hat{\mu}_{s_P}$ is given by

$$RE = \frac{Var(\hat{\mu}_{s_P})}{MSE(\hat{\mu}_{s_{k_{opt}}})}$$

$$RE = 1 + \frac{Var(\hat{\mu}_{s_P})}{w^2\mu_s^2}. \tag{3.17}$$

From (3.17), it can be seen that, optimum estimator $\hat{\mu}_{s_{k_{opt}}}$ is always more efficient than the proposed IST estimator $\hat{\mu}_{s_P}$.

## 4. Some alternative family of estimators, utilizing auxiliary variable

It is well known that the incorporation of auxiliary information in the estimation procedure yields efficient estimators. The use of such information is generally made through ratio, product and regression methods of estimation. [5] was the first to show the contribution of such known information, He introduced a ratio estimator of population mean. Murthy [32] introduced product method of estimation. Similarly, as in usual surveys, auxiliary information is also utilized in sensitive surveys where, our main purpose is to estimate population proportion/mean of stigmatizing attribute/variable. There are two ways of utilizing auxiliary information. Firstly it may be at the design stage and secondly at estimation stage. There are few studies, which utilizes auxiliary information in RRT at estimation stage. For example, Zaizai [40], Diana and Perri [9] and Diana and Perri [10] used auxiliary information at estimation stage. Motivated by Zaizai [40], Diana and Perri [10] and Hussain and Shabbir [20] we propose a class of estimators which utilizes the known auxiliary information. From (2.2) we have

$$\mu_s = \frac{E(Y_j) - \sum_{i=1}^{g}\mu_{t_i}}{g}$$

In order to estimate $\mu_s$, we can replace $E(Y_j)$ by $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_{lr}$ and $\bar{y}_{lm}$.

where $\bar{y}_R = \frac{\bar{y}}{\bar{x}}\bar{X}$, $\bar{y}_P = \frac{\bar{y}}{\bar{X}}\bar{x}$ , $\bar{y}_{lr} = \bar{y} + b[\bar{X} - \bar{x}]$, and

$\bar{y}_{lm} = \bar{y} + b_1[\bar{X} - \bar{x}] + b_2[M - m]$ and $m = n^{-1}\sum_{j=1}^{n} x^2$ = second raw moment of auxiliary variable. Let $q = x^2$, then $m = n^{-1}\sum_{j=1}^{n} q$, and $\bar{y}_{lm} = \bar{y} + b_1[\bar{X} - \bar{x}] + b_2[\bar{Q} - \bar{q}]$

Now, we study these methods of estimation one by one.

**4.1. Ratio method of estimation.** Replacing $E(Y_j)$ by $\bar{y}_R$ in (2.2), a new class of estimators is given by

$$\hat{\mu}_{s_{P1}} = \frac{\bar{y}_R - \sum_{i=1}^{g} \mu_{t_i}}{g}.$$

The bias and MSE of $\bar{y}_R$ are given by

$$Bias(\bar{y}_R) = \frac{1-f}{n}\bar{Y}(C_x^2 - \rho_{yx}C_yC_x),$$

and

$$MSE(\bar{y}_R) = \frac{1-f}{n}(\sigma_y^2 + R^2\sigma_x^2 - 2R\rho_{yx}\sigma_y\sigma_x),$$

where $R = \frac{\bar{Y}}{\bar{X}}$. Now, the MSE of $\hat{\mu}_{s_{P1}}$ is given by

$$MSE(\hat{\mu}_{s_{P1}}) = \frac{MSE(\bar{y}_R)}{g^2},$$

$$MSE(\hat{\mu}_{s_{P1}}) = \frac{\frac{1-f}{n}[\sigma_y^2 + R^2\sigma_x^2 - 2R\rho_{yx}\sigma_y\sigma_x]}{g^2}.$$

As we have $\sigma_y^2 = g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2$, and

$$\rho_{yx} = \frac{\rho_{sx}}{\sqrt{1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}}}$$

we can write

$$MSE(\hat{\mu}_{s_{P1}}) = \frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 - 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})], \tag{4.1}$$

where $R_1 = \frac{\mu_s}{\bar{X}}$.

**4.2. Product method of estimation.** From (2.2), Replacing $E(Y_j)$ by $\bar{y}_P$, a new class of estimators is given by

$$\hat{\mu}_{s_{P2}} = \frac{\bar{y}_P - \sum_{i=1}^{g} \mu_{t_i}}{g}.$$

The bias and MSE of $\bar{y}_R$ are given by

$$Bias(\bar{y}_P) = \frac{1-f}{n}\bar{Y}(C_x^2 + \rho_{yx}C_yC_x),$$

and

$$MSE(\bar{y}_P) = \frac{1-f}{n}(\sigma_y^2 + R^2\sigma_x^2 + 2R\rho_{yx}\sigma_y\sigma_x),$$

where $R = \frac{\bar{Y}}{\bar{X}}$. Now, and MSE of $\hat{\mu}_{s_{P2}}$ is given by

$$MSE(\hat{\mu}_{s_{P2}}) = \frac{MSE(\bar{y}_P)}{g^2},$$

$$MSE(\hat{\mu}_{s_{P2}}) = \frac{\frac{1-f}{n}[\sigma_y^2 + R^2\sigma_x^2 + 2R\rho_{yx}\sigma_y\sigma_x]}{g^2},$$

substituting $\sigma_y^2 = g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2$ in the above equation, we get

$$MSE(\hat{\mu}_{s_{P2}}) = \frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 + 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})]. \tag{4.2}$$

**4.3. Regression method of estimation.** As in section 4.1 and 4.2, replacing $E(Y_j)$ by $\bar{y}_{lr}$ in (2.2) , a new family of estimators may be proposed as

$$\hat{\mu}_{s_{P3}} = \frac{\bar{y}_{lr} - \sum_{i=1}^{g}\mu_{t_i}}{g}.$$

It is understood, that the regression coefficient $b$ may either be known or unknown. We consider both cases one by one.

*Case (i): when coefficient b is known.* As $\bar{y}_{lr} = \bar{y} + b[\bar{X} - \bar{x}]$, we have bias$(\bar{y}_{lr}) = 0$ and its variance is given by

$$Var(\bar{y}_{lr}) = \frac{1-f}{n}(\sigma_y^2 + b^2\sigma_x^2 - 2b\rho_{yx}\sigma_y\sigma_x). \tag{4.3}$$

As $b$ is a known regression coefficient, it must be

$$b = \frac{\sigma_{xy}}{\sigma_x^2}. \tag{4.4}$$

it is interesting to see that $b = \frac{\sigma_{xy}}{\sigma_x^2}$ also minimizing the $Var(\bar{y}_{lr})$.

Now substituting above value of $b$ in (4.3), the minimum variance of the estimator $\bar{y}_{lr}$ is given by

$$Var(\bar{y}_{lr})_{min} = \frac{1-f}{n}\sigma_y^2(1 - \rho_{yx}^2),$$

$$\rho_{yx} = \frac{\rho_{sx}}{\sqrt{1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}}}$$

The minimum variance of $\hat{\mu}_{s_{P3}}$ is now given by

$$Var(\hat{\mu}_{s_{P3}})_{min} = \frac{Var(\bar{y}_{lr})_{min}}{g^2}$$

$$Var(\hat{\mu}_{s_{P3}})_{min} = \frac{\frac{1-f}{n}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2][1 - \frac{\rho_{sx}^2 g^2\sigma_s^2}{g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2}]}{g^2}.$$

$$(4.5) \qquad Var(\hat{\mu}_{s_{P3}})_{min} = \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2} - \rho_{sx}^2).$$

*Case(ii): when $b$ is unknown.* As $b$ is the regression coefficient in the regression of $y = b_o + bx + \epsilon$, where $b_0$ is constant and $\epsilon$ is random error term, so the unbiased ordinary least square estimator of $b$ is $\hat{b} = \frac{\sigma_{xy}}{\sigma_x^2}$, which minimizes the error sum of squares. Thus, new class of estimators becomes

$$\hat{\mu}_{s_{N3}} = \frac{\hat{\bar{y}}_{lr} - \sum_{i=1}^{g}\mu_{t_i}}{g}.$$

Following the same steps as in case (i), it can be shown that MSE of $\hat{\mu}_{s_{N3}}$ is given by

$$MSE(\hat{\mu}_{s_{N3}}) = \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2} - \rho_{sx}^2).$$

It is clear that both the variance and MSE expressions are same in case (i) and (ii), so we do not need to consider both cases of regression coefficient separately, since the results in one case remain valid in other case.

**4.4. Regression method of estimation, where second raw moment is used.** From (2.2), Replacing $E(Y_j)$ by $\bar{y}_{lm}$, a new family of estimators is given by

$$\hat{\mu}_{s_{P4}} = \frac{\bar{y}_{lm} - \sum_{i=1}^{g}\mu_{t_i}}{g}.$$

As we mentioned above that the regression coefficients may be either known or unknown. But the results in both cases remain same. So we do not need to consider both cases separately. We have bias$(\bar{y}_{lm}) = 0$ and variance of $\bar{y}_{lm}$ is given by

$$(4.6) \qquad Var(\bar{y}_{lm}) = \frac{1-f}{n}(\sigma_y^2 + b_1^2\sigma_x^2 + b_2^2\sigma_q^2 - 2b_1\rho_{yx}\sigma_y\sigma_x - 2b_1\rho_{yq}\sigma_y\sigma_q + 2b_1b_2\rho_{xq}\sigma_x\sigma_q),$$

Differentiating (4.6) with respect to $b_1$ and $b_2$, and equating to zero, we obtain

$$(4.7) \qquad b_1 = \frac{\sigma_q^2\sigma_{yx} - \sigma_{xq}\sigma_{yq}}{\sigma_x^2\sigma_q^2 + \sigma_{xq}^2}.$$

$$(4.8) \qquad b_2 = \frac{\sigma_x^2\sigma_{yq} - \sigma_{xq}\sigma_{yx}}{\sigma_x^2\sigma_q^2 + \sigma_{xq}^2}.$$

Substituting optimum values of $b_1$ and $b_2$ in (4.6), the minimum variance of the estimator $\bar{y}_{lm}$ is given by

$$Var(\bar{y}_{lm})_{min} = \frac{1-f}{n}\sigma_y^2[1 - R_{y.xq}^2],$$

where

$$R^2_{y.xq} = \frac{\rho^2_{yx} + \rho^2_{yq} - 2\rho_{yx}\rho_{yq}\rho_{xq}}{1-\rho^2_{xq}}.$$

Now, the minimum variance of $\hat{\mu}_{sP4}$ is given by

$$Var(\hat{\mu}_{sP4})_{min} = \frac{Var(\bar{y}_{lm})_{min}}{g^2}$$

$$Var(\hat{\mu}_{sP4})_{min} = \frac{1-f}{n}\sigma^2_s(1 + \frac{\sum_{i=1}^{g}\sigma^2_{t_i}}{g^2\sigma^2_s} - \frac{\rho^2_{sx} + \rho^2_{sq} - 2\rho_{sx}\rho_{sq}\rho_{xq}}{1-\rho^2_{xq}}),$$

$$Var(\hat{\mu}_{sP4})_{min} = \frac{1-f}{n}\sigma^2_s(1 + \frac{\sum_{i=1}^{g}\sigma^2_{t_i}}{g^2\sigma^2_s} - R^2_{s.xq}). \tag{4.9}$$

Now, consider

$$\Psi^2 + \rho^2_{sx} = \frac{(\rho_{sq} - \rho_{sx}\rho_{xq})^2}{1-\rho^2_{xq}} + \rho^2_{sx} = \frac{\rho^2_{sx} + \rho^2_{sq} - 2\rho_{sx}\rho_{sq}\rho_{xq}}{1-\rho^2_{xq}} = R^2_{s.xq}. \tag{4.10}$$

Thus, by (4.9) and (4.10), we get

$$Var(\hat{\mu}_{sP4})_{min} = \frac{1-f}{n}\sigma^2_s(1 + \frac{\sum_{i=1}^{g}\sigma^2_{t_i}}{g^2\sigma^2_s} - \Psi^2 - \rho^2_{sx}). \tag{4.11}$$

**4.5. Efficiency comparison.**

(1) From (2.5)and (4.1), we have

$$Var(\hat{\mu}_{sP}) - MSE(\hat{\mu}_{sP1}) > 0$$

or

$$\frac{1-f}{ng^2}(g^2\sigma^2_s + \sum_{i=1}^{g}\sigma^2_{t_i}) - \frac{1-f}{ng^2}[g^2\sigma^2_s + \sum_{i=1}^{g}\sigma^2_{t_i} + \sigma^2_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2$$
$$- 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})] > 0$$

$$\rho_{sx} > \frac{\sigma_x(gR_1 + \bar{X}^{-1}\sum_{i=1}^{g}\mu_{t_i})}{2g\sigma_s} \tag{4.12}$$

We infer that estimator $\hat{\mu}_{sP1}$ is more efficient than estimator $\hat{\mu}_{sP}$, if the inequality (4.12) is satisfied.

(2) From (2.5)and (4.2), we have

$$Var(\hat{\mu}_{sP}) - MSE(\hat{\mu}_{sP2}) > 0$$

or

$$\frac{1-f}{ng^2}(g^2\sigma_s^2+\sum_{i=1}^{g}\sigma_{t_i}^2)-\frac{1-f}{ng^2}[g^2\sigma_s^2+\sum_{i=1}^{g}\sigma_{t_i}^2+\sigma_x^2(gR_1+\frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2+2g\rho_{sx}\sigma_s\sigma_x(gR_1+\frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})]>0$$

(4.13) $$\rho_{sx}<-\frac{\sigma_x(gR_1+\bar{X}^{-1}\sum_{i=1}^{g}\mu_{t_i})}{2g\sigma_s}.$$

We infer that estimator $\hat{\mu}_{s_{P2}}$ is more efficient than estimator $\hat{\mu}_{s_P}$, if above inequality(4.13) is satisfied

(3) From (2.5) and (4.5), the estimator $\hat{\mu}_{s_{P3}}$ is more efficient than the estimator $\hat{\mu}_{s_P}$ if

$Var(\hat{\mu}_{s_P})>Var(\hat{\mu}_{s_{P3}})_{min}$

$$\frac{1-f}{ng^2}(g^2\sigma_s^2+\sum_{i=1}^{g}\sigma_{t_i}^2)>\frac{1-f}{n}\sigma_s^2(1+\frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}-\rho_{sx}^2)$$

(4.14) $$\rho_{sx}>0.$$

(4) The estimator $\hat{\mu}_{s_{P4}}$ is more efficient than the estimator $\hat{\mu}_{s_P}$ if

$Var(\hat{\mu}_{s_P})>Var(\hat{\mu}_{s_{P4}})_{min}$

From (2.5) and (4.9), we have

$$\frac{1-f}{ng^2}(g^2\sigma_s^2+\sum_{i=1}^{g}\sigma_{t_i}^2)>\frac{1-f}{n}\sigma_s^2(1+\frac{\sum_{i=1}^{g}\sigma_{N_i}^2}{g^2\sigma_s^2}-R_{s.xq}^2)$$

(4.15) $$R_{s.xq}^2>0.$$

(5) The estimator $\hat{\mu}_{s_{P4}}$ is more efficient than the estimator $\hat{\mu}_{s_{P3}}$ if

$Var(\hat{\mu}_{s_{P3}})_{min}-Var(\hat{\mu}_{s_{P4}})_{min}\geq 0$

From (4.5) and (4.11), we have

$$\frac{1-f}{n}\sigma_s^2[1+\frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}-\rho_{sx}^2]-\frac{1-f}{n}\sigma_s^2(1+\frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}-\Psi^2-\rho_{sx}^2)\geq 0$$

(4.16) $$\Psi^2\geq 0,$$

which is always a non-negative quantity.

(6) The estimator $\hat{\mu}_{s_{P4}}$ is more efficient than the estimator $\hat{\mu}_{s_{P2}}$ if

$MSE(\hat{\mu}_{s_{P2}}) \geq Var(\hat{\mu}_{s_{P4}})_{min}$

From (4.2) and (4.11), we have

$$\frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 + 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})] - \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2} - \Psi^2 - \rho_{sx}^2) \geq 0$$

$$g^2\sigma_s^2\Psi^2 + (g\sigma_s\rho_{sx} + \sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}}))^2 \geq 0, \tag{4.17}$$

which is always true.

(7) The estimator $\hat{\mu}_{s_{P4}}$ is more efficient than the estimator $\hat{\mu}_{s_{P1}}$ if

$MSE(\hat{\mu}_{s_{P1}}) \geq Var(\hat{\mu}_{s_{P4}})_{min}$

From (4.1) and (4.11), we have

$$\frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 - 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})] - \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2} - \Psi^2 - \rho_{sx}^2) \geq 0$$

$$g^2\sigma_s^2\Psi^2 + (g\sigma_s\rho_{sx} - \sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}}))^2 \geq 0, \tag{4.18}$$

which is again always true.

(8) The estimator $\hat{\mu}_{s_{P3}}$ is more efficient than the estimator $\hat{\mu}_{s_{P1}}$ if

$MSE(\hat{\mu}_{s_{P1}}) \geq Var(\hat{\mu}_{s_{P3}})_{min}$.

From (4.1) and (4.5) we have

$$\frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 - 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})] - \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2} - \rho_{sx}^2) \geq 0$$

$$(g\sigma_s\rho_{sx} - \sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}}))^2 \geq 0, \tag{4.19}$$

which is always true.

(9) The estimator $\hat{\mu}_{s_{P3}}$ is more efficient than the estimator $\hat{\mu}_{s_{P2}}$ if

$MSE(\hat{\mu}_{s_{P2}}) \geq Var(\hat{\mu}_{s_{P3}})_{min}$.

From (4.2) and (4.5), we have

$$\frac{1-f}{ng^2}[g^2\sigma_s^2 + \sum_{i=1}^{g}\sigma_{t_i}^2 + \sigma_x^2(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})^2 + 2g\rho_{sx}\sigma_s\sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}})] - \frac{1-f}{n}\sigma_s^2(1 + \frac{\sum_{i=1}^{g}\sigma_{t_i}^2}{g^2\sigma_s^2}\rho_{sx}^2) \geq 0$$

$$(g\sigma_s\rho_{sx} + \sigma_x(gR_1 + \frac{\sum_{i=1}^{g}\mu_{t_i}}{\bar{X}}))^2 \geq 0, \tag{4.20}$$

which is always true.

One may easily see that regression method of estimation which utilized higher order moments of auxiliary variable is more efficient than other classes of estimators which utilizes auxiliary information. Furthermore, regression method of estimation is efficient than product and ratio methods of estimation.

## 5. Conclusion

In this article, we proposed one sample version of IST (alternative IST) to estimate the mean of a sensitive variable. The proposed IST is based on the idea of combining the sensitive item with each of the unrelated item. The main fruitful feature of this technique is that unlike the usual IST, it does not require two subsamples and finding optimum subsample sizes. It is established that the proposed estimator is better than the usual estimator of IST and it may be made more efficient if a prior guess about the population mean is available. In case when auxiliary information is available, ratio, product and regression method of estimation are considered. Through algebraic comparisons, it is concluded that the regression method of estimation utilizing the second order moment of auxiliary variable is the most efficient than all the estimators considered in this article. As an overall conclusion it is stated that when studying the sensitive variable the application of one sample version of IST (proposed estimator) is more advantageous in terms of ease in practice and providing accurate and precise estimators. The application of the proposed IST can be even more advantageous if some auxiliary information is available and regression method of estimation, utilizing the second raw moment of auxiliary information, is applied.

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## Appendix

**Table 1.** PRE of $\hat{\mu}_{s_P}$ with respect to $\hat{\mu}_{s_1}$ for $n = 50$, $n_1 = 25$, $n_2 = 25$ and different values of $g$, $\sigma_s^2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2$.

| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 0.1$ | | | | |
|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 327.2727 | 344.6809 | 354.3307 | 358.5657 |
| 0.5 | 266.6667 | 273.9130 | 277.7778 | 279.4411 |
| 1 | 234.1463 | 237.3626 | 239.0438 | 239.7602 |
| 2 | 217.2840 | 218.7845 | 219.5609 | 219.8901 |
| 3 | 211.5702 | 212.5461 | 213.0493 | 213.2622 |
| 4 | 208.6957 | 209.4183 | 209.7902 | 209.9475 |
| 5 | 206.9652 | 207.5388 | 207.8337 | 207.9584 |
| 10 | 203.4913 | 203.7736 | 203.9184 | 203.9796 |
| 20 | 201.7478 | 201.8878 | 201.9596 | 201.9899 |
| 50 | 200.6997 | 200.7554 | 200.7839 | 200.7960 |
| 100 | 200.3499 | 200.3777 | 200.3920 | 200.3980 |
| 200 | 200.1750 | 200.1889 | 200.1960 | 200.1990 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| o.25 | 900.000 | 1246.154 | 1551.724 | 1730.769 |
| 0.5 | 666.6667 | 818.1818 | 925.9259 | 980.3922 |
| 1 | 480.0000 | 540.0000 | 576.9231 | 594.0594 |
| 2 | 355.5556 | 378.9474 | 392.1569 | 398.0100 |
| 3 | 307.6923 | 321.4286 | 328.9474 | 332.2259 |
| 4 | 282.3529 | 291.8919 | 297.0297 | 299.2519 |
| 5 | 266.6667 | 273.9130 | 277.7778 | 279.4411 |
| 10 | 234.1463 | 237.3626 | 239.0438 | 239.760 |
| 20 | 217.2840 | 218.7845 | 219.5609 | 219.8901 |
| 50 | 206.9652 | 207.5388 | 207.8337 | 207.9584 |
| 100 | 203.4913 | 203.7736 | 203.9184 | 203.9796 |
| 200 | 201.7478 | 201.8878 | 201.9596 | 201.9899 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 100$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| o.25 | 1586.139 | 3525.183 | 9423.529 | 32040.000 |
| o.5 | 1572.549 | 3453.589 | 8911.111 | 26733.333 |
| 1 | 1546.154 | 3319.266 | 8040.000 | 20100.00 |
| 2 | 1496.296 | 3081.356 | 6733.333 | 13466.667 |
| 3 | 1450.000 | 2877.165 | 5800.000 | 10150.000 |
| 4 | 1406.897 | 2700.000 | 5100.000 | 8160.00 |
| 5 | 1366.667 | 2544.828 | 4555.556 | 6833.33 |
| 10 | 1200.000 | 1989.474 | 3000.000 | 3818.18 |
| 20 | 977.7778 | 1414.2857 | 1833.3333 | 2095.2381 |
| 50 | 666.6667 | 818.1818 | 925.9259 | 980.3922 |
| 100 | 480.0000 | 540.0000 | 576.9231 | 594.0594 |
| 200 | 355.5556 | 378.9474 | 392.1569 | 398.0100 |

**Table 2.** PRE of $\hat{\mu}_{s_P}$ with respect to $\hat{\mu}_{s_1}$ for $n = 50$, $n_1 = 20$, $n_2 = 30$ and different values of $g$, $\sigma_s^2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2$.

| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 0.1$ | | | | |
|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 378.7879 | 398.9362 | 410.1050 | 415.0066 |
| 0.5 | 317.4603 | 326.0870 | 330.6878 | 332.6680 |
| 1 | 284.5528 | 288.4615 | 290.5046 | 291.3753 |
| 2 | 267.4897 | 269.3370 | 270.2927 | 270.6980 |
| 3 | 261.7080 | 262.9151 | 263.5375 | 263.8010 |
| 4 | 258.7992 | 259.6953 | 260.1565 | 260.3516 |
| 5 | 257.0481 | 257.7605 | 258.1268 | 258.2817 |
| 10 | 253.5328 | 253.8846 | 254.0650 | 254.1413 |
| 20 | 251.7686 | 251.9434 | 252.0329 | 252.0707 |
| 50 | 250.7080 | 250.7776 | 250.8133 | 250.8283 |
| 100 | 250.3541 | 250.3888 | 250.4067 | 250.4142 |
| 200 | 250.1771 | 250.1944 | 250.2033 | 250.2071 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 958.3333 | 1326.9231 | 1652.2989 | 1842.9487 |
| 0.5 | 722.2222 | 886.3636 | 1003.0864 | 1062.0915 |
| 1 | 533.3333 | 600.0000 | 641.0256 | 660.0660 |
| 2 | 407.4074 | 434.2105 | 449.3464 | 456.0531 |
| 3 | 358.9744 | 375.0000 | 383.7719 | 387.5969 |
| 4 | 333.3333 | 344.5946 | 350.6601 | 353.2835 |
| 5 | 317.4603 | 326.0870 | 330.6878 | 332.6680 |
| 10 | 284.5528 | 288.4615 | 290.5046 | 291.3753 |
| 20 | 267.4897 | 269.3370 | 270.2927 | 270.6980 |
| 50 | 257.0481 | 257.7605 | 258.1268 | 258.2817 |
| 100 | 253.5328 | 253.8846 | 254.0650 | 254.1413 |
| 200 | 251.7686 | 251.9434 | 252.0329 | 252.0707 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 100$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 1652.640 | 3672.983 | 9818.627 | 33383.333 |
| 0.5 | 1638.889 | 3599.282 | 9287.037 | 27861.111 |
| 1 | 1612.179 | 3461.009 | 8383.333 | 20958.333 |
| 2 | 1561.728 | 3216.102 | 7027.778 | 14055.556 |
| 3 | 1514.881 | 3005.906 | 6059.524 | 10604.167 |
| 4 | 1471.264 | 2823.529 | 5333.333 | 8533.333 |
| 5 | 1430.556 | 2663.793 | 4768.519 | 7152.778 |
| 10 | 1261.905 | 2092.105 | 3154.762 | 4015.152 |
| 20 | 1037.037 | 1500.000 | 1944.444 | 2222.222 |
| 50 | 722.2222 | 886.3636 | 1003.0864 | 1062.0915 |
| 100 | 533.3333 | 600.0000 | 641.0256 | 660.0660 |
| 200 | 407.4074 | 434.2105 | 449.3464 | 456.0531 |

**Table 3.** PRE of $\hat{\mu}_{s_P}$ with respect to $\hat{\mu}_{s_1}$ for $n = 50$, $n_1 = 30$, $n_2 = 20$ and different values of $g$, $\sigma_s^2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2$.

| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 0.1$ | | | | |
|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 303.0303 | 319.1489 | 328.0840 | 332.0053 |
| 0.5 | 238.0952 | 244.5652 | 248.0159 | 249.5010 |
| 1 | 203.2520 | 206.0440 | 207.5033 | 208.1252 |
| 2 | 185.1852 | 186.4641 | 187.1257 | 187.4063 |
| 3 | 179.0634 | 179.8893 | 180.3151 | 180.4954 |
| 4 | 175.9834 | 176.5928 | 176.9064 | 177.0391 |
| 5 | 174.1294 | 174.6120 | 174.8601 | 174.9650 |
| 10 | 170.4073 | 170.6437 | 170.7650 | 170.8163 |
| 20 | 168.5393 | 168.6563 | 168.7163 | 168.7416 |
| 50 | 167.4163 | 167.4628 | 167.4866 | 167.4967 |
| 100 | 167.0416 | 167.0648 | 167.0767 | 167.0817 |
| 200 | 166.8541 | 166.8657 | 166.8717 | 166.8742 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 916.6667 | 1269.2308 | 1580.4598 | 1762.8205 |
| 0.5 | 666.6667 | 818.1818 | 925.9259 | 980.3922 |
| 1 | 466.6667 | 525.0000 | 560.8974 | 577.5578 |
| 2 | 333.3333 | 355.2632 | 367.6471 | 373.1343 |
| 3 | 282.0513 | 294.6429 | 301.5351 | 304.5404 |
| 4 | 254.9020 | 263.5135 | 268.1518 | 270.1579 |
| 5 | 238.0952 | 244.5652 | 248.0159 | 249.5010 |
| 10 | 203.2520 | 206.0440 | 207.5033 | 208.1252 |
| 20 | 185.1852 | 186.4641 | 187.1257 | 187.4063 |
| 50 | 174.1294 | 174.6120 | 174.8601 | 174.9650 |
| 100 | 170.4073 | 170.6437 | 170.7650 | 170.8163 |
| 200 | 168.5393 | 168.6563 | 168.7163 | 168.7416 |
| $\sum_{i=1}^{g} \sigma_{t_i}^2 = 100$ | | | | |
| $\sigma_s^2 \downarrow$ | $g = 2$ | $g = 3$ | $g = 5$ | $g = 10$ |
| 0.25 | 1651.815 | 3671.149 | 9813.725 | 33366.667 |
| 0.5 | 1637.255 | 3595.694 | 9277.778 | 27833.333 |
| 1 | 1608.974 | 3454.128 | 8366.667 | 20916.667 |
| 2 | 1555.556 | 3203.390 | 7000.000 | 14000.000 |
| 3 | 1505.952 | 2988.189 | 6023.810 | 10541.667 |
| 4 | 1459.770 | 2801.471 | 5291.667 | 8466.667 |
| 5 | 1416.667 | 2637.931 | 4722.222 | 7083.333 |
| 10 | 1238.095 | 2052.632 | 3095.238 | 3939.394 |
| 20 | 1000.000 | 1446.429 | 1875.000 | 2142.857 |
| 50 | 666.6667 | 818.1818 | 925.9259 | 980.3922 |
| 100 | 466.6667 | 525.0000 | 560.8974 | 577.5578 |
| 200 | 333.3333 | 355.2632 | 367.6471 | 373.1343 |

**Table 4.** Ranges of $\lambda$ for different values of $n$, $\mu_s$ and $\sigma_s^2$ when $g = 2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$.

| $\mu_s = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0.28571 \sim 1$ | $0.45946 \sim 1$ | $0.56522 \sim 1$ | $0.80000 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0.09091 \sim 1$ | $0.28571 \sim 1$ | $0.41176 \sim 1$ | $0.71429 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0.14894 \sim 1$ | $0.28571 \sim 1$ | $0.63636 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0.03846 \sim 1$ | $0.18032 \sim 1$ | $0.56522 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.33333 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.16129 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.02857 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.42857 \sim 1$ | $0.66667 \sim 1$ | $0.76471 \sim 1$ | $0.81818 \sim 1$ | $0.92307 \sim 1$ |
| 0.5 | $0.2500 \sim 1$ | $0.53846 \sim 1$ | $0.66667 \sim 1$ | $0.73913 \sim 1$ | $0.88679 \sim 1$ |
| 0.75 | $0.11111 \sim 1$ | $0.42857 \sim 1$ | $0.57895 \sim 1$ | $0.66667 \sim 1$ | $0.85185 \sim 1$ |
| 1 | $0 \sim 1$ | $0.33333 \sim 1$ | $0.50000 \sim 1$ | $0.60000 \sim 1$ | $0.81818 \sim 1$ |
| 2 | $0 \sim 1$ | $0.052632 \sim 1$ | $0.25000 \sim 1$ | $0.37931 \sim 1$ | $0.69492 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0.07143 \sim 1$ | $0.21212 \sim 1$ | $0.58730 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.08108 \sim 1$ | $0.49254 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.40845 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.09890 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 1$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.81818 \sim 1$ | $0.90476 \sim 1$ | $0.93548 \sim 1$ | $0.95122 \sim 1$ | $0.98019 \sim 1$ |
| 0.5 | $0.73913 \sim 1$ | $0.86046 \sim 1$ | $0.90476 \sim 1$ | $0.92771 \sim 1$ | $0.97044 \sim 1$ |
| 0.75 | $0.66667 \sim 1$ | $0.81818 \sim 1$ | $0.87500 \sim 1$ | $0.90476 \sim 1$ | $0.96078 \sim 1$ |
| 1 | $0.60000 \sim 1$ | $0.77778 \sim 1$ | $0.84615 \sim 1$ | $0.88235 \sim 1$ | $0.95122 \sim 1$ |
| 2 | $0.37931 \sim 1$ | $0.63265 \sim 1$ | $0.73913 \sim 1$ | $0.79775 \sim 1$ | $0.91388 \sim 1$ |
| 3 | $0.21212 \sim 1$ | $0.50943 \sim 1$ | $0.64384 \sim 1$ | $0.72043 \sim 1$ | $0.8779 \sim 1$ |
| 4 | $0.08108 \sim 1$ | $0.40351 \sim 1$ | $0.55844 \sim 1$ | $0.64948 \sim 1$ | $0.84331 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0.18812 \sim 1$ | $0.32231 \sim 1$ | $0.65975 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.42349 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 5.** Ranges of $\lambda$ for different values of $n$, $\mu_s$and$\sigma_s^2$ when $g = 5$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$.

| $\mu_s = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.26582 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.21622 \sim 1$ | $0.51261 \sim 1$ | $0.64634 \sim 1$ | $0.72249 \sim 1$ | $0.87891 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0.25000 \sim 1$ | $0.42857 \sim 1$ | $0.53846 \sim 1$ | $0.78571 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0.06509 \sim 1$ | $0.26168 \sim 1$ | $0.38996 \sim 1$ | $0.70132 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0.12971 \sim 1$ | $0.26761 \sim 1$ | $0.62455 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.37615 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.19363 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.05386 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $\sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.62338 \sim 1$ | $0.79211 \sim 1$ | $0.85644 \sim 1$ | $0.89035 \sim 1$ | $0.95465 \sim 1$ |
| 0.5 | $0.39665 \sim 1$ | $0.64474 \sim 1$ | $0.74825 \sim 1$ | $0.80505 \sim 1$ | $0.91717 \sim 1$ |
| 0.75 | $0.22549 \sim 1$ | $0.51976 \sim 1$ | $0.65198 \sim 1$ | $0.72712 \sim 1$ | $0.88111 \sim 1$ |
| 1 | $0.09170 \sim 1$ | $0.41243 \sim 1$ | $0.56576 \sim 1$ | $0.65563 \sim 1$ | $0.8464 \sim 1$ |
| 2 | $0 \sim 1$ | $0.10132 \sim 1$ | $0.29534 \sim 1$ | $0.42045 \sim 1$ | $0.71939 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0.10457 \sim 1$ | $0.24378 \sim 1$ | $0.60875 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.10619 \sim 1$ | $0.51149 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.42531 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.10914 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.71939 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $\mu_s = 1$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.89036 \sim 1$ | $0.94363 \sim 1$ | $0.96207 \sim 1$ | $0.97141 \sim 1$ | $0.98847 \sim 1$ |
| 0.5 | $0.80505 \sim 1$ | $0.89753 \sim 1$ | $0.93050 \sim 1$ | $0.94741 \sim 1$ | $0.97863 \sim 1$ |
| 0.75 | $0.72712 \sim 1$ | $0.85357 \sim 1$ | $0.89993 \sim 1$ | $0.92400 \sim 1$ | $0.96889 \sim 1$ |
| 1 | $0.65563 \sim 1$ | $0.81159 \sim 1$ | $0.87032 \sim 1$ | $0.90114 \sim 1$ | $0.95925 \sim 1$ |
| 2 | $0.42045 \sim 1$ | $0.66113 \sim 1$ | $0.76056 \sim 1$ | $0.81488 \sim 1$ | $0.92159 \sim 1$ |
| 3 | $0.24378 \sim 1$ | $0.53374 \sim 1$ | $0.66297 \sim 1$ | $0.73611 \sim 1$ | $0.88537 \sim 1$ |
| 4 | $0.10619 \sim 1$ | $0.424501 \sim 1$ | $0.57563 \sim 1$ | $0.66389 \sim 1$ | $0.85048 \sim 1$ |
| 5 | $0 \sim 1$ | $0.329787 \sim 1$ | $0.49701 \sim 1$ | $0.59744 \sim 1$ | $0.81686 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0.19809 \sim 1$ | $0.33156 \sim 1$ | $0.66556 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0.76056 \sim 1$ | $0 \sim 1$ | $0.42775 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 6.** Ranges of $K$ for different values of $n$, $w$ and $\sigma_s^2$ when $\mu_s = 0.1$ , $g = 5$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$.

| $w = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.9$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 7.** Ranges of $K$ for different values of $n$, $w$ and $\sigma_s^2$ when $\mu_s = 0.5$, $g = 5$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$.

| $w = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.21622 \sim 1$ | $0.59011 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.35135 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.17493 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.03926 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.0373 \sim 1$ | $0.36612 \sim 1$ | $0.52749 \sim 1$ | $0.62338 \sim 1$ | $0.83016 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0.07296 \sim 1$ | $0.26903 \sim 1$ | $0.39665 \sim 1$ | $0.70532 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0.08538 \sim 1$ | $0.22549 \sim 1$ | $0.59642 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.09170 \sim 1$ | $0.50060 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.21006776 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.01378751 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.9$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.55470 \sim 1$ | $0.74946 \sim 1$ | $0.82569 \sim 1$ | $0.86636 \sim 1$ | $0.94431 \sim 1$ |
| 0.5 | $0.30435 \sim 1$ | $0.57894 \sim 1$ | $0.69811 \sim 1$ | $0.76471 \sim 1$ | $0.89873 \sim 1$ |
| 0.75 | $0.12344 \sim 1$ | $0.43872 \sim 1$ | $0.58719 \sim 1$ | $0.67355 \sim 1$ | $0.85525 \sim 1$ |
| 1 | $0 \sim 1$ | $0.32137 \sim 1$ | $0.48988 \sim 1$ | $0.59136 \sim 1$ | $0.81370 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0.19646 \sim 1$ | $0.33005 \sim 1$ | $0.66461 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.14245 \sim 1$ | $0.53817 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.00124 \sim 1$ | $0.42958 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.3353 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.00422 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 8.** Range of $K$ for different values of $n$, $w$ and $\sigma_s^2$ when $\mu_s = 1$, $g = 5$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$

| $w = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.26582 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.21622 \sim 1$ | $0.512606 \sim 1$ | $0.64634 \sim 1$ | $0.72249 \sim 1$ | $0.87891 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0.25000 \sim 1$ | $0.42857 \sim 1$ | $0.53846 \sim 1$ | $0.78571 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0.06509 \sim 1$ | $0.26168 \sim 1$ | $0.38996 \sim 1$ | $0.70132 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0.12971 \sim 1$ | $0.26761 \sim 1$ | $0.62455 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.37615 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.19363 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.05386 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.62338 \sim 1$ | $0.79211 \sim 1$ | $0.85644 \sim 1$ | $0.89036 \sim 1$ | $0.95465 \sim 1$ |
| 0.5 | $0.39665 \sim 1$ | $0.64473 \sim 1$ | $0.74825 \sim 1$ | $0.80505 \sim 1$ | $0.91718 \sim 1$ |
| 0.75 | $0.22549 \sim 1$ | $0.51976 \sim 1$ | $0.65198 \sim 1$ | $0.72712 \sim 1$ | $0.88111 \sim 1$ |
| 1 | $0.09170 \sim 1$ | $0.41243 \sim 1$ | $0.56576 \sim 1$ | $0.65563 \sim 1$ | $0.84638 \sim 1$ |
| 2 | $0 \sim 1$ | $0.10132 \sim 1$ | $0.29534 \sim 1$ | $0.42045 \sim 1$ | $0.71939 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0.10457 \sim 1$ | $0.24378 \sim 1$ | $0.60875 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.10619 \sim 1$ | $0.51149 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.42531 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.10913 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.9$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.86636 \sim 1$ | $0.93087 \sim 1$ | $0.95338 \sim 1$ | $0.96482 \sim 1$ | $0.98578 \sim 1$ |
| 0.5 | $0.76471 \sim 1$ | $0.87500 \sim 1$ | $0.91489 \sim 1$ | $0.93548 \sim 1$ | $0.97368 \sim 1$ |
| 0.75 | $0.67355 \sim 1$ | $0.82227 \sim 1$ | $0.87789 \sim 1$ | $0.90700 \sim 1$ | $0.96173 \sim 1$ |
| 1 | $0.59136 \sim 1$ | $0.77242 \sim 1$ | $0.84230 \sim 1$ | $0.87935 \sim 1$ | $0.94992 \sim 1$ |
| 2 | $0.33004 \sim 1$ | $0.59763 \sim 1$ | $0.71247 \sim 1$ | $0.77632 \sim 1$ | $0.90409 \sim 1$ |
| 3 | $0.14245 \sim 1$ | $0.45422 \sim 1$ | $0.59974 \sim 1$ | $0.68399 \sim 1$ | $0.86036 \sim 1$ |
| 4 | $0.00124 \sim 1$ | $0.33443 \sim 1$ | $0.50093 \sim 1$ | $0.60079 \sim 1$ | $0.81859 \sim 1$ |
| 5 | $0 \sim 1$ | $0.2328767 \sim 1$ | $0.41362 \sim 1$ | $0.5254237 \sim 1$ | $0.7786561 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0.09509 \sim 1$ | $\sim 1$ | $0.60269 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.33796 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 9.** Range of $K$ for different values of $n$, $w$ and$\sigma_s^2$ when $\mu_s = 0.1$ , $g = 2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$

| $w = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.9$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |

**Table 10.** Range of $K$ for different values of $n$, $w$ and$\sigma_s^2$ when $\mu_{t_i} = 1$ , $g = 2$ and $\sum_{i=1}^{g} \sigma_{t_i}^2 = 1$

| $w = 0.1$ | | | | | |
|---|---|---|---|---|---|
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.3$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0 \sim 1$ | $0.28571 \sim 1$ | $0.45946 \sim 1$ | $0.56521 \sim 1$ | $0.80000 \sim 1$ |
| 0.5 | $0 \sim 1$ | $0.09091 \sim 1$ | $0.28571 \sim 1$ | $0.41177 \sim 1$ | $0.71429 \sim 1$ |
| 0.75 | $0 \sim 1$ | $0 \sim 1$ | $0.14894 \sim 1$ | $0.28571 \sim 1$ | $0.63636 \sim 1$ |
| 1 | $0 \sim 1$ | $0 \sim 1$ | $0.03846 \sim 1$ | $0.18033 \sim 1$ | $0.56522 \sim 1$ |
| 2 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.16129 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.02857 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.5$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.42857 \sim 1$ | $0.666667 \sim 1$ | $0.76471 \sim 1$ | $0.81818 \sim 1$ | $0.92307 \sim 1$ |
| 0.5 | $0.25000 \sim 1$ | $0.53846 \sim 1$ | $0.66667 \sim 1$ | $0.73913 \sim 1$ | $0.88679 \sim 1$ |
| 0.75 | $0.11111 \sim 1$ | $0.42857 \sim 1$ | $0.57895 \sim 1$ | $0.66667 \sim 1$ | $0.85185 \sim 1$ |
| 1 | $0 \sim 1$ | $0.33333 \sim 1$ | $0.50000 \sim 1$ | $0.60000 \sim 1$ | $0.81818 \sim 1$ |
| 2 | $0 \sim 1$ | $0.05263 \sim 1$ | $0.25000 \sim 1$ | $0.37931 \sim 1$ | $0.69492 \sim 1$ |
| 3 | $0 \sim 1$ | $0 \sim 1$ | $0.07143 \sim 1$ | $0.21212 \sim 1$ | $0.58730 \sim 1$ |
| 4 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.08108 \sim 1$ | $0.49253 \sim 1$ |
| 5 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.40845 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.09890 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |
| $w = 0.9$ | | | | | |
| $\sigma_s^2 \downarrow$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ | $n = 50$ |
| 0.25 | $0.78022 \sim 1$ | $0.88372 \sim 1$ | $0.92095 \sim 1$ | $0.94012 \sim 1$ | $0.97561 \sim 1$ |
| 0.5 | $0.6875 \sim 1$ | $0.83051 \sim 1$ | $0.88372 \sim 1$ | $0.91150 \sim 1$ | $0.96364 \sim 1$ |
| 0.75 | $0.60396 \sim 1$ | $0.78022 \sim 1$ | $0.84791 \sim 1$ | $0.88372 \sim 1$ | $0.95180 \sim 1$ |
| 1 | $0.52830 \sim 1$ | $0.73262 \sim 1$ | $0.81343 \sim 1$ | $0.85673 \sim 1$ | $0.94012 \sim 1$ |
| 2 | $0.28571 \sim 1$ | $0.56521 \sim 1$ | $0.687500 \sim 1$ | $0.75600 \sim 1$ | $0.89473 \sim 1$ |
| 3 | $0.10959 \sim 1$ | $0.42731 \sim 1$ | $0.57792 \sim 1$ | $0.66581 \sim 1$ | $0.85142 \sim 1$ |
| 4 | $0 \sim 1$ | $0.31174 \sim 1$ | $0.48171 \sim 1$ | $0.58435 \sim 1$ | $0.81006 \sim 1$ |
| 5 | $0 \sim 1$ | $0.21348 \sim 1$ | $0.39655 \sim 1$ | $0.51049 \sim 1$ | $0.77049 \sim 1$ |
| 10 | $0 \sim 1$ | $0 \sim 1$ | $0.08482 \sim 1$ | $0.22495 \sim 1$ | $0.59606 \sim 1$ |
| 20 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0.33333 \sim 1$ |
| 50 | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ | $0 \sim 1$ |