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Properties of future lifetime distributions and estimation

Harmanpreet Singh Kapoor* and Kanchan Jain†‡

Abstract

Distributional properties of continuous future lifetime of an individual aged x have been studied. The expressions for moments, coefficients of skewness and kurtosis have been derived when newborn's lifetime follows different distributions. The maximum likelihood (ML) estimation of the parameters of future lifetime's distributions has been explored for whole life and term assurance contracts. Simulations have been carried out to find the ML estimates and the corresponding root mean square errors.

Keywords: Future Lifetime, Whole life assurance, Term assurance, Maximum Likelihood Estimation, Censoring.

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1. Introduction

Financial and actuarial models are built around the future lifetime random variable of individuals. The distribution of future lifetime is required for statistical measurements, analysis and inference. There are two approaches for modeling this distribution. In the first approach, a theoretical statistical distribution is assumed for the future lifetime associated with a stochastic status. In the second, empirical or observed survival and mortality data are used for constructing life mortality tables [1, 3, 4, 12].

In Actuarial Science, the future lifetime of a life aged x plays a significant role. The importance of its distribution is highlighted when dealing with continuous whole life and term assurance contracts where the death benefit is paid at the moment of death. It also plays a role while studying continuous n-year temporary and n-year deferred whole life annuities. The distribution of future lifetime and its properties are utilized for finding the

*Department of Statistics, Panjab University, Chandigarh-160014 (India), Email: harman.pu.87@gmail.com

†Department of Statistics, Panjab University, Chandigarh-160014 (India), Email: jaink14@gmail.com

‡Corresponding Author.

expected present values (Actuarial values) and the variances of the present value random variables for benefits or payments in case of continuous life annuities/assurances. The force of mortality, alternatively known as failure rate/hazard rate function in reliability is defined through its density function [1, 2, 3]. Let X be the new born's age at death and $f(x)$, $F(x)$ and $S(x)$ be the corresponding probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) respectively. (x) symbolizes an individual aged x and $T(x)$ denotes the future lifetime of (x) at the inception of an insurance contract. The survival function of $T(x)$ is written as

$$(1.1) \qquad {}_tp_x = P[T(x) > t] = P(X > x + t \mid X > x) = \frac{S(x+t)}{S(x)}.$$

${}_tp_x$ gives the survival probability that (x) survives for more than t years.
For $t \geq 0, {}_tq_x = 1 - {}_tp_x$ gives the distribution function of $T(x)$ and represents the probability that (x) does not survive beyond age t. The force of mortality is an important and fundamental concept in modelling future lifetime and is defined as

$$(1.2) \qquad \mu_{x+t} = lim_{dt \to 0^+} \frac{1}{dt} P[t < T(x) \leq t + dt \mid T(x) > t] = \frac{f(x+t)}{S(x+t)}.$$

In reliability theory, μ_x is known as failure rate or hazard rate function and $T(x)$ is termed as residual life. One can refer to [2, 5] for the notations given above.
If the policies are not surrendered or do not lapse as in case of whole life assurance, then the observed future lifetimes are uncensored. The censored observations arise if the policyholders surrender the policy before the expiry of the contract. In some cases, the policy lapses due to non-payment of regular premium amount. In term assurance contract or n-year temporary policy, the death benefit is paid only if the insured dies within the term of the contract. In this case, the future lifetime of those individuals who survive the term of the policy, will not be known to the company. The same is true for an n-year temporary life annuity where annuity payments stop when an individual dies within or after n years, whichever is earlier.
In literature, many authors have explored the ageing properties of $T(x)$ [11]. The distributional and ageing properties of curtate future lifetime $K(x)$, the greatest integer less than or equal to $T(x)$ have been already studied [6]. To the best of our knowledge, the distributional properties of $T(x)$ have not been studied so far. Hence, we study the distributional forms of $T(x)$ for different distributions of X and explore the Maximum Likelihood Estimation (MLE) for uncensored and censored cases. The distributions chosen for X are Gompertz, exponential, Weibull, Pareto Type I, exponentiated exponential and burr [7, 8]. Gompertz and Weibull are considered to be very good models for studying human life-length. Pareto is a particular case of Benktander Gibrat distribution [9] and Burr distribution is considered since Pareto type II is its special case.
The expressions for pdfs, n^{th} order moments, coefficients of skewness and kurtosis of different distributions of $T(x)$ are derived in Section 2. In Sections 3 and 4, we discuss the maximum likelihood (ML) estimation of parameters future lifetime's distributions for uncensored and censored observations in case of whole life and n-year term assurance contracts. Section 5 consists of simulation results for ML estimation using numerical techniques. The root mean square errors (RMSEs) are also reported.

2. Distributional properties of future lifetime

In this section, we derive the n^{th} order moment of $T(x)$ which helps in finding its expectation, variance, coefficients of skewness and kurtosis. The probability density function of $T(x)$ is given by

$$f_{T(x)}(t) = -\frac{d}{dt}\,{}_tp_x = {}_tp_x\mu_{x+t}. \tag{2.1}$$

The cdf of $T(x)$ is ${}_tq_x = \int_0^t {}_sp_x\mu_{x+s}\,ds$ [2].

If we assume that the n^{th} order moment of $T(x)$,that is, $E[(T(x))^n]$ exists, then

$$\begin{aligned} E[(T(x))^n] &= \int_0^\infty t^n\,{}_tp_x\mu_{x+t}\,dt \\ &= \int_0^t y^n\,{}_yp_x\mu_{x+y}\,dy + \int_t^\infty y^n\,{}_yp_x\mu_{x+y}\,dy \\ &\geq \int_0^t y^n\,{}_yp_x\mu_{x+y}\,dy + (t^n)\int_t^\infty {}_yp_x\mu_{x+y}\,dy \end{aligned}$$

This implies that

$$\begin{aligned} E[(T(x))^n] - \int_0^t y^n\,{}_yp_x\mu_{x+y}\,dy &\geq (t^n)\int_t^\infty {}_yp_x\mu_{x+y}\,dy \\ &\geq (t^n)\,{}_tp_x \end{aligned} \tag{2.2}$$

As $t \to \infty$, LHS of (2.2) tends to zero if $E[(T(x))^n]$ exists and is finite.
Hence $lim_{t\to\infty} t^n\,{}_tp_x \leq 0$.
On the other hand, $t^n\,{}_tp_x \geq 0$. Therefore $0 \leq lim_{t\to\infty} t^n\,{}_tp_x \leq 0$.
This gives that $lim_{t\to\infty} t^n\,{}_tp_x = 0$. Hence

$$E[(T(x))^n] = \int_0^\infty t^n\,d(-{}_tp_x) = \int_0^\infty nt^{n-1}\,{}_tp_x\,dt. \tag{2.3}$$

For $n = 1$, if the expected value of $T(x)$ is assumed to exist, then

$$E[T(x)] = \int_0^\infty {}_tp_x dt$$

as the existence of $E[T(x)]$ implies that $\lim_{t\to\infty}\ t(-{}_tp_x) = 0$. ([2], Chapter 3, pp 68).

$E[T(x)]$ is called the complete expectation of life and is denoted by $\mathring{e}_x$. In reliability theory, this is termed as the mean residual life function.
We also get

$$V[T(x)] = 2\int_0^\infty t({}_tp_x)\,dt - \left(\int_0^\infty {}_tp_x\right)^2. \tag{2.4}$$

The expectation and variance of $T(x)$ are same as derived in [2, 3].
If μ_k denotes the k^{th} order central moment of $T(x)$, then coefficient of skewness is $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$ where $\mu_2 = V[T(x)]$ and

$$\mu_3 = \int_0^\infty 3t^2\,{}_tp_x\,dt - 3\int_0^\infty {}_tp_x\,dt\int_0^\infty 2t\,{}_tp_x\,dt + 2\left(\int_0^\infty {}_tp_x\,dt\right)^3. \tag{2.5}$$

A measure of kurtosis is given by $\gamma_2 = \frac{\mu_4}{\mu_2^2}$
where

$$\begin{aligned} \mu_4 = &\int_0^\infty 4t^3\,{}_tp_x\,dt - 4\left(\int_0^\infty {}_tp_x\,dt\right)\left(\int_0^\infty 3t^2\,{}_tp_x\,dt\right) \\ &+ 6\left(\int_0^\infty 2t\,{}_tp_x\,dt\right)\left(\int_0^\infty {}_tp_x\,dt\right)^2 - 3\left(\int_0^\infty {}_tp_x\,dt\right)^4. \end{aligned} \tag{2.6}$$

Using (1.1) and (2.1), we can find the probability density function (pdf) and force of mortality of $T(x)$ for different distributions of X. For x, $t > 0$, the expressions for survival probability $({}_tp_x)$ and force of mortality (μ_{x+t}) are listed in Table 1.

Table 1: Survival function and force of mortality of $T(x)$

Distribution of X	$\mathbf{{}_tp_x}$	$\boldsymbol{\mu_{x+t}}$
$Exp(\lambda)$, $\lambda > 0$	$e^{-\lambda t}$	λ
Gompertz (α, λ), $\alpha > 0,\ \lambda > 1$	$e^{-(\alpha\lambda^x(\lambda^t-1))/log\lambda}$	$\alpha\lambda^{(x+t)}$
Weibull (α, λ), $\alpha > 0,\ \lambda > 0$	$\dfrac{e^{-(\lambda(x+t))^\alpha}}{e^{-(\lambda x)^\alpha}}$	$\alpha\lambda^\alpha(x+t)^{\alpha-1}$
Pareto (θ), $\theta > 0$	$\dfrac{x^\theta}{(x+t)^\theta}$	$\dfrac{\theta}{(x+t)}$
Burr (γ, θ), $\gamma > 0,\ \theta > 0$	$\dfrac{(1+x^\gamma)^\theta}{(1+(x+t)^\gamma)^\theta}$	$\dfrac{\gamma\theta(1+x^\gamma)^\theta(x+t)^{\gamma-1}}{(1+(x+t)^\gamma)}$
Exponentiated Exp. (α, λ), $\alpha > 0,\ \lambda > 0$	$\dfrac{1-(1-e^{-\lambda(x+t)})^\alpha}{1-(1-e^{-\lambda x})^\alpha}$	$\dfrac{\alpha(1-e^{-\lambda(x+t)})^{\alpha-1}\lambda e^{-\lambda(x+t)}}{1-(1-e^{-\lambda(x+t)})^\alpha}$

From Table 1, we can conclude that :

- For $X \sim$ Exp (λ) and all λ, ${}_tp_x$ is independent of x, the initial age;
- For $X \sim$ Gompertz (α, λ), ${}_tp_x$ is a decreasing function of x for all $\alpha > 0$ and $\lambda > 1$;
- For $X \sim$ Weibull (α, λ), ${}_tp_x$ is an increasing (decreasing) function of x for all λ and $0 < \alpha < 1$ $(\alpha > 1)$;
- For $X \sim$ Pareto (θ), ${}_tp_x$ is an increasing function of x for all θ;
- For $X \sim$ Burr (γ, θ), ${}_tp_x$ is an increasing function of x for all θ, γ;
- For $X \sim$ Exponentiated Exp (α, λ), ${}_tp_x$ is an increasing (decreasing) function of x for all α and $\lambda > 1$ $(0 < \lambda < 1)$.

Since ${}_tp_x = P[T(x) > t]$ is the survival function of $T(x)$, hence it is decreasing in t for the distribution under consideration. The pdfs of future lifetime distributions of X mentioned in Table 1 can be obtained by multiplying ${}_tp_x$ and μ_{x+t}.

In the sequel, the distributions of $T(x)$ shall be labeled as future lifetime distributions.

3. Estimation of the parameters for uncensored case

In insurance sector, a whole life assurance policy provides cover till the death of the insured. We assume that the insured takes out life assurance by paying either a single premium or a series of premiums and does not surrender the policy till his death. Suppose factory workers buy a whole life assurance policy from the insurer, by paying a single premium. Insurer is liable to pay the sum assured at the moment of death of the individual. Hence, it is important for the insurer to predict the future lifetime of the insured for deciding the premium amount so that the assurance contract is mutually advantageous. The future lifetime $T(x)$ of the insured aged x can be modeled by assuming some continuous distribution of X and for doing this accurately, the parameters of the distribution of $T(x)$ need to be estimated and this is done through method of maximum likelihood (ML) estimation. As the insurance companies are more interested in future lifetime modelling, it will be observed that the MLEs based on future lifetime can be directly computed by knowing the age of the policyholder at the time of initiation of the policy.

In the following subsections, we discuss the Maximum Likelihood (ML) estimation of parameters of future lifetime distributions when X follows exponential, Weibull, Pareto, burr, exponentiated exponential and Gompertz distributions. For a sample of n individuals aged x, let $t_i(x)$ denote the future lifetime of the i^{th} individual. For exponential and Pareto future lifetime distributions, the ML estimators of the parameters can be written in a closed form whereas for other distributions under study, the non-linear equations result while maximizing the log-likelihood. As it is difficult to solve these equations analytically, the estimates are obtained in Section 5 by using numerical approximation through BFGS in R.

3.1. Exponential future lifetime distribution. If X follows Exp (λ), then the pdf of $T(x)$ is

$$f_T(x)(t(x),\lambda) = \lambda e^{-\lambda t(x)}, \lambda > 0.$$

The corresponding likelihood function is

$$L(t(x),\lambda) = \prod_{i=1}^{n} f_{T(x)}(t_i(x),\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i(x)}.$$

Maximizing $LogL(t(x),\lambda)$ with respect to λ gives the MLE of λ as

$$\hat{\lambda} = \frac{1}{\overline{t(x)}} \text{ where } \overline{t(x)} = \frac{1}{n}\sum_{i=1}^{n} t_i(x).$$

Hence ML estimator of λ for exponential future lifetime distribution is of similar type as for exponential distribution.

3.2. Pareto type I future lifetime distribution. Let X follow Pareto (θ), then the likelihood function for T(x) is

$$L(t(x),\theta) = \prod_{i=1}^{n} \frac{\theta x^{\theta}}{(x+t_i(x))^{\theta+1}}, \theta > 0, \text{ and } x > 0.$$

Taking log on both sides and differentiating with respect to θ, we get

$$\frac{dlog(L(t(x),\theta))}{d\theta} = n\left(\frac{1}{\theta} + logx\right) - \sum_{i=1}^{n} log(x+t_i(x)). \tag{3.1}$$

Equating (3.1) to zero and solving, the MLE of θ is found to be

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} log(\dfrac{x+t_i(x)}{x})}.$$

3.3. Weibull future lifetime distribution. For X following Weibull (α, λ), the pdf of $T(x)$ is given by

$$f_T(x)(t(x), \lambda, \alpha) = \alpha\lambda^{\alpha}(x+t(x))^{(\alpha-1)}e^{-\lambda^{\alpha}((x+t(x))^{\alpha}-x^{\alpha})}, \alpha, \lambda > 0, t(x) > 0.$$

The corresponding log-likelihood can be written as

$$log(L(t(x), \lambda, \alpha)) = \sum_{i=1}^{n} log\{\alpha\lambda^{\alpha}(x+t_i(x))^{\alpha-1}e^{-\lambda^{\alpha}((x+t_i(x))^{\alpha}-x^{\alpha})}\}.$$

ML estimators of λ and α can be obtained by equating the derivatives of loglikelihood with respect to λ and α to zero and solving where

$$\begin{aligned}\frac{dlog(L(t(x), \lambda, \alpha))}{d\lambda} &= \frac{n\alpha}{\lambda} - \alpha\lambda^{\alpha-1}\left(\sum_{i=1}^{n}(x+t_i(x))^{\alpha} - nx^{\alpha}\right);\\ \frac{dlog(L(t(x), \lambda, \alpha))}{d\alpha} &= n\left(\frac{1}{\alpha} + log\lambda + (\lambda x)^{\alpha}log(\lambda x)\right)\\ &\quad + \sum_{i=1}^{n} log(x+t_i(x))\left(1-(\lambda(x+t_i(x)))^{\alpha}\right)\\ &\quad - log\lambda\sum_{i=1}^{n}(\lambda(x+t_i(x)))^{\alpha}.\end{aligned}$$

3.4. Burr future lifetime distribution. If X follows Burr (γ, θ) then

$$f_{T(x)}(t(x), \theta, \gamma) = \frac{\gamma\theta(1+x^{\gamma})^{\theta}(x+t(x))^{\gamma-1}}{(1+(x+t(x))^{\gamma})^{\theta+1}}, \gamma > 0 \text{ and } \theta > 0$$

If $L(t(x), \theta, \gamma)$ denotes the likelihood function,then

$$\begin{aligned}\frac{dlog(L(t(x), \theta, \gamma))}{d\gamma} &= \frac{n}{\gamma} + \frac{n\theta x^{\gamma}logx}{(1+x^{\gamma})}\\ &\quad + \sum_{i=1}^{n} log(x+t_i(x))\left(1 - \frac{(x+t_i(x))^{\gamma}(\theta+1)}{1+(x+t_i(x))^{\gamma}}\right);\end{aligned} \tag{3.2}$$

$$\frac{dlog(L(t(x), \theta, \gamma))}{d\theta} = \frac{n}{\theta} + nlog(1+x^{\gamma}) - \sum_{i=1}^{n} log(1+(x+t_i(x))^{\gamma}). \tag{3.3}$$

The ML estimators of θ and γ shall be obtained by equating (3.2) and (3.3) to zero and finding numerical solutions to the non-linear equations in Section 5.

3.5. Exponentiated exponential future lifetime distribution. Let X follow Exponentiated Exponential (α, λ), then the corresponding likelihood function is

$$L(t(x), \alpha, \lambda) = \prod_{i=1}^{n} \frac{\alpha\lambda(1-e^{-\lambda(x+t_i(x))})^{\alpha-1}e^{-\lambda(x+t_i(x))}}{1-(1-e^{-\lambda x})^{\alpha}}.$$

The ML estimators of α and λ can be obtained using the non-linear expressions given by

$$\begin{aligned}\frac{dlog(L(t(x), \lambda, \alpha))}{d\lambda} =& n\left(\frac{1}{\lambda} + \frac{\alpha x e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}}{1-(1-e^{-\lambda x})^{\alpha}}\right)\\ &+ \sum_{i=1}^{n}(x+t_i(x))\left(\frac{(\alpha-1)e^{-\lambda(x+t_i(x))}}{1-e^{-\lambda(x+t_i(x))}} - 1\right);\end{aligned} \tag{3.4}$$

$$\frac{dlog(L(t(x),\lambda,\alpha))}{d\alpha} = n\left(\frac{1}{\alpha} + \frac{(1-e^{-\lambda x})^{\alpha} log(1-e^{-\lambda x})}{1-(1-e^{-\lambda x})^{\alpha}}\right) + \sum_{i=1}^{n} log(1-e^{-\lambda(x+t_i(x))}). \tag{3.5}$$

3.6. Gompertz future lifetime distribution. The pdf of $T(x)$, when X follows Gompertz (α,λ), is

$$f_T(x)(t(x)) = \alpha\lambda^{(x+t(x))} e^{\frac{-\alpha\lambda^x(\lambda^{t(x)}-1)}{log\lambda}} \quad \alpha>0, \lambda>1, t(x) \text{ and } x>0.$$

$$L(t(x),\alpha,\lambda) = \prod_{i=1}^{n} \alpha\lambda^{x+t_i(x)} e^{\frac{-\alpha\lambda^x(\lambda^{t_i(x)}-1)}{log\lambda}}$$

is the corresponding likelihood function.
Hence

$$\frac{dlog(L(t(x),\alpha,\lambda))}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{\lambda^x(\lambda^{t_i(x)}-1)}{log\lambda}; \tag{3.6}$$

$$\frac{dlog(L(t(x),\alpha,\lambda))}{d\lambda} = \frac{\alpha\lambda^{x-1}}{log\lambda}\sum_{i=1}^{n}\left((\lambda^{t_i(x)}-1)(\frac{1}{log\lambda}-x) - \lambda^{t_i(x)}t_i(x)\right) + \sum_{i=1}^{n}\frac{(x+t_i(x))}{\lambda}. \tag{3.7}$$

Equating (3.6) and (3.7) to zero, the MLEs of α and λ are obtained in simulations section using numerical approximation.

The following section discusses the estimation of parameters of different continuous future lifetime distributions when some observations are censored in a given set of observations.

4. Estimation of parameters in censored case

In insurance sector, it is possible that some insured persons do not continue the contract due to some reasons. They surrender the policy or don't pay the renewal premium after few years to draw benefit from the contract. Such individuals are considered to be censored and their time of death is not known. The time of occurrence of such events is not known in advance and is a random variable, named as censoring variable. This type of censoring is known as right censoring.
For an individual aged x, let the random variable $C(x)$ be the time at which an individual either leaves the contract or stops paying premium and is randomly censored. $T(x)$ and $C(x)$ are assumed to be independent. In the subsequent discussion, we study the estimation under term assurance contract and whole life assurance contract in case of censoring.

4.1. Estimation for term assurance contract. We consider a term assurance contract, tenable for t_0 years. Under this contract, an insurer pays the sum assured only if insured dies within the term of the contract. This means that at the time of inception of the policy, a policy holder aged x gets the benefit if he dies between x and $x+t_0$. Hence, the insured ones surviving at time t_0 and those who surrender the policy due to one or the other reason are lost to further follow up by the company. Surviving persons are right censored at a predetermined time t_0 and those who surrender the policy are randomly

censored. We attempt to estimate the unknown parameters of the distributionsof future lifetime. For this type of situation, we attempt to estimate the unknown parameters of the distributions of future lifetime.
Let the random variable Z be defined as

$$Z = min(T(x), C(x), t_0) = \begin{cases} T(x) & if \;\; T(x) < C(x) \;\; and \;\; T(x) < t_0; \\ C(x) & if \;\; C(x) < T(x) \;\; and \;\; C(x) < t_0; \\ t_0 & if \;\; t_0 < T(x) \;\; and \;\; t_0 < C(x). \end{cases}$$

For i^{th} life aged x, we take $t_i(x)$ to be the future lifetime and $c_i(x)$, the censoring time. This means that the policy is either surrendered or gets lapsed. We assume that, there are n individuals aged x in the cohort and a total of m leave the cohort due to death or some other reason. Out of m, r is assumed to be the random no. of individuals that leave the cohort due to death. Hence, there will be $(n-m)$ observations in the cohort at time t_0 and the likelihood function [10] can be written as

$$L(z) = \prod_{i=1}^{r} f_T(x)(t_i(x))\bar{G}_{C(x)}(t_i(x)) \prod_{i=r+1}^{m} g_{C(x)}(c_i(x))\bar{F}_{T(x)}(c_i(x))$$
$$\left[\bar{F}_{T(x)}(t_0)\bar{G}_{C(x)}(t_0)\right]^{n-m} \tag{4.1}$$

where
$\bar{F}_{T(x)}(.)$: survival function of $T(x)$,
$\bar{G}_{C(x)}(.)$: survival function of $C(x)$,
$g_{C(x)}(.)$: probability density function of $C(x)$.
In the following discussion, we explore the maximum likelihood estimation of the parameters of exponential, Weibull, Pareto, burr and Gompertz future lifetime distributions when some observations are censored. For all the cases, the distribution of $C(x)$ is assumed to follow $U(0,\eta)$ where the value of η depends on the percentage of censoring.

4.1.1. *Exponential future lifetime distribution.* If λ is the parameter of exponential distribution followed by X, the likelihood function for $T(x)$ using (4.1), is written as

$$L(z,\lambda) = \lambda^r e^{-\lambda\sum_{i=1}^{r} t_i(x)} \left(\prod_{i=1}^{r}\left(1-\frac{t_i(x)}{\eta}\right)\right)\left(\frac{e^{-\lambda\sum_{i=r+1}^{m} c_i(x)}}{\eta^{m-r}}\right)$$
$$\left(e^{-\lambda t_0}\left(1-\frac{t_0(x)}{\eta}\right)\right)^{n-m}.$$

Maximizing the loglikelihood, we have

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} t_i(x) + \sum_{i=r+1}^{m} c_i(x) + (n-m)t_0}.$$

4.1.2. *Weibull future lifetime distribution.* Let X follow Weibull (α,λ) and $C(x)$ follow $U(0,\eta)$, then

$$L(z,\alpha,\lambda) = e^{n(\lambda x)^\alpha} \prod_{i=1}^{r} \alpha\lambda^\alpha \, (x+t_i(x))^{\alpha-1} \, e^{-(\lambda(x+t_i(x)))^\alpha}\left(1-\frac{t_i(x)}{\eta}\right)$$
$$\prod_{i=r+1}^{m} \frac{e^{-(\lambda(x+c_i(x)))^\alpha}}{\eta}\left(e^{-(\lambda(x+t_0))^\alpha}\left(1-\frac{t_0(x)}{\eta}\right)\right)^{n-m}.$$

Hence

$$\frac{dLog(L(z,\alpha,\lambda))}{d\alpha} = \frac{r}{\alpha} + rlog\lambda + n(\lambda x)^{\alpha} log(\lambda x)$$

$$+ \sum_{i=1}^{r} (log(x + t_i(x)) - \lambda^{\alpha}(x + t_i(x))^{\alpha} log(\lambda(x + t_i(x))))$$

$$- \sum_{i=r+1}^{m} (\lambda(x + c_i(x)))^{\alpha} log(\lambda(x + c_i(x)))$$

$$- (n-m)(\lambda(x + t_0))^{\alpha} log(\lambda(x + t_0)); \tag{4.2}$$

$$\frac{dLog(L(z,\alpha,\lambda))}{d\lambda} = \frac{\alpha r}{\lambda} + \alpha\lambda^{\alpha-1}(nx^{\alpha} - (n-m)(x + t_0)^{\alpha})$$

$$- \alpha\lambda^{\alpha-1}(\sum_{i=1}^{r}(x + t_i(x))^{\alpha} + \sum_{i=r+1}^{m}(x + c_i(x))^{\alpha}) \tag{4.3}$$

ML estimates of α and λ are found by solving the non-linear equations obtained as a result of equating (4.2) and (4.3) to zero. This is done using numerical approximation and corresponding RMSEs are calculated in Section 5.

4.1.3. *Pareto type I future lifetime distribution.* If X follows Pareto (θ), $\theta > 0$, the MLE of θ is

$$\hat{\theta} = \frac{r}{\sum_{i=1}^{r} log(x + t_i(x)) - nlogx + \sum_{i=r+1}^{m} log(x + c_i(x)) + (n-m)log(x + t_0)}.$$

4.1.4. *Burr future lifetime distribution.* For $X \sim$Burr(γ, θ) $\gamma, \theta > 0$, the ML estimators of θ and γ can be obtained using non-linear equations given by

$$\frac{dLog(L(z,\gamma,\theta))}{d\theta} = \frac{r}{\theta} + nlog(1 + x^{\gamma}) - \sum_{i=1}^{r} log(1 + (x + t_i(x))^{\gamma})$$

$$- \sum_{i=r+1}^{m} log(1 + (x + c_i(x))^{\gamma}) - (n-m)log(1 + (x + t_0)^{\gamma});$$

$$\frac{dLog(L(z,\gamma,\theta))}{d\gamma} = \frac{r}{\gamma} + \frac{n\theta x^{\gamma} logx}{(1 + x^{\gamma})} + \sum_{i=1}^{r} log(x + t_i(x)) \left(\frac{1 - \theta(x + t_i(x))^{\gamma}}{(1 + (x + t_i(x))^{\gamma})}\right)$$

$$- \theta \sum_{i=r+1}^{m} \frac{(x + c_i(x))^{\gamma} log(x + c_i(x))}{(1 + (x + c_i(x))^{\gamma})}$$

$$- \frac{(n-m)\theta(x + t_0)^{\gamma} log(x + t_0)}{(1 + (x + t_0)^{\gamma})}.$$

The estimation of γ and θ shall be explored in Section 5 by using numerical methods.

4.1.5. *Gompertz future lifetime distribution.* If X follows Gompertz (α, λ), then using the log-likelihood function, we have

$$\begin{aligned}\frac{dLog(L(z,\alpha,\lambda))}{d\alpha} =& \frac{n\lambda^x}{log\lambda} + \frac{r}{\alpha} + \sum_{i=1}^{r}(x+t_i(x))log\lambda \\ & - \frac{\lambda^x}{log\lambda}\left(\sum_{i=1}^{r}\lambda^{t_i(x)} + \sum_{i=r+1}^{m}\lambda^{c_i(x)}\right) - (n-m)\frac{\lambda^{x+t_0}}{log\lambda}.\end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{dLog(L(z,\alpha,\lambda))}{d\lambda} =& n\alpha\lambda^{x-1}\frac{xlog\lambda - 1}{(log\lambda)^2} - (n-m)\alpha\lambda^{x+t_0-1}\frac{log\lambda(x+t_0)-1}{(log\lambda)^2} \\ & + \sum_{i=1}^{r}\frac{x+t_i(x)}{\lambda} - \sum_{i=1}^{r}\alpha\lambda^{x+t_i(x)-1}\frac{log\lambda(x+t_i(x))-1}{(log\lambda)^2} \\ & - \sum_{i=r+1}^{m}\alpha\lambda^{x+c_i(x)-1}\frac{log\lambda(x+c_i(x))-1}{(log\lambda)^2};\end{aligned} \tag{4.5}$$

By applying Newton-Raphson Method and using (4.4) and (4.5), the estimates are found in Section 5.

4.2. Estimation for whole life assurance contracts in case of censoring. In insurance, there are some policies which pay benefit till or at the time of the death of the insured, known as whole life assurance contracts. This happens provided the insured does not leave the cohort due to any other reason except death. To evaluate benefits accurately under this situation, the insurance company takes account of all the conditions under which insured may leave the contract. For example, the insured may surrender the policy at a random time giving rise to a censored observation. We take account of this censoring for estimation of parameters of the future lifetime distributions under whole life assurance contract. We write $Z = min(T(x), C(x)) = \begin{cases} T(x) & if \quad T(x) < C(x) \\ C(x) & if \quad C(x) < T(x) \end{cases}$

where $T(x)$ and $C(x)$ are defined in Section 4.1.
The likelihood function, under random right censoring [10] is written as

$$L(z) = \prod_{i=1}^{r} f_T(x)(t_i(x))\bar{G}_{C(x)}(t_i(x)) \prod_{i=r+1}^{n} g_{C(x)}(c_i(x))\bar{F}_{T(x)}(c_i(x)). \tag{4.6}$$

In the following subsections, ML estimation of unknown parameters of different future lifetime distributions shall be discussed by assuming that $C(x) \sim U(0, \eta)$ where the value of η depends upon the assumed percentage of censoring.

4.2.1. *Exponential future lifetime distribution.* Let X follow Exponential (λ), then using (4.6)

$$L(z,\lambda) = \lambda^r e^{-\lambda\sum_{i=1}^{r}t_i(x)}\prod_{i=1}^{r}\left(1 - \frac{t_i(x)}{\eta}\right)\frac{e^{-\lambda\sum_{i=r+1}^{n}c_i(x)}}{\eta^{n-r}}.$$

Differentiating the log-likelihood with respect to λ and solving the normal equation, we get

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r}t_i(x) + \sum_{i=r+1}^{n}c_i(x)}.$$

4.2.2. *Weibull future lifetime distribution.* If X follows Weibull (α, λ), then considering the log likelihood and differentiating with respect to α and λ, we get

$$\begin{aligned}\frac{dLog(L(z,\alpha,\lambda))}{d\alpha} &= \frac{r}{\alpha} + rlog\lambda + n(\lambda x)^{\alpha} log(\lambda x) + \sum_{i=1}^{r} log(x + t_i(x)) \\ &- \sum_{i=1}^{r} (x + t_i(x))^{\alpha} \lambda^{\alpha} log(\lambda(x + t_i(x))) \\ &- \sum_{i=r+1}^{n} (\lambda(x + c_i(x)))^{\alpha} log(\lambda(x + c_i(x))); \end{aligned} \tag{4.7}$$

$$\frac{dLog(L(z,\alpha,\lambda))}{d\lambda} = \frac{\alpha r}{\lambda} - \alpha\lambda^{\alpha-1}\left(nx^{\alpha} + \sum_{i=1}^{r}(x + t_i(x))^{\alpha} - \sum_{i=r+1}^{n}(x + c_i(x))^{\alpha}\right). \tag{4.8}$$

When (4.7) and (4.8) are equated to zero, the resulting non-linear equations can be solved using numerical methods. The results are demonstrated in Section 5.

4.2.3. *Pareto type I future lifetime distribution.* If X follows Pareto (θ), $\theta > 0$, then differentiation of log likelihood with respect to θ and equating the resulting expressions to zero, gives

$$\hat{\theta} = \frac{r}{\sum_{i=1}^{r} log(x + t_i(x)) - nlogx + \sum_{i=r+1}^{n} log(x + c_i(x))}.$$

4.2.4. *Burr future lifetime distribution.* Let X follow Burr $(\gamma, \theta), \gamma, \theta > 0$, then

$$\begin{aligned}\frac{dLog(L(z,\gamma,\theta))}{d\theta} =& \frac{r}{\theta} + nlog(1 + x^{\gamma}) - \sum_{i=1}^{r} log(1 + (x + t_i(x))^{\gamma}) \\ &- \sum_{i=r+1}^{n} log(1 + (x + c_i(x))^{\gamma}); \end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{dLog(L(z,\gamma,\theta))}{d\gamma} =& \frac{r}{\gamma} + \frac{r\theta x^{\gamma} log(x)}{(1 + x^{\gamma})} + \sum_{i=1}^{r} log(x + t_i(x)) - (\theta + 1) \\ &\sum_{i=1}^{r} \frac{(x + t_i(x))^{\gamma} log(x + t_i(x))}{(1 + (x + t_i(x))^{\gamma})} + \frac{\theta(n - r)x^{\gamma} logx}{(1 + x^{\gamma})} \\ &- \theta \sum_{i=r+1}^{n} \frac{(x + c_i(x))^{\gamma} log(x + c_i(x))}{(1 + (x + c_i(x))^{\gamma})} \\ =& \frac{r}{\gamma} + \frac{n\theta x^{\gamma} logx}{(1 + x^{\gamma})} \\ &+ \sum_{i=1}^{r} log(x + t_i(x)) \left[1 - (\theta + 1)\frac{(x + t_i(x))^{\gamma}}{(1 + (x + t_i(x))^{\gamma})}\right] \\ &- \theta \sum_{i=r+1}^{n} \frac{(x + c_i(x))^{\gamma} log(x + c_i(x))}{(1 + (x + c_i(x))^{\gamma})}. \end{aligned} \tag{4.10}$$

In Section 5, the estimates of γ and θ are obtained by equating (4.9) and (4.10) to zero and solving the resulting equations using numerical methods.

4.2.5. *Gompertz future lifetime distribution.* If X follows Gompertz (α, λ),

$$\begin{aligned}\frac{dLog(L(z,\alpha,\lambda))}{d\alpha} =& \frac{n\lambda^x}{log\lambda} + \frac{r}{\alpha} + \sum_{i=1}^{r}(x + t_i(x))log\lambda \\ &- \frac{\lambda^x}{log\lambda}\left[\sum_{i=1}^{r}\lambda^{t_i(x)} + \sum_{i=r+1}^{n}\lambda^{c_i(x)}\right]. \\ \frac{dLog(L(z,\alpha,\lambda))}{d\lambda} =& n\alpha\lambda^{x-1}\frac{xlog\lambda - 1}{(log\lambda)^2} + \sum_{i=1}^{r}\frac{x + t_i(x)}{\lambda} \\ &- \frac{\alpha\lambda^{x-1}}{(log\lambda)^2}\sum_{i=1}^{r}\lambda^{t_i(x)}(log\lambda(x + t_i(x)) - 1) \\ &+ \frac{\alpha\lambda^{x-1}}{(log\lambda)^2}\lambda^{c_i(x)}(log\lambda(x + c_i(x)) - 1);\end{aligned}$$

The maximum likelihood estimates are presented in Section 5.

5. Simulations

In this section, we generate random samples of sizes 50, 100 and 500 from different continuous future lifetime distributions and estimate their parameters, using BFGS method in R. The root mean square errors (RMSEs) are evaluated to check the extent of the deviation of the estimate from the true value. The number of replications is 10000 for starting age $x = 20, 40$ and 60.

5.1. Uncensored case for whole life assurance contract. Tables 2-5 give the estimates and the corresponding RMSEs when the data sets are generated from Weibull (α = 1.5 and λ = 2.5), Burr (θ = 3 and γ = 2), exponentiated exponential (α = 1.5 and λ = 0.5) and Gompertz (α = 0.0005 and λ = 1.08) future lifetime distributions.

Table 2: Estimates and RMSEs for Weibull future lifetime dist.

x	$n = 50$		$n = 100$		$n = 500$	
20	1.5223 (0.0626)	2.4262 (0.2134)	1.5034 (0.0435)	2.4856 (0.1947)	1.4987 (0.0112)	2.5371 (0.1255)
40	1.4963 (0.0173)	2.5496 (0.1034)	1.4978 (0.0097)	2.5064 (0.0891)	1.4989 (0.0055)	2.5072 (0.0350)
60	1.4979 (0.0085)	2.5390 (0.0820)	1.4999 (0.0044)	2.4969 (0.0543)	1.5005 (0.0039)	2.4979 (0.0131)

Table 3: Estimates and RMSEs for Burr future lifetime dist.

x	$n = 50$		$n = 100$		$n = 500$	
20	2.8290 (0.2532)	2.2758 (0.2430)	3.0099 (0.2232)	2.2939 (0.2014)	2.9358 (0.1532)	2.1314 (0.1232)
40	2.8971 (0.1917)	2.1205 (0.1859)	2.8451 (0.1603)	2.0950 (0.1215)	2.9291 (0.0951)	2.0868 (0.0843)
60	2.9680 (0.0959)	2.1660 (0.1123)	3.0494 (0.0751)	2.1077 (0.0825)	3.0121 (0.0275)	2.0750 (0.0372)

Table 4: Estimates and RMSEs for Exponentiated Exp. future lifetime dist.

x	$n = 50$		$n = 100$		$n = 500$	
20	1.5995 (0.2468)	0.5102 (0.0740)	1.5990 (0.1024)	0.5047 (0.0512)	1.5991 (0.0991)	0.5008 (0.0223)
40	1.5999 (0.2215)	0.5109 (0.0706)	1.5985 (0.0985)	0.5048 (0.0511)	1.5925 (0.0925)	0.5013 (0.0220)
60	1.5212 (0.2076)	0.4873 (0.0685)	1.5002 (0.0832)	0.4997 (0.0434)	1.5001 (0.0604)	0.5071 (0.0198)

Table 5: Estimates and RMSEs for Gompertz future lifetime dist.

x	$n = 50$		$n = 100$		$n = 500$	
20	1.0828 (0.0117)	0.0005 (0.0003)	1.0813 (0.0080)	0.0005 (0.0002)	1.0802 (0.0034)	0.0005 (0.0000)
40	1.0834 (0.0150)	0.0005 (0.0005)	1.0819 (0.0103)	0.0005 (0.0003)	1.0802 (0.0044)	0.0005 (0.0001)
60	1.0869 (0.0256)	0.0009 (0.0020)	1.0838 (0.0170)	0.0006 (0.0008)	1.0811 (0.0078)	0.0005 (0.0002)

On the basis of values in Tables 2-5, we conclude that for all future lifetime distributions under study

- RMSEs decrease as sample size increases, that is, the estimates get closer to the assumed values of the parameters for large samples;
- RMSEs decrease as initial age increases, except when X follows Gompertz distribution.

5.2. Censored case. As mentioned in Section 4.1, $C(x)$, the censoring variable is assumed to follow $U(0, \eta)$. The parameter η for $C(x)$ is determined by assuming 30% censoring in the data, that is, by solving

$$P[T(x) > C(x)] = 0.30. \tag{5.1}$$

We find the values of η for different combinations of parameters of $T(x)$. The values are generated from the future lifetime and Uniform distributions. These generated sets of values are used to estimate the values of parameters when $x = 20,\ 40,\ 60;\quad n = 50,\ 100,\ 500$ and $t_0 = 10,\ 20,\ 30,\ 40$ and 50.

5.3. Term life assurance contract. Tables $6 - 8$ display the values of η, estimates of parameters of future lifetime distributions and the corresponding RMSEs when the random samples are generated from Weibull ($\alpha = 1.5$ and $\lambda = 2.5$), Burr ($\theta = 3$ and $\gamma = 2$), and Gompertz ($\alpha = 0.0005$ and $\lambda = 1.08$) future lifetime distributions. The values in the parentheses give the RMSEs.

Table 6: Estimates and RMSEs for censored Weibull future lifetime dist.

t_0	$n=50$		$n=100$		$n=500$	
t_0			$x=20,\quad \eta=0.120$			
10	1.5126 (0.0505)	2.4146 (0.2500)	1.5005 (0.0339)	2.4996 (0.1596)	1.4946 (0.0120)	2.5400 (0.0779)
20	1.5090 (0.0462)	2.4321 (0.2243)	1.5010 (0.0281)	2.5086 (0.1375)	1.5008 (0.0097)	2.5122 (0.0561)
30	1.5050 (0.0451)	2.4374 (0.2072)	1.4962 (0.0262)	2.5332 (0.1359)	1.4966 (0.0096)	2.5189 (0.0453)
40	1.5051 (0.0415)	2.4573 (0.1689)	1.4995 (0.0259)	2.5228 (0.1283)	1.4994 (0.0060)	2.5199 (0.0319)
50	1.5025 (0.0399)	2.4737 (0.1374)	1.5111 (0.0160)	2.4781 (0.0932)	1.4960 (0.0058)	2.5076 (0.0109)
t_0			$x=40,\quad \eta=0.085$			
10	1.5044 (0.0223)	2.4819 (0.0288)	1.4963 (0.0172)	2.5165 (0.0271)	1.4992 (0.0073)	2.5104 (0.0215)
20	1.5041 (0.0221)	2.4809 (0.0283)	1.4978 (0.0155)	2.5149 (0.0269)	1.5014 (0.0050)	2.5121 (0.0200)
30	1.5019 (0.0198)	2.4842 (0.0251)	1.4975 (0.0143)	2.5151 (0.0250)	1.4989 (0.0046)	2.5121 (0.0188)
40	1.5066 (0.0189)	2.4867 (0.0198)	1.5003 (0.0140)	2.5129 (0.0225)	1.4996 (0.0031)	2.5025 (0.0074)
50	1.5007 (0.0166)	2.4891 (0.0152)	1.4969 (0.0140)	2.5115 (0.0178)	1.5022 (0.0027)	2.5044 (0.0056)
t_0			$x=60,\quad \eta=0.070$			
10	1.5045 (0.0225)	2.4745 (0.0354)	1.4961 (0.0165)	2.5112 (0.0185)	1.4988 (0.0068)	2.5094 (0.0154)
20	1.5024 (0.0188)	2.4803 (0.0321)	1.4987 (0.0141)	2.5090 (0.0163)	1.4978 (0.0063)	2.5086 (0.0146)
30	1.5043 (0.0179)	2.4798 (0.0313)	1.4983 (0.0140)	2.5085 (0.0154)	1.4992 (0.0062)	2.5090 (0.0145)
40	1.5005 (0.0145)	2.4834 (0.0292)	1.4971 (0.0138)	2.5082 (0.0153)	1.4996 (0.0058)	2.5078 (0.0122)
50	1.5008 (0.0097)	2.4930 (0.0112)	1.4981 (0.0075)	2.5083 (0.0132)	1.4978 (0.0055)	2.5059 (0.0094)

Table 7: Estimates and RMSEs for censored Burr future lifetime dist.

	$n=50$		$n=100$		$n=500$	
t_0			$x=20$, $\eta=12.5$			
10	2.8388 (0.4877)	2.1497 (0.4392)	2.7930 (0.4562)	2.1671 (0.4278)	2.9601 (0.4223)	1.9129 (0.3762)
20	2.7799 (0.4583)	2.1659 (0.4363)	2.7020 (0.4303)	2.1905 (0.3959)	2.7613 (0.3686)	2.1185 (0.3444)
30	2.8021 (0.4409)	2.1510 (0.4301)	2.7854 (0.4184)	2.1594 (0.3630)	2.8359 (0.3455)	2.0595 (0.3442)
40	2.7928 (0.4223)	2.8010 (0.4246)	2.8010 (0.3799)	2.0806 (0.3507)	2.8289 (0.2966)	1.9735 (0.3100)
50	2.7881 (0.4164)	2.1392 (0.4047)	2.8250 (0.3375)	2.0729 (0.3427)	2.8856 (0.2114)	2.0961 (0.2697)
t_0			$x=40$, $\eta=24.5$			
10	3.0250 (0.1755)	2.0045 (0.2133)	2.9921 (0.1284)	1.9985 (0.1396)	3.0059 (0.1194)	1.9823 (0.0880)
20	3.0125 (0.1612)	1.9932 (0.1959)	2.9800 (0.1260)	2.0007 (0.1232)	2.9889 (0.0845)	2.0159 (0.0669)
30	3.0040 (0.1542)	2.0299 (0.1897)	2.9796 (0.1243)	2.0163 (0.1089)	2.9725 (0.0749)	2.0095 (0.0648)
40	3.0106 (0.1486)	2.0249 (0.1751)	3.0105 (0.1212)	2.0480 (0.1089)	2.9850 (0.0609)	2.0053 (0.0521)
50	3.0094 (0.1468)	2.0327 (0.1678)	3.0456 (0.0789)	2.0641 (0.1086)	2.9677 (0.0322)	1.9673 (0.0326)
t_0			$x=60$, $\eta=36.5$			
10	2.9772 (0.1561)	2.0113 (0.2291)	2.9767 (0.1002)	1.9349 (0.1775)	2.9865 (0.0394)	1.9384 (0.0801)
20	2.9883 (0.1548)	1.9486 (0.2113)	2.9665 (0.0952)	1.9524 (0.1261)	2.9826 (0.0206)	1.9800 (0.0294)
30	3.0091 (0.1240)	1.9647 (0.1948)	2.9534 (0.0925)	1.9456 (0.1100)	3.0122 (0.0122)	2.0202 (0.0202)
40	2.9685 (0.1206)	1.9464 (0.1938)	2.9505 (0.0778)	1.9368 (0.0951)	2.9842 (0.0117)	1.9778 (0.0200)
50	2.9497 (0.0730)	1.9234 (0.1317)	3.0084 (0.0599)	2.0135 (0.0848)	3.0086 (0.0086)	2.0171 (0.0171)

Table 8: Estimates and RMSEs for censored Gompertz future lifetime dist.

t_0	$n=50$		$n=100$		$n=500$	
	$x=20,\ \eta=130$					
10	1.1917 (0.1395)	0.0001 (0.0003)	1.1688 (0.1155)	0.0001 (0.0003)	1.1332 (0.0812)	0.0002 (0.0003)
20	1.1302 (0.0725)	0.0002 (0.0003)	1.1172 (0.0575)	0.0002 (0.0003)	1.0901 (0.0268)	0.0004 (0.0002)
30	1.1078 (0.0453)	0.0003 (0.0003)	1.0980 (0.0345)	0.0003 (0.0002)	1.0822 (0.0122)	0.0005 (0.0001)
40	1.0961 (0.0296)	0.0003 (0.0003)	1.0900 (0.0203)	0.0004 (0.0002)	1.0807 (0.0065)	0.0005 (0.0001)
50	1.0874 (0.0165)	0.0004 (0.0002)	1.0830 (0.0104)	0.0004 (0.0002)	1.0807 (0.0050)	0.0004 (0.0001)
t_0	$x=40,\ \eta=70$					
10	1.1394 (0.0786)	0.0001 (0.0003)	1.1362 (0.0716)	0.0002 (0.0003)	1.1082 (0.0420)	0.0002 (0.0003)
20	1.1044 (0.0366)	0.0002 (0.0003)	1.1029 (0.0333)	0.0002 (0.0003)	1.0840 (0.0128)	0.0004 (0.0002)
30	1.0982 (0.0257)	0.0002 (0.0003)	1.0891 (0.0169)	0.0003 (0.0002)	1.0811 (0.0069)	0.0005 (0.0001)
40	1.0869 (0.0143)	0.0004 (0.0002)	1.0842 (0.0110)	0.0004 (0.0002)	1.0805 (0.0052)	0.0005 (0.0001)
50	1.0903 (0.0170)	0.0003 (0.0002)	1.0854 (0.0096)	0.0004 (0.0002)	1.0801 (0.0048)	0.0005 (0.0001)
t_0	$x=60,\ \eta=34$					
10	1.1143 (0.0471)	0.0002 (0.0003)	1.1108 (0.0424)	0.0002 (0.0003)	1.0882 (0.0161)	0.0003 (0.0002)
20	1.1038 (0.0330)	0.0002 (0.0003)	1.0947 (0.0228)	0.0003 (0.0003)	1.0851 (0.0106)	0.0004 (0.0002)
30	1.1033 (0.0327)	0.0002 (0.0003)	1.0941 (0.0197)	0.0002 (0.0003)	1.0822 (0.0077)	0.0004 (0.0002)
40	1.1040 (0.0324)	0.0002 (0.0003)	1.0955 (0.0247)	0.0003 (0.0003)	1.0836 (0.0084)	0.0004 (0.0002)
50	1.1039 (0.0314)	0.0002 (0.0003)	1.0948 (0.0228)	0.0003 (0.0003)	1.0827 (0.0078)	0.0004 (0.0002)

From Tables $6-8$, it can be observed that RMSEs decrease with an increase in sample size n, initial age x and term of the contract t_0.

5.4. Whole life assurance contract. Tables $9-11$ show the estimates and the corresponding RMSEs (in parantheses) when the data sets are generated from Weibull ($\alpha=1.5$ and $\lambda=2.5$), Burr ($\theta=3$ and $\gamma=2$) and Gompertz ($\alpha=0.0005$ and $\lambda=1.08$) future lifetime distributions.

Table 9: Estimates and RMSEs for censored Weibull future lifetime dist.

	$n = 50$		$n = 100$		$n = 500$	
$\eta = 0.120$ for $x = 20$	1.5003 (0.0182)	2.4969 (0.0097)	1.4984 (0.0114)	2.5011 (0.0075)	1.4999 (0.0075)	2.5003 (0.0030)
$\eta = 0.085$ for $x = 40$	1.5051 (0.0224)	2.4978 (0.0141)	1.4980 (0.0154)	2.5012 (0.0088)	1.4975 (0.0079)	2.5006 (0.0046)
$\eta = 0.070$ for $x = 60$	1.5022 (0.0264)	2.4823 (0.1188)	1.5005 (0.0356)	2.5001 (0.0511)	1.4948 (0.0124)	2.5017 (0.0156)

Table 10: Estimates and RMSEs for censored Burr future lifetime dist.

	$n = 50$		$n = 100$		$n = 500$	
$\eta = 12.5$ for $x = 20$	2.9836 (0.4440)	2.1824 (0.4117)	2.9318 (0.4418)	2.1222 (0.4477)	2.9960 (0.4643)	2.0209 (0.3849)
$\eta = 24.5$ for $x = 40$	3.1175 (0.2199)	2.1302 (0.2688)	3.0823 (0.2062)	2.1006 (0.1936)	3.0376 (0.1705)	2.0309 (0.1221)
$\eta = 36.5$ for $x = 60$	3.1277 (0.2152)	2.1311 (0.2609)	3.0889 (0.1436)	2.1180 (0.1845)	3.0501 (0.0931)	2.0598 (0.0946)

Table 11: Estimates and RMSEs for censored Gompertz future lifetime dist.

	$n = 50$		$n = 100$		$n = 500$	
$\eta = 130$ for $x = 20$	1.0825 (0.0135)	0.0005 (0.0003)	1.0815 (0.0093)	0.0005 (0.0002)	1.0804 (0.0041)	0.0005 (0.0001)
$\eta = 75$ for $x = 40$	1.0846 (0.0189)	0.0006 (0.0007)	1.0824 (0.0128)	0.0005 (0.0003)	1.0806 (0.0056)	0.0005 (0.0001)
$\eta = 34$ for $x = 60$	1.0884 (0.0339)	0.0020 (0.0083)	1.0837 (0.0232)	0.0010 (0.0023)	1.0791 (0.0096)	0.0006 (0.0004)

On the basis of Tables $9 - 11$, it is concluded that

- as sample size increases, RMSEs decrease;
- as initial age increases, RMSEs increase except for Burr future lifetime distribution.

6. Conclusions

Estimation of the parameters of future lifetime distributions is helpful in modeling the remaining lifetime of the insured. This is helpful to insurer for setting the premium in order to make the insured policy mutually advantageous. We study the distributional properties of the complete future lifetime and explore the maximum likelihood estimation of the parameters when the insured does not leave the cohort till the end of the contract or leaves at a random time.

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