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Review and classifications of the ridge parameter estimation techniques

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Abstract

Ridge parameter estimation techniques under the influence of multicollinearity in Linear regression model were reviewed and classified into different forms and various types. The different forms are Fixed Maximum (FM), Varying Maximum (VM), Arithmetic Mean (AM), Geometric Mean (GM), Harmonic Mean (HM) and Median (M) and the various types are Original (O), Reciprocal (R), Square Root (SR) and Reciprocal of Square Root (RSR). These classifications resulted into proposing some other techniques of Ridge parameter estimation. Investigation of the existing and proposed ones were done by conducting 1000 Monte-Carlo experiments under five (5) levels of multicollinearity ($\rho = 0.8, 0.9, 0.95, 0.99, 0.999$), three (3) levels of error variance ($\sigma^2 = 0.25, 1, 25$) and five levels of sample size ($n = 10, 20, 30, 40, 50$). The relative efficiency ($RF \leq 0.75$) of the techniques resulting from the ratio of their mean square error and that of the ordinary least square was used to compare the techniques.

Results show that the proposed techniques perform better than the existing ones in some situations; and that the best technique is generally the ridge parameter in the form of Harmonic Mean, Fixed Maximum and Varying Maximum in their Original and Square Root types.

Keywords: Linear Regression Model, Multicollinearity, Ridge Parameter Estimation Techniques, Relative Efficiency.

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1. Introduction

Regression analysis is a study on relationship among variables often classified as dependent and independent variables. The dependent variable is modeled on one or more explanatory variables. The aim is often to estimate the mean value of the dependent variable in terms of the known or fixed value of the explanatory variables. The Ordinary Least Squares (OLS) estimator is the most popularly used estimator to estimate the parameters in a regression model. The estimator under certain assumptions has some very attractive statistical properties which have made it one of the most powerful estimators. One of the assumptions is that the explanatory variables are independent. However, in practice, there may be strong or perfect linear relationships among the explanatory variables. This problem is often referred to as multicollinearity problem. It is well known that the performance of OLS estimator is unsatisfactory in the presence of multicollinearity in that the regression coefficients possess large standard errors and some will even have the wrong sign (Gujarati, 1995). In literature, there are various methods existing to solve this problem. Among them is the ridge regression estimator first introduced by Hoerl and Kennard (1970). Ridge estimator has a smaller mean square error (MSE) than OLS estimator (Vinod and Ullah, 1981).
Consider the standard regression model:

$$(1.1) \qquad Y = X\beta + U$$

where X is an $n \times p$ matrix with full rank, Y is a $n \times 1$ vector of dependent variable, β is a $p \times 1$ vector of unknown parameters, and U is the error term such that $E(U) = 0$ and $E(UU') = \sigma^2 I_n$.
The ridge estimator is defined as:

$$(1.2) \qquad \hat{\beta} = \left(X'X + KI\right)^{-1} X'Y$$

where K is a non-negative constant. It is called biasing or ridge parameter. It is observed that when $K = 0$, (2) returns OLS estimates. Ridge regression estimators are biased as K increases but give more precise estimates than OLS estimator (Mardikyan and Cetin, 2008). Hoerl *et al.* (1975) suggested that the value of K should be chosen small enough such that the mean squared error of ridge estimator is less than the mean squared error of OLS estimator. Different techniques of estimation had been proposed or suggested by various researchers. Hoerl and Kennard (1970) suggested a graphical method called ridge trace to select the value of the ridge parameter K. This is a plot of the values of individual components of $\hat{\beta}_k$ against a range of values of K $(0 < K < 1)$. The minimum value for which $\hat{\beta}_k$ become stable and the wrong signs in the regression coefficient is corrected is used. Also, one can select the value of K for which the residual sum of square is not too large. The performance of the Ridge estimator with different Ks had often been compared through simulation study. Most of the researchers have generated data from a normal population as explanatory variables with different number of regressors. The mean squared error (MSE) has been used severally as a performance criterion.
Several techniques had been developed to estimate K. These were reviewed and classified into different forms and various types in this paper. These classifications resulted into proposing new techniques for the ridge parameter estimation. The organization of this paper is as follows: Theoretical background of ridge regression, different methods of estimations are reviewed and proposed estimators are presented in section 2. The model, details of the Monte Carlo simulation study and performance criterion is given in Section 3. Results and discussions are presented in section 4. Some concluding remarks are given in Section 5.

2. Ridge regression and ridge estimator

2.1. Background of ridge regression. Ridge regression centers on the use of the biased parameter K which yields estimation with a smaller mean square error. Hoerl and Kennard (1970) suggested the optimum ridge parameter as:

$$(2.1) \qquad K_i = \frac{\sigma^2}{\sigma_i^2}, i = 1, 2, 3, \ldots, p$$

Since σ^2 and α_i^2 are generally unknown and the K_i needs to be estimated. Consequently, they suggested the replacement of σ^2 and α_i^2 by their corresponding unbiased estimators $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$.
Therefore, the estimator of K_i is given as:

$$(2.2) \qquad K_i = \frac{\sigma^2}{\sigma_i^2}$$

where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} e_i^2}{n-p}$ and α_i is the i^{th} element of the vector $\hat{\alpha} = Q'\hat{\beta}$ where Q is an orthogonal matrix.

2.2. Review of methods of estimating the ridge parameter. The existing ridge parameters are reviewed as follows:

2.2.1. *Estimators based on Hoerl and Kennard (1970).* Hoerl and Kennard (1970) proposed $K_{HK_i} = \frac{\sigma^2}{\alpha_i^2}$. They suggested estimating ridge parameter by taking the maximum (Fixed Maximum) of α_i^2 such that the estimator of K is:

$$(2.3) \qquad \hat{K}_{HK}^{FM} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha}_i^2)}$$

Hoerl *et al.* (1975) proposed a different estimator of K by taking the Harmonic Mean of the ridge parameter K_{HK_i}. This estimator is given as:

$$(2.4) \qquad \hat{K}_{HK}^{FM} = \frac{P\hat{\sigma}^2}{\sum_{i=1}^{p} \alpha_i^2}$$

Kibria (2003) proposed some new estimators of K by taking the Geometric Mean, Arithmetic Mean and Median ($p \geq 3$) of the ridge parameter K_{HK_i}. These estimators are respectively defined as:

$$(2.5) \qquad \hat{K}_{HK}^{GM} = \frac{\hat{\alpha}^2}{(\prod_{i=1}^{p} \hat{\alpha}_i^2)^{\frac{1}{p}}}$$

$$(2.6) \qquad \hat{K}_{HK}^{AM} = \frac{1}{p}\sum_{i=1}^{p}(\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2})$$

$$(2.7) \qquad \hat{K}_{HK}^{M} = Median(\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2})$$

Furthermore, Muniz and Kibria (2009) proposed some estimators of K in form of Square root of Geometric Mean of K_{HK_i} and its reciprocal, the Median of the Square root of K_{HK_i} and its reciprocal, and Varying Maximum of the Square root of K_{HK_i} and its reciprocal. These estimators are respectively defined as:

$$(2.8) \qquad \hat{K}_{HK}^{GMSR} = \sqrt{\frac{\hat{\sigma}^2}{(\prod_{i=1}^{p} \hat{\alpha}_i^2)^{\frac{1}{p}}}}$$

$$(2.9) \quad \hat{K}_{HK}^{GMRSR} = \frac{1}{\sqrt{\frac{\hat{\sigma}^2}{(\prod_{i=1}^{p} \hat{\alpha}_i^2)^{\frac{1}{p}}}}}$$

$$(2.10) \quad \hat{K}_{HK}^{MSR} = Median(\sqrt{\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2}})$$

$$(2.11) \quad \hat{K}_{HK}^{MRSR} = Median(\frac{1}{\sqrt{\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2}}})$$

$$(2.12) \quad \hat{K}_{HK}^{MRSR} = Max(\sqrt{\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2}})$$

$$(2.13) \quad \hat{K}_{HK}^{MRSR} = Max(\frac{1}{\sqrt{\frac{\hat{\alpha}^2}{\hat{\alpha}_i^2}}})$$

2.2.2. *Estimators based on Lawless and Wang (1976).* Lawless and Wang (1976) proposed a different estimator of K resulting from taking the Harmonic Mean of the ridge parameter $K_{LW_i} = \frac{\sigma^2}{\lambda_i \alpha_i^2}$. The estimator is defined as:

$$(2.14) \quad \hat{K}_{LW}^{HM} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \lambda_i \hat{\alpha}_i^2}$$

where λ_i is the eigenvalue of the matrix $X'X$.

2.2.3. *Estimators based on Alkhamisi* et al. *(2006).* Alkhamisi *et al.* (2006) proposed another ridge parameter $K_{AKS_i} = \frac{\sigma^2 \lambda_i}{(n-p)\sigma^2 + \lambda_i \alpha_i^2}$. They proposed estimators of K as the Arithmetic Mean and Median of the ridge parameter K_{AKS_i}. These estimators are respectively defined as:

$$(2.15) \quad \hat{K}_{AKS}^{AM} = \frac{1}{p} \sum_{i=1}^{p} (\frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})$$

$$(2.16) \quad \hat{K}_{AKS}^{M} = Median(\frac{\hat{\sigma}^2 \lambda_i}{(n-p)\sigma^2 + \lambda_i \hat{\alpha}_i^2})$$

However, a new approach of choosing the ridge parameter K suggested by Khalaf and Shukur (2005) can be seen in form of Fixed Maximum of the ridge parameter K_{AKS_i}. The estimator is defined as:

$$(2.17) \quad \hat{K}_{AKS}^{FM} = \frac{\hat{\sigma}^2 max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i) Max(\hat{\alpha}_i^2)}$$

Muniz and Kibria (2009) proposed the estimator of the ridge parameter K as the Geometric Mean of the ridge parameter K_{AKS_i}. The estimator is given as:

$$(2.18) \quad \hat{K}_{AKS}^{GMO} = (\prod_{i=1}^{p} \frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})^{\frac{1}{p}}$$

Muniz *et al.* (2012) proposed the estimator of the ridge parameter K as the Varying Maximum and Arithmetic Mean of the ridge parameter K_{AKS_i}. These estimators are defined respectively as:

$$(2.19) \quad \hat{K}_{AKS}^{VMO} = Max(\frac{\lambda_i \hat{\alpha}^2}{(n-p)\hat{\alpha}^2 + \lambda_i \hat{\alpha}_i^2})$$

$$(2.20) \quad \hat{K}_{AKS}^{AMO} = \frac{1}{p} \sum_{i=1}^{p} (\frac{\lambda_i \hat{\alpha}^2}{(n-p)\hat{\alpha}^2 + \lambda_i \hat{\alpha}_i^2})$$

2.2.4. *Estimators based on Muniz* et al. *(2012).* Muniz *et al.* (2012) proposed another ridge parameter $K_{MAOi} = \frac{\hat{\sigma}^2 Max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\alpha_i^2}$. They proposed estimators of K as Varying Maximum of K_{MAO_i} and its reciprocal, its square root and reciprocal of its square root, Median of the reciprocal of K_{MAO_i}, Median of the reciprocal of the square root of K_{MAO_i}, and the Geometric Mean of K_{MAO_i}, its square root and reciprocal of its square root. These estimators are defined respectively as:

$$K_{MAOi} = Max(\frac{\hat{\sigma}^2 Max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\alpha_i^2}) \tag{2.21}$$

$$\hat{K}_{MAO}^{VMR} = Max(\frac{1}{\hat{K}_{MAO}}) \tag{2.22}$$

$$\hat{K}_{MAO}^{VMSR} = Max(\sqrt{\hat{K}_{MAO}}) \tag{2.23}$$

$$\hat{K}_{MAO}^{VMRSR} = Max(\frac{1}{\sqrt{\hat{K}_{MAO}}}) \tag{2.24}$$

$$\hat{K}_{MAO}^{MR} = Median(\frac{1}{\hat{K}_{MAO}}) \tag{2.25}$$

$$\hat{K}_{MAO}^{MRSR} = Median(\frac{1}{\sqrt{\hat{K}_{MAO}}}) \tag{2.26}$$

$$K_{MAOi} = (\prod_{i=1}^{p} \frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\alpha_i^2})^{\frac{1}{p}} \tag{2.27}$$

$$\hat{K}_{MAO}^{GMSR} = \sqrt{\hat{K}_{MAO}^{GMO}} \tag{2.28}$$

$$\hat{K}_{MAO}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{MAO}^{GMO}}} \tag{2.29}$$

It should be noted that the Fixed Maximum of K_{MAO_i} gives the ridge parameter proposed by Khalaf and Shukur (2005) already defined in (19).

2.3. Summary of the existing and proposed ridge parameters. Following the review in section 2.2, it is observed that the parameters follow some different forms and various types. These are explained and summarized as follows:

A. Different forms of K

1. Fixed Maximum (FM): This is obtained by taking the highest value of the estimated regression coefficient or the eigenvalue or both.
2. Varying Maximum (VM): This allows the estimated regression coefficient and the eigenvalue to vary, and eventually the maximum of the estimated ridge parameter is chosen. That is, the ridge parameter with the highest estimated ridge parameters or eigenvalues or both.
3. Arithmetic Mean (AM): It involves taking the average of the estimated ridge parameters.
4. Harmonic Mean (HM): The ridge parameter is expressed as Harmonic Mean of the estimated ridge parameters.
5. Geometric Mean (GM): The ridge parameter is expressed as the Geometric Mean of the estimated ridge parameters.
6. Median (M): This involves taking the Median of the estimated ridge parameters.

B. Various types of K

1. Original form(O)
2. Reciprocal form(R)
3. Square root form(SR)

Table 1. Summary of Different Forms and Various Types for $\hat{K}_{HK_i} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$

	Various types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{HK}^{FMO} = \frac{\hat{\sigma}^2}{Max(\hat{\alpha}_i^2)}$	$\hat{K}_{HK}^{FMR} = \frac{1}{\hat{K}_{HK}^{FMO}}$	$\hat{K}_{HK}^{FMSR} = \sqrt{\hat{K}_{HK}^{FMO}}$	$\hat{K}_{HK}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{FMO}}}$
	Hoerl *et al.* (1970)	proposed	proposed	proposed
VM	$\hat{K}_{HK}^{VMO} = Max(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})$	$\hat{K}_{HK}^{VMR} = Max(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$	$\hat{K}_{HK}^{VMSR} = Max(\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}})$	$\hat{K}_{HK}^{VMRSR} = Max(\frac{1}{\sqrt{\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}}})$
	Proposed	Proposed	Proposed	Proposed
AM	$\hat{K}_{HK}^{AMO} = \frac{1}{p}\sum_{i=1}^{p} \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$	$\hat{K}_{HK}^{AMR} = \frac{1}{\hat{K}_{HK}^{AMO}}$	$\hat{K}_{HK}^{AMSR} = \sqrt{\hat{K}_{HK}^{AMO}}$	$\hat{K}_{HK}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{AMO}}}$
	Proposed	Proposed	Proposed	Proposed
HM	$\hat{K}_{HK}^{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \hat{\alpha}_i^2}$	$\hat{K}_{HK}^{HMR} = \frac{1}{\hat{K}_{HK}^{HMO}}$	$\hat{K}_{HK}^{HMSR} = \sqrt{\hat{K}_{HK}^{HMO}}$	$\hat{K}_{HK}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{HMO}}}$
	Hoerl *et al.* (1975)	Proposed	Proposed	Proposed
GM	$\hat{K}_{HK}^{GMO} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p} \hat{\alpha}_i^2)^{\frac{1}{p}}}$	$\hat{K}_{HK}^{GMR} = \frac{1}{\hat{K}_{HK}^{GMO}}$	$\hat{K}_{HK}^{GMSR} = \sqrt{\hat{K}_{HK}^{GMO}}$	$\hat{K}_{HK}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{HK}^{GMO}}}$
	Kibria (2003)	Proposed	Proposed	Proposed
M	$\hat{K}_{HK}^{MO} = Median(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})$	$\hat{K}_{HK}^{MR} = Median(\frac{1}{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$	$\hat{K}_{HK}^{MSR} = Median(\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})})$	$\hat{K}_{HK}^{MRSR} = Median(\frac{1}{\sqrt{(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2})}})$
	Kibria (2003)	Proposed	Proposed	Muniz and Kibria (2009)

4. Reciprocal of Square root(RSR)

In the light of this knowledge, the existing and newly proposed ridge parameters are summarized based on the previous works as follows:

2.3.1. *Ridge Parameter Proposed by Hoerl and Kennard (1970).* Hoerl and Kennard (1970) proposed the ridge parameter $K_{HK_i} = \frac{\sigma^2}{\alpha_i^2}$. Its estimator in the light of different forms and various types are summarized in Table 1.
Table 1 gives the summary of the different forms and various types of the Hoerl and Kennard (1970) ridge parameter. Those already in existence and the proposed ones are specified.

2.3.2. *Ridge Parameter Proposed by Lawless and Wang (1976).* Lawless and Wang (1976) proposed ridge parameter resulting from the Harmonic Mean of $K_{LW_i} = \frac{\sigma^2}{\lambda_i \alpha_i^2}$. Its estimator in the light of different forms and various types are summarized in Table 2.

Table 2 gives the summary of the different forms and various types of the Lawless and Wang (1976) ridge parameter. The one already in existence and the proposed are specified.

2.3.3. *Ridge Parameter Proposed by Alkhamisi* et al. *(2006).* Alkhamisi *et al.* (2006) proposed the ridge parameter $K_{AKS_i} = \frac{\sigma^2 \lambda_i}{(n-p)\sigma^2 + \lambda_i \alpha_i^2}$. Its estimator in the light of different forms and various types are summarized in Table 3.
Table 3 gives the summary of the different forms and various types of the Alkhamisi *et al* (2006) ridge parameter. Those already in existence and the proposed ones are also specified.

2.3.4. *Ridge Parameter Proposed by Muniz* et al. *(2012).* Muniz *et al.*(2012) proposed the ridge parameter $K_{MAO_i} = \frac{\hat{\sigma}^2 Max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\alpha_i^2}$. Its estimator in the light of different forms and various types are summarized in Table 4.
Table 4 gives the summary of the different forms and various types of the Muniz *et al.* (2012) ridge parameter. Those already in existence and the proposed are specified.

Table 2. Summary of Different Forms and Various Types for $\hat{K}_{LW_i} = \frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2}$

	Various types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{LW}^{FMO} = \frac{\hat{\sigma}^2}{Max(\lambda_i)Max(\hat{\alpha}_i^2)}$	$\hat{K}_{LW}^{FMR} = \frac{1}{\hat{K}_{LW}^{FMO}}$	$\hat{K}_{LW}^{FMSR} = \sqrt{\hat{K}_{LW}^{FMO}}$	$\hat{K}_{LW}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{LW}^{FMO}}}$
	Proposed	proposed	proposed	proposed
VM	$\hat{K}_{LW}^{VMO} = Max[\frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2}]$	$\hat{K}_{LW}^{VMR} = Max(\frac{1}{\hat{K}_{LW}})$	$\hat{K}_{LW}^{VMSR} = Max(\sqrt{\hat{K}_{LW}})$	$\hat{K}_{LW}^{VMRSR} = Max(\frac{1}{\sqrt{\hat{K}_{LW}}})$
	Proposed	Proposed	Proposed	Proposed
AM	$\hat{K}_{LW}^{AMO} = \frac{1}{p}\sum_{i=1}^{p} \frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2}$	$\hat{K}_{LW}^{AMR} = \frac{1}{\hat{K}_{LW}^{AMO}}$	$\hat{K}_{LW}^{AMSR} = \sqrt{\hat{K}_{LW}^{AMO}}$	$\hat{K}_{LW}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{LW}^{AMO}}}$
	Proposed	Proposed	Proposed	Proposed
HM	$\hat{K}_{LW}^{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \lambda_i \hat{\alpha}_i^2}$	$\hat{K}_{LW}^{HMR} = \frac{1}{\hat{K}_{LW}^{HMO}}$	$\hat{K}_{LW}^{HMSR} = \sqrt{\hat{K}_{LW}^{HMO}}$	$\hat{K}_{LW}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{LW}^{HMO}}}$
	Lawless and Wang (1976)	Proposed	Proposed	Proposed
GM	$\hat{K}_{LW}^{GMO} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^{p} \lambda_i \hat{\alpha}_i^2)^{\frac{1}{p}}}$	$\hat{K}_{LW}^{GMR} = \frac{1}{\hat{K}_{LW}^{GMO}}$	$\hat{K}_{LW}^{GMSR} = \sqrt{\hat{K}_{LW}^{GMO}}$	$\hat{K}_{LW}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{LW}^{GMO}}}$
	Proposed	Proposed	Proposed	Proposed
M	$\hat{K}_{LW}^{MO} = Median(\frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2})$	$\hat{K}_{LW}^{MR} = Median(\frac{1}{\hat{K}_{LW}})$	$\hat{K}_{LW}^{MSR} = Median(\sqrt{\hat{K}_{LW}})$	$\hat{K}_{LW}^{MRSR} = Median(\frac{1}{\sqrt{\hat{K}_{LW}}})$
	Proposed	Proposed	Proposed	Proposed

Table 3. Summary of Different Forms and Various Types for $\hat{K}_{AKS_i} = \frac{\sigma^2 \lambda_i}{(n-p)\sigma^2 + \lambda_i \alpha_i^2}$

	Various types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{AKS}^{FM} = \frac{\hat{\sigma}^2 Max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)Max(\hat{\alpha}_i^2)}$	$\hat{K}_{AKS}^{FMR} = \frac{1}{\hat{K}_{AKS}^{FMO}}$	$\hat{K}_{AKS}^{FMSR} = \sqrt{\hat{K}_{AKS}^{FMO}}$	$\hat{K}_{AKS}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{AKS}^{FMO}}}$
	Khalaf and Shukur (2005)	proposed	proposed	proposed
VM	$\hat{K}_{AKS}^{VMO} = Max(\frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})$	$\hat{K}_{AKS}^{VMR} = Max(\frac{1}{\hat{K}_{AKS}})$	$\hat{K}_{AKS}^{VMSR} = Max(\sqrt{\hat{K}_{AKS}})$	$\hat{K}_{AKS}^{VMRSR} = Max(\frac{1}{\sqrt{\hat{K}_{AKS}}})$
	Muniz *et al.* (2012)	Proposed	Proposed	Proposed
AM	$\hat{K}_{AKS}^{AMO} = \frac{1}{p}\sum_{i=1}^{p}(\frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})$	$\hat{K}_{AKS}^{AMR} = \frac{1}{\hat{K}_{AKS}^{AMO}}$	$\hat{K}_{AKS}^{AMSR} = \sqrt{\hat{K}_{AKS}^{AMO}}$	$\hat{K}_{AKS}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{AKS}^{AMO}}}$
	Muniz *et al.* (2012)	Proposed	Proposed	Proposed
HM	$\hat{K}_{AKS}^{HMO} = p\sum_{i=1}^{p}(\frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})$	$\hat{K}_{AKS}^{HMR} = \frac{1}{\hat{K}_{AKS}^{HMO}}$	$\hat{K}_{AKS}^{HMSR} = \sqrt{\hat{K}_{AKS}^{HMO}}$	$\hat{K}_{AKS}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{AKS}^{HMO}}}$
	Lawless and Wang (1976)	Proposed	Proposed	Proposed
GM	$\hat{K}_{AKS}^{GMO} = (\prod_{i=1}^{p} \frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})^{\frac{1}{p}}$	$\hat{K}_{AKS}^{GMR} = \frac{1}{\hat{K}_{AKS}^{GMO}}$	$\hat{K}_{AKS}^{GMSR} = \sqrt{\hat{K}_{AKS}^{GMO}}$	$\hat{K}_{AKS}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{AKS}^{GMO}}}$
	Muniz and Kibria (2009)	Proposed	Proposed	Proposed
M	$\hat{K}_{AKS}^{MO} = (\frac{\lambda_i \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2})$	$\hat{K}_{AKS}^{MR} = Median(\frac{1}{\hat{K}_{AKS}})$	$\hat{K}_{AKS}^{MSR} = Median(\sqrt{\hat{K}_{AKS}})$	$\hat{K}_{AKS}^{MRSR} = Median(\frac{1}{\sqrt{\hat{K}_{AKS}}})$
	Alkhamisi *et al.*(2006)	Proposed	Proposed	Proposed

Table 4. Summary of Different Forms and Various Types for $K_{MAO_i} = \frac{\hat{\sigma}^2 Max(\lambda_i)}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\alpha_i^2}$

	Various types of K			
Different Forms	O	R	SR	RSR
FM	$\hat{K}_{MAO}^{FM} = \frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)Max(\hat{\alpha}_i^2)}$	$\hat{K}_{MAO}^{FMR} = \frac{1}{\hat{K}_{MAO}^{FMO}}$	$\hat{K}_{MAO}^{FMSR} = \sqrt{\hat{K}_{MAO}^{FMO}}$	$\hat{K}_{MAO}^{FMRSR} = \frac{1}{\sqrt{\hat{K}_{MAO}^{FMO}}}$
	Khalaf and Shukur (2005)	proposed	proposed	proposed
VM	$\hat{K}_{MAO}^{VMO} = Max(\frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2})$	$\hat{K}_{MAO}^{VMR} = Max(\frac{1}{\hat{K}_{MAO}})$	$\hat{K}_{MAO}^{VMSR} = Max(\sqrt{\hat{K}_{MAO}})$	$\hat{K}_{MAO}^{VMRSR} = Max(\frac{1}{\sqrt{\hat{K}_{MAO}}})$
	Muniz *et al.* (2010)	Muniz *et al.* (2010)	Muniz *et al.* (2012)	Muniz *et al.* (2012)
AM	$\hat{K}_{MAO}^{AMO} = \frac{1}{p}\sum_{i=1}^{p}(\frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2})$	$\hat{K}_{MAO}^{AMR} = \frac{1}{\hat{K}_{MAO}^{AMO}}$	$\hat{K}_{MAO}^{AMSR} = \sqrt{\hat{K}_{MAO}^{AMO}}$	$\hat{K}_{MAO}^{AMRSR} = \frac{1}{\sqrt{\hat{K}_{MAO}^{AMO}}}$
	Proposed	Proposed	Proposed	Proposed
HM	$\hat{K}_{MAO}^{HMO} = p\sum_{i=1}^{p}(\frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2})$	$\hat{K}_{MAO}^{HMR} = \frac{1}{\hat{K}_{MAO}^{HMO}}$	$\hat{K}_{MAO}^{HMSR} = \sqrt{\hat{K}_{MAO}^{HMO}}$	$\hat{K}_{MAO}^{HMRSR} = \frac{1}{\sqrt{\hat{K}_{MAO}^{HMO}}}$
	Lawless and Wang (1976)	Proposed	Proposed	Proposed
GM	$\hat{K}_{MAO}^{GMO} = (\prod_{i=1}^{p} \frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2})^{\frac{1}{p}}$	$\hat{K}_{MAO}^{GMR} = \frac{1}{\hat{K}_{MAO}^{GMO}}$	$\hat{K}_{MAO}^{GMSR} = \sqrt{\hat{K}_{MAO}^{GMO}}$	$\hat{K}_{MAO}^{GMRSR} = \frac{1}{\sqrt{\hat{K}_{MAO}^{GMO}}}$
	Muniz *et al.* (2010)	Proposed	Muniz *et al.* (2012)	Muniz *et al.* (2012)
M	$\hat{K}_{MAO}^{MO} = Median(\frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2})$	$\hat{K}_{MAO}^{MR} = Median(\frac{1}{\hat{K}_{MAO}})$	$\hat{K}_{MAO}^{MSR} = Median(\sqrt{\hat{K}_{MAO}})$	$\hat{K}_{MAO}^{MRSR} = Median(\frac{1}{\sqrt{\hat{K}_{MAO}}})$
		Muniz *et al.* (2010)	Proposed	Muniz *et al.* (2010)

3. Model formulation and procedure for data generation for simulation study

Consider a linear regression model of the form:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + ... + \beta_p X_{pt} + U_t \quad (3.1)$$

$$t = 1, 2, ..., n; p = 3, 7$$

where $U_t \sim N(0, \sigma^2)$.
The model was studied with fixed regressors, $X_{it}, i = 1, 2, ..., p; t = 1, 2, ..., n$ such that there exist different levels of multicollinearity among the regressors.

3.1. Procedure for generating the error terms. The error term U_t was generated to be normally distributed with mean zero and variance $U_t \sim N(0, \sigma^2)$. In this study, σ values were 0.5, 1 and 5.

3.2. Procedure for generating the explanatory variables. The procedure used by McDonald and Galarneau (1975), Wichern and Churchill (1978), Gibbons (1981) and Kibria (2003) was also used to generate the explanatory variables in this study. This is given as:

$$X_{ti} = (1 - \rho^2)^{\frac{1}{2}} Z_{ti} + \rho Z_{tp} \quad (3.2)$$

$$t = 1, 2, 3, ..., n; i = 1, 2, ..., p$$

where Z_{ti} is independent standard normal distribution with mean zero and unit variance, ρ is the correlation between any two explanatory variables and p is the number of explanatory variables. The values of ρ were taken as $0.8, 0.9, 0.95, 0.99$ and 0.999. Thus, the inter-correlation between the variables was the same. In this study, the number of explanatory variable (p) was taken to be three (3) and seven (7).

3.3. Procedure for generating the dependent variable. The regression model is

$$Y_t = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + ... + \beta_p X_{ip} + U_t \quad (3.3)$$

$$t = 1, 2, ..., n; p = 3, 7$$

β_0 was taken to be identically zero. When $p = 3$, the values of β were chosen to be: $\beta_1 = 0.8$, $\beta_2 = 0.1$, $\beta_3 = 0.6$. When $p = 7$, the values of β were chosen to be: $\beta_1 = 0.4, \beta_2 = 0.1, \beta_3 = 0.6, \beta_4 = 0.2, \beta_5 = 0.25, \beta_6 = 0.3, \beta_7 = 0.53$. The parameter values were chosen such that $\beta'\beta = 1$ which is a common restriction in simulation studies of this type (Muniz and Kibria, 2009). We varied the sample sizes between $10, 20, 30, 40$ and 50. Three different values of $\sigma : 0.5, 1$ and 5 were also used. At a specified value of n, p and σ, the fixed Xs are first generated; followed by the U, and the values of $Y$are then obtained using the regression model. The experiment is repeated 1000 times.

3.4. Criterion for investigation. Several authors in literatures had applied the mean square error (MSE) to evaluate and compare the performance of ridge regression estimator with that of the ordinary least square estimator when there is multicollinearity. Some of these were Hoerl and Kennard (1970), Lawless and Wang (1976), Saleh and Kibria (1993), Kibria (2003), Khalaf and Shukur (2005), Alkhamisi *et al.* (2006), Mansson *et al.* (2010). To investigate whether the ridge estimator is better than the OLS estimator, the MSE is calculated using equation defined already:

$$MSE(\hat{\beta}_{ridge}) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + \hat{K})^2} + \hat{K}^2 \sum_{i=1}^{p} \frac{\hat{\alpha}_i^2}{(\lambda_i + \hat{K})^2} \quad (3.4)$$

$$(3.5) \qquad MSE(\hat{\beta}_{OLS}) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are the eigenvalues of $X'X, \hat{K}$ is the estimator of the ridge parameter $K, \hat{\alpha}_i$ is the i^{th} element of the vector $\hat{\alpha} = Q'\hat{\beta}$ where Q is an orthogonal matrix.
This is further examined by computing the relative efficiency of the ridge regression estimator relative to OLS estimator.

$$(3.6) \qquad RelativeEfficiency(RE) = \frac{MSE(\hat{\beta}_{ridge})}{\hat{\beta}(OLS)}$$

Thus, the smaller the efficiency value the better the ridge parameter. Consequently, ridge parameter estimates whose efficiency was not more than 0.75 are preferred and selected. That is, the ridge estimators whose MSE were better than that of OLS by at least 25% of OLS MSE. Furthermore, the number of times they were preferred (RE$\leq$ 0.75) over the five (5) levels of multicollinearity and three (3) error variance was counted so as to know the frequency of their efficiency at each level of sample size. Thus, a maximum of fifteen (15) counts was expected.

4. Results and discussion

4.1. Summary of results with $\hat{K}_{HK_i} = \frac{\hat{\sigma}^2}{\hat{\alpha}_1^2}$. A sample of the relative efficiency of the ridge parameter based on $\hat{K}_{HK_i}$ of Hoerl and Kennard (1970) at different forms and various types when n=10 is given in Appendix 1. The details are contained in the work of Lukman (2015). The frequency of the relative efficiency over the levels of multicollinearity and error variance is summarized in Table 5.
From Table 5, all the best methods in p=7 are also best in p=3. However, their order of performance differs and GMO which is best in p=3 is not among the best in p=7. It is also noted that two of the newly proposed techniques, **FMSR and HMSR**, are among the best five at each number of regressors. Consequently, the best techniques are: HMO, MO, FMO, **FMSR** and **HMSR**.

4.2. Summary of Results with $\hat{K}_{LW_i} = \frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2}$. A sample of the relative efficiency of the ridge parameter based on $\hat{K}_{LW_i}$ of Lawless and Wang (1975) at different forms and various types when n=30 is given in Appendix 2. The details are contained in the work of Lukman (2015). The frequency of the relative efficiency over the levels of multicollinearity and error variance is summarized in Table 6.
From Table 6, the best methods are **MO, GMO, AMSR, GMSR** and **MSR**. All these are newly proposed techniques.

4.3. Summary of Results with $\hat{K}_{AKS_i} = \frac{\hat{\sigma}^2 \lambda_i}{(n-p)\hat{\sigma}^2 + \lambda_i \hat{\alpha}_i^2}$. The relative efficiency of the ridge parameter based on $\hat{K}_{AKS_i}$ of Alkhamisi *et al.* (2006) at different forms and various types when n=40 is given in Appendix 3. The details are contained in the work of Lukman (2015). The frequency of the relative efficiency over the levels of multicollinearity and error variance is summarized in Table 7.
From Table 7, the preferred techniques are FMO, **FMSR, HMSR, MSR** and **GMSR**. It is also noted that four of the newly proposed techniques, **FMSR, HMSR, MSR** and **GMSR,** are among the best five at each number of regressors.

4.4. Summary of Results with $\hat{K}_{MAO_i} = \frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + Max(\lambda_i)\hat{\alpha}_i^2}$. The relative efficiency of the ridge parameter based on $\hat{K}_{MAO_i}$of Muniz *et al.* (2012) at different forms and various types when $n = 50$ is given in Appendix 4. The details are contained in the work of Lukman (2015). The frequency of the relative efficiency over the levels of multicollinearity and error variance is summarized in Table 8.

From Table 8, the results of $p = 3$ and $p = 7$, it can be concluded that the best methods are VMO, FMO, **FMSR, VMSR, AMSR, AMO, MSR** and **GMSR**. Among these, two are newly proposed.

Table 5: Frequency of the relative efficiency of ridge parameters based on $\hat{K}_{HK_i} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$ estimator with Multicollinearity (5 Levels) and Error Variances (3 levels) effect partial out

Different Forms	Various Types	methods	p=3							p=7						
			n							n						
			10	20	30	40	50	Total	Rank	10	20	30	40	50	Total	Rank
Fixed Maximum	Original	**FMO**	15	15	15	13	13	71	3.5	15	12	15	13	13	68	1
	Reciprocal	**FMR**	5	2	6	7	4	24	18.5	0	4	6	7	4	21	15.5
	Square root	**FMSR**	14	14	15	13	12	68	5	7	10	13	13	13	56	3
	Reciprocal of Square root	**FMRSR**	9	8	10	8	5	40	12.5	0	5	9	8	6	28	10.5
Varying Maximum	Original	**VMO**	8	2	5	4	3	22	20	0	0	0	3	0	3	24
	Reciprocal	**VMR**	5	2	6	7	4	24	18.5	0	4	6	7	4	21	15.5
	Square root	**VMSR**	11	8	11	13	11	54	10	0	3	0	6	3	12	21
	Reciprocal of Square root	**VMRSR**	9	8	10	8	5	40	12.5	0	5	9	8	6	28	10.5
Arithmetic Mean	Original	**AMO**	11	6	9	8	8	42	11	0	1	0	4	0	5	23
	Reciprocal	**AMR**	2	0	2	2	2	8	24	1	1	1	4	1	8	22
	Square root	**AMSR**	11	11	11	13	11	57	9	0	6	4	11	10	31	9
	Reciprocal of Square root	**AMRSR**	8	5	7	4	4	28	17	2	3	6	4	3	18	17
Harmonic Mean	Original	**HMO**	15	15	15	15	15	75	1	10	10	15	15	15	65	2
	Reciprocal	**HMR**	4	0	4	4	2	14	22	0	2	6	4	1	13	20
	Square root	**HMSR**	13	12	13	15	13	66	6	4	9	12	14	13	52	4
	Reciprocal of Square root	**HMRSR**	10	5	8	5	6	34	16	0	6	6	7	4	23	14
Geometric Mean	Original	**GMO**	15	13	13	15	15	71	3.5	2	9	6	15	13	45	6
	Reciprocal	**GMR**	3	0	3	2	2	10	23	1	3	3	4	3	14	19
	Square root	**GMSR**	13	11	13	13	13	63	7	1	7	9	13	12	42	8
	Reciprocal of Square root	**GMRSR**	10	6	8	7	4	35	15	1	5	6	7	6	25	12.5
Median	Original	**MO**	15	14	13	15	15	72	2	2	9	8	15	13	47	5
	Reciprocal	**MR**	4	2	7	4	4	21	21	1	4	3	4	3	15	18
	Square root	**MSR**	13	11	12	13	11	60	8	1	7	9	13	13	43	7
	Reciprocal of Square root	**MRSR**	11	6	8	7	7	39	14	1	5	6	7	6	25	12.5

Table 6: Frequency of the relative efficiency of ridge parameters based on $\hat{K}_{LW_i} = \frac{\hat{\sigma}^2}{\lambda_i \hat{\alpha}_i^2}$ estimator with multicollinearity (5 levels) and error variances (3 levels) effect partial out

Different Forms	Various Types	methods	p=3							p=7						
			n							n						
			10	20	30	40	50	Total	Rank	10	20	30	40	50	Total	Rank
Fixed Maximum	Original	**FMO**	0	0	0	0	0	0	24	0	1	0	0	0	1	24
	Reciprocal	**FMR**	2	0	3	2	1	8	22	0	1	0	2	0	3	23
	Square root	**FMSR**	10	6	6	6	3	31	13	13	5	6	3	3	30	11
	Reciprocal of Square root	**FMRSR**	5	1	5	5	5	21	17	0	2	0	7	4	13	18
Varying Maximum	Original	**VMO**	11	7	9	6	6	39	11	0	1	0	5	0	6	21
	Reciprocal	**VMR**	3	0	4	2	1	10	21	0	1	0	3	0	4	22
	Square root	**VMSR**	11	9	11	13	10	54	1.5	0	6	4	12	9	31	10
	Reciprocal of Square root	**VMRSR**	6	3	6	6	5	26	15	0	3	3	7	5	18	16
Arithmetic Mean	Original	**AMO**	11	11	10	10	8	50	5	0	4	0	8	3	15	17
	Reciprocal	**AMR**	2	0	2	2	1	7	23	0	3	1	3	2	9	20
	Square root	**AMSR**	11	11	11	11	10	54	1.5	0	7	8	11	10	36	4
	Reciprocal of Square root	**AMRSR**	6	4	4	5	4	23	16	2	7	5	4	4	22	15
Harmonic Mean	Original	**HMO**	9	5	6	4	3	27	14	7	5	5	5	4	26	13
	Reciprocal	**HMR**	5	0	5	4	2	16	19	0	2	0	6	2	10	19
	Square root	**HMSR**	11	6	8	6	6	37	12	4	7	6	6	6	29	12
	Reciprocal of Square root	**HMRSR**	8	7	9	9	9	42	10	0	6	6	12	9	33	8
Geometric Mean	Original	**GMO**	13	9	11	8	7	48	7	2	8	8	11	10	39	2
	Reciprocal	**GMR**	5	0	5	4	3	17	18	0	8	7	9	8	32	9
	Square root	**GMSR**	12	11	11	8	7	49	6	1	8	8	10	9	36	4
	Reciprocal of Square root	**GMRSR**	9	8	11	10	8	46	8.5	1	8	8	10	8	35	6.5
Median	Original	**MO**	13	11	11	8	8	51	3.5	2	8	8	10	12	40	1
	Reciprocal	**MR**	4	0	5	4	2	15	20	0	7	5	7	6	25	14
	Square root	**MSR**	11	11	11	10	8	51	3.5	1	8	8	10	9	36	4
	Reciprocal of Square root	**MRSR**	9	8	11	10	8	46	8.5	1	8	8	10	8	35	6.5

Note: Proposed methods of ridge parameters are in bold form.

Table 7: Frequency of the relative efficiency of ridge parameters based on $\hat{K}_{AKS_i} = \frac{\hat{\sigma}\lambda_i}{(n-p)\hat{\sigma}^2+\lambda_i\hat{\alpha}_i^2}$ estimator with multicollinearity (5 levels) and error variances (3 levels) effect partial out

Different Forms	Various Types	methods	p=3							p=7						
			n							n						
			10	20	30	40	50	Total	Rank	10	20	30	40	50	Total	Rank
Fixed Maximum	Original	**FMO**	15	15	15	13	11	69	1	15	12	15	13	13	68	1
	Reciprocal	**FMR**	7	5	8	8	5	33	21	0	4	6	8	5	23	21
	Square root	**FMSR**	14	14	15	12	10	65	2	7	10	13	13	12	55	2
	Reciprocal of Square root	**FMRSR**	9	8	11	8	5	41	17	0	5	9	8	6	28	17
Varying Maximum	Original	**VMO**	11	10	11	11	10	53	3.5	1	7	9	11	10	38	3.5
	Reciprocal	**VMR**	6	3	6	7	5	27	22	0	1	2	8	4	15	22
	Square root	**VMSR**	11	11	11	10	10	53	3.5	1	7	9	11	9	37	3.5
	Reciprocal of Square root	**VMRSR**	9	5	9	11	9	43	15.5	0	5	6	11	9	31	15.5
Arithmetic Mean	Original	**AMO**	13	9	11	7	5	45	13.5	3	9	10	7	7	36	13.5
	Reciprocal	**AMR**	9	8	11	10	8	46	11.5	0	5	7	11	10	33	11.5
	Square root	**AMSR**	11	11	13	8	7	50	7	2	8	12	8	7	37	7
	Reciprocal of Square root	**AMRSR**	11	9	11	8	10	49	9	1	6	9	11	10	37	9
Harmonic Mean	Original	**HMO**	15	0	0	0	0	15	24	10	4	0	0	0	14	24
	Reciprocal	**HMR**	7	4	7	9	8	35	20	0	3	3	9	7	22	20
	Square root	**HMSR**	15	11	12	6	6	50	7	5	10	12	9	9	45	7
	Reciprocal of Square root	**HMRSR**	9	8	9	11	8	45	13.5	0	6	9	9	8	32	13.5
Geometric Mean	Original	**GMO**	15	9	10	6	3	43	15.5	7	10	9	1	3	30	15.5
	Reciprocal	**GMR**	8	5	9	10	7	39	18.5	0	4	6	11	9	30	18.5
	Square root	**GMSR**	13	11	11	9	6	50	7	2	9	13	9	9	42	7
	Reciprocal of Square root	**GMRSR**	9	8	11	10	8	46	11.5	0	6	9	11	9	35	11.5
Median	Original	**MO**	15	0	7	0	0	22	23	6	10	10	6	0	32	23
	Reciprocal	**MR**	7	5	9	9	9	39	18.5	0	4	6	9	9	28	18.5
	Square root	**MSR**	13	11	13	9	6	52	5	2	10	13	9	9	43	5
	Reciprocal of Square root	**MRSR**	9	8	11	11	8	47	10	0	6	9	11	9	35	10

Note: Proposed methods of ridge parameters are in bold form.

Table 8: Frequency of the relative efficiency of ridge parameters based on $\hat{K}_{MAO_i} = \frac{Max(\lambda_i)\hat{\sigma}^2}{(n-p)\hat{\sigma}^2+Max(\lambda_i)\hat{\alpha}_i^2}$ estimator with multicollinearity (5 levels) and error variances (3 levels) effect partial out

Different Forms	Various Types	methods	p=3							p=7						
			n							n						
			10	20	30	40	50	Total	Rank	10	20	30	40	50	Total	Rank
Fixed Maximum	Original	**FMO**	15	15	15	13	11	69	1.5	15	12	15	13	13	68	1.5
	Reciprocal	**FMR**	9	7	10	8	5	39	18	0	5	8	7	6	26	18.5
	Square root	**FMSR**	13	11	13	10	10	57	4	5	9	12	13	12	51	3.5
	Reciprocal of Square root	**FMRSR**	11	9	11	9	7	47	11	0	6	9	8	6	29	16
Varying Maximum	Original	**VMO**	15	15	15	13	11	69	1.5	15	12	15	13	13	68	1.5
	Reciprocal	**VMR**	6	3	6	7	5	27	22	0	1	2	8	4	15	23
	Square root	**VMSR**	13	11	13	10	10	57	4	5	9	12	13	12	51	3.5
	Reciprocal of Square root	**VMRSR**	9	5	9	11	9	43	16	0	5	6	11	9	31	13
Arithmetic Mean	Original	**AMO**	15	10	13	9	7	54	6.5	10	10	12	7	7	46	7
	Reciprocal	**AMR**	8	8	9	11	8	44	14.5	0	4	6	11	9	30	15
	Square root	**AMSR**	13	12	13	10	9	57	4	5	10	13	10	10	48	5.5
	Reciprocal of Square root	**AMRSR**	9	8	11	8	8	44	14.5	0	5	9	11	10	35	10
Harmonic Mean	Original	**HMO**	13	0	0	0	0	13	24	10	2	0	0	0	12	24
	Reciprocal	**HMR**	7	5	7	9	8	36	21	0	3	3	9	7	22	22
	Square root	**HMSR**	15	11	12	6	6	50	9	6	10	12	6	9	43	9
	Reciprocal of Square root	**HMRSR**	9	8	9	11	8	45	13	0	5	6	11	9	31	13
Geometric Mean	Original	**GMO**	15	9	9	4	1	38	20	10	8	6	0	0	24	20.5
	Reciprocal	**GMR**	7	5	9	9	9	39	18	0	3	5	9	7	24	20.5
	Square root	**GMSR**	14	11	13	9	6	53	8	5	10	12	9	9	45	8
	Reciprocal of Square root	**GMRSR**	9	8	11	11	8	47	11	0	5	6	11	9	31	13
Median	Original	**MO**	15	0	6	0	0	21	23	10	8	9	0	0	27	17
	Reciprocal	**MR**	7	5	9	9	9	39	18	0	4	6	9	7	26	18.5
	Square root	**MSR**	15	11	13	9	6	54	6.5	5	10	15	9	9	48	5.5
	Reciprocal of Square root	**MRSR**	9	8	11	11	8	47	11	0	5	8	11	9	33	11

Note: Proposed methods of ridge parameters are in bold form.

5. Summary and concluding remarks

Table 9 presents the summary of the five best ridge parameters estimation techniques having obtained their relative efficiency, counting the relative efficiency (RE$\leq$ 0.75) over the five (5) levels of multicollinearity, three (3) levels of error variances and the five (5) levels of sample sizes. Thus, a maximum of seventy five (75) counts is expected.

Table 9: Summary of best ridge parameters

Ridge parameter	p=3			p=7		
	Best Method	**Frequency**	**Best**	**Best Method**	**Frequency**	**Best**
K_{HK}	HMO	75	HMO	HMO	65	FMO
	MO	72		MO	47	
	FMO	71		FMO	68	
	GMO	71		GMO	45	
	FMSR	68		**FMSR**	56	
	HMSR	66		**HMSR**	52	
K_{LW}	**MO**	51	MSR	**MO**	40	MO
	GMO	48		**GMO**	39	
	AMSR	54		**AMSR**	36	
	GMSR	49		**GMSR**	36	
	MSR	51		**MSR**	36	
K_{LS}	VMO	69	VMO/FMO	VMO	68	VMO/FMO
	FMO	69		FMO	68	
	FMSR	57		**FMSR**	51	
	VMSR	57		VMSR	51	
	AMSR	57		AMSR	48	
	AMO	54		AMO	46	
	MSR	54		**MSR**	48	
	GMSR	53		**GMSR**	45	
K_{ALK}	FMO	69	FMO	FMO	68	FMO
	FMSR	65		**FMSR**	55	
	HMSR	50		**FMSR**	45	
	MSR	52		**MSR**	43	
	GMSR	50		**GMSR**	42	

Source: Table 5, 6, 7 and 8

From Table 9, the best estimators of the ridge parameter techniques are of the different forms and various types. These are generally Fixed Maximum Original, Varying Maximum Original, Harmonic Mean Original, Geometric Mean Original and Arithmetic Mean Square root. The best ridge parameter techniques consist of both the existing ones and newly proposed. The proposed techniques also perform better than the existing ones in some cases. Moreover, with $\hat{K}_{LW_i}$, the existing Harmonic Mean version is not even among the best five.

The conclusions of this paper are restricted to the simulation study that has been conducted in this paper. To make a definite statement, one might need more data from different kind of populations. However, the findings of this paper can be generalized for a great population with high confidence.

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Appendix 1: MSE of OLS and Relative efficiency of the ridge parameter based on $\hat{K}_{HK}$

Method	n=10 p=3														
	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.920	1.908	1.889	22.661	248.616	0.985	2.043	4.315	24.262	266.182	3.034	6.292	13.291	74.738	819.954
FMO	0.520	0.438	0.401	0.380	0.374	0.512	0.433	0.397	0.376	0.370	0.475	0.431	0.405	0.373	0.354
FMR	0.771	0.840	0.935	1.003	0.998	0.773	0.831	0.915	0.974	0.966	0.739	0.705	0.724	0.708	0.667
FMSR	0.534	0.447	0.428	0.538	0.764	0.529	0.443	0.424	0.529	0.744	0.508	0.444	0.422	0.463	0.557
FMRSR	0.662	0.692	0.811	0.975	0.996	0.664	0.687	0.795	0.947	0.964	0.652	0.605	0.642	0.691	0.666
VMO	0.866	0.732	0.729	0.878	0.980	0.846	0.722	0.721	0.859	0.949	0.818	0.729	0.695	0.678	0.663
VMR	0.771	0.840	0.935	1.003	0.998	0.773	0.831	0.915	0.974	0.966	0.739	0.705	0.724	0.708	0.667
VMSR	0.587	0.598	0.671	0.880	0.982	0.571	0.590	0.663	0.859	0.951	0.615	0.621	0.637	0.667	0.662
VMRSR	0.662	0.692	0.8117	0.975	0.996	0.664	0.687	0.795	0.947	0.964	0.652	0.605	0.642	0.691	0.666
AMO	0.703	0.614	0.617	0.773	0.952	0.682	0.605	0.611	0.760	0.925	0.725	0.664	0.643	0.646	0.656
AMR	0.897	0.847	0.834	0.921	0.988	0.896	0.840	0.822	0.898	0.957	0.933	0.859	0.776	0.703	0.676
AMSR	0.537	0.542	0.607	0.830	0.972	0.522	0.535	0.600	0.811	0.942	0.569	0.578	0.601	0.649	0.659
AMRSR	0.791	0.708	0.711	0.887	0.985	0.789	0.704	0.705	0.864	0.954	0.832	0.716	0.637	0.626	0.650
HMO	0.443	0.407	0.397	0.397	0.398	0.439	0.404	0.394	0.394	0.395	0.455	0.432	0.416	0.392	0.375
HMR	0.803	0.809	0.889	0.993	0.997	0.805	0.804	0.872	0.964	0.965	0.804	0.707	0.696	0.704	0.667
HMSR	0.490	0.445	0.468	0.627	0.832	0.486	0.442	0.464	0.616	0.809	0.478	0.448	0.456	0.519	0.592
HMRSR	0.689	0.669	0.764	0.961	0.995	0.691	0.665	0.750	0.933	0.963	0.695	0.602	0.612	0.681	0.666
GMO	0.487	0.476	0.497	0.595	0.736	0.479	0.472	0.493	0.588	0.721	0.573	0.559	0.556	0.562	0.584
GMR	0.858	0.817	0.845	0.968	0.996	0.857	0.812	0.834	0.940	0.964	0.889	0.788	0.705	0.671	0.665
GMSR	0.470	0.474	0.536	0.748	0.927	0.467	0.471	0.531	0.733	0.899	0.496	0.506	0.536	0.604	0.641
GMRSR	0.740	0.675	0.724	0.932	0.993	0.738	0.670	0.715	0.905	0.961	0.768	0.644	0.590	0.647	0.663
MO	0.455	0.459	0.513	0.628	0.667	0.451	0.458	0.514	0.621	0.654	0.549	0.551	0.568	0.569	0.545
MR	0.823	0.767	0.804	0.951	0.995	0.818	0.760	0.785	0.922	0.963	0.846	0.741	0.663	0.661	0.665
MSR	0.470	0.471	0.544	0.750	0.900	0.468	0.470	0.542	0.736	0.875	0.501	0.507	0.543	0.604	0.630
MRSR	0.716	0.654	0.709	0.925	0.992	0.714	0.647	0.694	0.898	0.961	0.737	0.620	0.572	0.645	0.663
	n=10 p=7														
MSE OLS	18.560	41.405	91.811	549.321	6229.414	19.622	43.773	97.063	580.746	6585.779	52.209	116.469	258.2579	1545.204	17522.960
FMO	0.379	0.369	0.362	0.353	0.348	0.402	0.391	0.385	0.376	0.371	0.679	0.671	0.666	0.658	0.653
FMR	0.874	0.881	0.883	0.883	0.881	1.017	1.029	1.033	1.035	1.035	2.631	2.687	2.716	2.745	2.759
FMSR	0.493	0.550	0.609	0.721	0.809	0.550	0.624	0.701	0.842	0.951	1.245	1.519	1.782	2.242	2.559
FMRSR	0.829	0.857	0.871	0.881	0.881	0.970	1.003	1.021	1.033	1.035	2.563	2.658	2.704	2.744	2.759
VMO	0.879	0.868	0.866	0.871	0.879	1.018	1.013	1.013	1.022	1.033	2.614	2.662	2.694	2.737	2.757
VMR	0.874	0.881	0.883	0.883	0.881	1.017	1.029	1.033	1.035	1.035	2.631	2.687	2.716	2.745	2.759
VMSR	0.831	0.845	0.856	0.871	0.880	0.969	0.989	1.003	1.022	1.033	2.546	2.624	2.674	2.733	2.757
VMRSR	0.829	0.857	0.871	0.881	0.881	0.970	1.003	1.021	1.033	1.035	2.563	2.658	2.704	2.744	2.759
AMO	0.827	0.832	0.839	0.852	0.871	0.965	0.975	0.984	1.003	1.024	2.553	2.613	2.656	2.716	2.753
AMR	0.759	0.750	0.777	0.853	0.882	0.822	0.847	0.902	1.007	1.026	1.338	1.810	2.182	2.607	2.748
AMSR	0.793	0.820	0.839	0.863	0.877	0.929	0.962	0.984	1.014	1.031	2.483	2.581	2.645	2.721	2.755
AMRSR	0.669	0.727	0.780	0.859	0.880	0.762	0.849	0.921	1.013	1.033	1.697	2.153	2.423	2.694	2.755
HMO	0.471	0.476	0.480	0.485	0.488	0.517	0.524	0.529	0.535	0.538	1.107	1.131	1.146	1.164	1.173
HMR	0.842	0.863	0.875	0.882	0.881	0.983	1.010	1.025	1.034	1.035	2.573	2.668	2.710	2.745	2.759
HMSR	0.607	0.664	0.712	0.788	0.841	0.698	0.771	0.832	0.926	0.990	1.780	2.027	2.220	2.495	2.661
HMRSR	0.796	0.837	0.861	0.879	0.881	0.934	0.983	1.010	1.032	1.035	2.484	2.624	2.690	2.743	2.759
GMO	0.718	0.745	0.765	0.794	0.815	0.844	0.877	0.901	0.937	0.964	2.351	2.440	2.502	2.588	2.647
GMR	0.754	0.805	0.845	0.878	0.881	0.865	0.940	0.992	1.030	1.035	2.017	2.449	2.628	2.739	2.759
GMSR	0.724	0.769	0.801	0.844	0.869	0.852	0.906	0.944	0.993	1.022	2.330	2.475	2.570	2.683	2.740
GMRSR	0.729	0.794	0.836	0.875	0.881	0.850	0.933	0.983	1.027	1.035	2.169	2.478	2.623	2.733	2.758
MO	0.714	0.751	0.777	0.801	0.811	0.842	0.887	0.917	0.947	0.959	2.409	2.495	2.549	2.609	2.638
MR	0.736	0.795	0.839	0.877	0.881	0.848	0.926	0.983	1.029	1.035	1.872	2.373	2.604	2.738	2.759
MSR	0.723	0.772	0.805	0.845	0.868	0.851	0.910	0.950	0.996	1.021	2.364	2.501	2.587	2.687	2.739
MRSR	0.726	0.791	0.833	0.874	0.881	0.846	0.928	0.979	1.026	1.035	2.113	2.447	2.610	2.732	2.758

Appendix 2: MSE of OLS and Relative efficiency of the ridge parameter based on $\hat{K}_{LW}$

Method	n=10 p=3														
	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.386	0.804	1.700	9.536	104.240	0.396	0.826	1.746	9.794	107.059	0.732	1.526	3.226	18.094	197.792
FMO	0.993	0.988	0.980	0.946	0.909	0.993	0.988	0.979	0.945	0.908	0.984	0.972	0.953	0.899	0.865
FMR	1.940	1.568	1.348	1.125	1.056	1.877	1.524	1.313	1.099	1.032	0.941	0.814	0.728	0.630	0.594
FMSR	0.946	0.898	0.823	0.595	0.421	0.945	0.897	0.821	0.592	0.418	0.920	0.854	0.758	0.508	0.321
FMSR	0.946	0.898	0.823	0.595	0.421	0.945	0.897	0.821	0.592	0.418	0.920	0.854	0.758	0.508	0.321
FMRSR	0.809	0.956	1.041	1.064	1.051	0.790	0.932	1.016	1.040	1.026	0.525	0.552	0.585	0.599	0.591
VMO	0.777	0.629	0.647	0.860	1.017	0.765	0.621	0.638	0.843	0.995	0.546	0.437	0.429	0.517	0.579
VMR	1.864	1.473	1.258	1.087	1.051	1.802	1.430	1.225	1.061	1.026	0.890	0.753	0.674	0.607	0.591
VMRSR	0.606	0.509	0.569	0.854	1.020	0.597	0.504	0.560	0.836	0.997	0.484	0.382	0.379	0.501	0.577
VMSR	0.762	0.872	0.954	1.029	1.045	0.744	0.850	0.931	1.005	1.021	0.500	0.504	0.531	0.574	0.588
AMO	0.775	0.593	0.557	0.755	0.991	0.7587	0.586	0.551	0.741	0.969	0.570	0.429	0.382	0.466	0.568
AMR	0.915	0.921	0.931	0.965	1.025	0.908	0.912	0.919	0.951	1.001	0.853	0.818	0.780	0.686	0.613
AMSR	0.651	0.515	0.520	0.795	1.009	0.643	0.510	0.514	0.779	0.986	0.543	0.405	0.362	0.473	0.572
AMRSR	0.791	0.763	0.766	0.865	1.006	0.789	0.758	0.757	0.848	0.977	0.785	0.706	0.635	0.546	0.562
HMO	0.981	0.960	0.916	0.671	0.653	0.980	0.959	0.914	0.666	0.645	0.952	0.904	0.817	0.494	0.429
HMR	1.276	1.191	1.127	1.064	1.049	1.234	1.157	1.098	1.039	1.024	0.660	0.625	0.607	0.593	0.589
HMSR	0.910	0.827	0.701	0.556	0.900	0.909	0.824	0.697	0.550	0.881	0.867	0.754	0.597	0.388	0.522
HMRSR	0.649	0.747	0.859	1.001	1.042	0.637	0.730	0.838	0.977	1.018	0.505	0.472	0.492	0.5577	0.585
GMO	0.843	0.701	0.555	0.536	0.905	0.841	0.696	0.552	0.529	0.887	0.731	0.558	0.408	0.364	0.532
GMR	0.875	0.902	0.925	0.995	1.038	0.858	0.884	0.903	0.974	1.012	0.704	0.661	0.627	0.578	0.581
GMSR	0.779	0.620	0.503	0.669	0.977	0.777	0.616	0.499	0.657	0.955	0.704	0.521	0.387	0.417	0.557
GMRSR	0.684	0.688	0.743	0.923	1.030	0.679	0.679	0.728	0.902	1.005	0.654	0.569	0.521	0.526	0.577
MO	0.827	0.643	0.500	0.594	0.944	0.823	0.638	0.495	0.586	0.924	0.703	0.497	0.366	0.394	0.547
MR	0.826	0.853	0.903	0.998	1.042	0.812	0.837	0.886	0.976	1.018	0.698	0.681	0.662	0.616	0.588
MSR	0.771	0.592	0.490	0.704	0.989	0.768	0.588	0.485	0.691	0.967	0.686	0.487	0.368	0.432	0.562
MRSR	0.664	0.669	0.728	0.911	1.027	0.661	0.662	0.716	0.890	1.002	0.656	0.587	0.542	0.532	0.575
	n=10 p=7														
MSE OLS	1.124	2.347	4.961	27.782	303.180	1.152	2.405	5.085	28.472	310.719	2.050	4.280	9.047	50.660	552.848
FMO	0.996	0.994	0.992	0.988	0.986	0.996	0.994	0.992	0.988	0.986	0.996	0.995	5.549	0.991	0.989
FMR	1.311	1.212	1.149	1.073	1.050	1.307	1.213	1.151	1.074	1.051	1.310	1.245	6.099	1.089	1.044
FMSR	0.956	0.920	0.869	0.721	0.534	0.955	0.920	0.870	0.722	0.534	0.953	0.923	4.139	0.739	0.532
FMRSR	0.772	0.908	0.990	1.043	1.048	0.776	0.912	0.994	1.045	1.049	0.865	0.999	5.990	1.070	1.043
VMO	0.993	0.975	0.984	1.025	1.045	0.985	0.970	0.988	1.027	1.045	1.074	1.042	5.888	1.051	1.040
VMR	1.271	1.154	1.095	1.053	1.048	1.265	1.153	1.096	1.055	1.049	1.189	1.153	6.006	1.073	1.043
VMRSR	0.657	0.740	0.828	0.979	1.040	0.665	0.734	0.831	0.980	1.040	0.755	0.809	5.624	1.004	1.035
VMSR	0.739	0.855	0.936	1.021	1.045	0.741	0.857	0.938	1.022	1.045	0.744	0.862	5.783	1.033	1.039
AMO	0.761	0.793	0.848	0.976	1.038	0.761	0.784	0.851	0.976	1.038	0.861	0.861	5.611	1.002	1.034
AMR	0.904	0.860	0.816	0.777	0.872	0.904	0.861	0.821	0.783	0.880	0.918	0.864	4.151	0.741	0.832
AMSR	0.606	0.650	0.734	0.932	1.032	0.615	0.642	0.737	0.933	1.033	0.670	0.698	5.337	0.953	1.028
AMRSR	0.837	0.760	0.698	0.733	0.936	0.835	0.758	0.703	0.732	0.934	0.852	0.762	3.898	0.696	0.905
HMO	0.974	0.948	0.895	0.657	0.708	0.974	0.947	0.893	0.653	0.708	0.946	0.897	3.235	0.578	0.756
HMR	0.878	0.957	0.999	1.035	1.047	0.879	0.959	1.001	1.037	1.047	0.829	0.932	5.848	1.044	1.040
HMSR	0.889	0.796	0.677	0.626	0.903	0.888	0.795	0.675	0.625	0.904	0.851	0.741	3.490	0.623	0.921
HMRSR	0.609	0.712	0.822	0.983	1.041	0.611	0.713	0.823	0.984	1.041	0.617	0.694	5.487	0.980	1.032
GMO	0.638	0.565	0.582	0.803	1.003	0.639	0.563	0.583	0.804	1.003	0.632	0.577	4.649	0.830	1.004
GMR	0.737	0.699	0.709	0.849	1.007	0.735	0.698	0.708	0.849	1.007	0.758	0.688	4.567	0.816	0.991
GMSR	0.649	0.565	0.588	0.815	1.009	0.649	0.564	0.588	0.816	1.010	0.638	0.565	4.652	0.831	1.007
GMRSR	0.701	0.640	0.653	0.835	1.011	0.701	0.639	0.652	0.835	1.011	0.716	0.629	4.584	0.819	1.000
MO	0.623	0.558	0.590	0.813	1.006	0.625	0.559	0.589	0.812	1.006	0.637	0.583	4.691	0.838	1.005
MR	0.756	0.718	0.720	0.842	1.000	0.756	0.717	0.719	0.843	1.001	0.760	0.695	4.553	0.813	0.989
MSR	0.638	0.560	0.590	0.821	1.011	0.638	0.560	0.590	0.821	1.011	0.638	0.566	4.675	0.835	1.007
MRSR	0.713	0.651	0.657	0.830	1.008	0.713	0.650	0.656	0.831	1.008	0.717	0.633	4.568	0.816	1.000

Note: Proposed techniques are bolded.

Appendix 3: MSE of OLS and Relative efficiency of the ridge parameter based on $\hat{K}_{AKS}$

n=40 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.100	0.429	0.891	4.870	52.452	0.215	0.438	0.909	4.968	53.506	0.352	0.716	1.486	8.123	87.482
FMO	0.818	0.695	0.571	0.430	0.407	0.816	0.689	0.567	0.427	0.404	0.746	0.598	0.465	0.334	0.314
FMR	0.817	0.712	0.691	0.960	1.031	0.818	0.689	0.684	0.942	1.013	0.857	0.744	0.624	0.598	0.643
FMSR	0.818	0.693	0.565	0.465	0.656	0.817	0.690	0.562	0.461	0.647	0.781	0.635	0.488	0.347	0.438
FMRSR	0.817	0.699	0.623	0.855	1.022	0.818	0.691	0.619	0.839	1.004	0.836	0.709	0.569	0.536	0.636
VMO	0.817	0.691	0.575	0.648	0.957	0.815	0.691	0.571	0.640	0.940	0.742	0.588	0.457	0.448	0.606
VMR	0.646	0.852	1.089	1.104	1.045	0.641	0.837	1.069	1.084	1.026	0.544	0.555	0.678	0.688	0.652
VMSR	0.818	0.692	0.575	0.656	0.961	0.817	0.690	0.572	0.647	0.944	0.779	0.630	0.485	0.442	0.603
VMRSR	0.694	0.589	0.712	1.006	1.031	0.693	0.583	0.701	0.987	1.013	0.665	0.480	0.473	0.626	0.644
AMO	0.891	0.822	0.719	0.534	0.847	0.890	0.820	0.716	0.529	0.834	0.856	0.761	0.629	0.400	0.548
AMR	0.727	0.601	0.607	0.863	1.009	0.727	0.598	0.598	0.846	0.991	0.755	0.591	0.482	0.520	0.624
AMSR	0.858	0.761	0.639	0.576	0.917	0.857	0.759	0.637	0.569	0.901	0.835	0.719	0.572	0.409	0.579
AMRSR	0.773	0.633	0.563	0.770	0.992	0.773	0.632	0.558	0.755	0.974	0.784	0.628	0.490	0.480	0.615
HMO	0.929	0.921	0.915	0.909	0.905	0.929	0.920	0.915	0.908	0.905	0.925	0.917	0.912	0.907	0.904
HMR	0.666	0.662	0.925	1.072	1.042	0.664	0.652	0.908	1.052	1.024	0.630	0.478	0.580	0.668	0.651
HMSR	0.884	0.831	0.767	0.611	0.545	0.883	0.830	0.767	0.610	0.538	0.879	0.824	0.757	0.576	0.416
HMRSR	0.737	0.589	0.634	0.975	1.030	0.736	0.585	0.625	0.958	1.011	0.724	0.518	0.445	0.607	0.643
GMO	0.913	0.883	0.849	0.754	0.610	0.912	0.883	0.848	0.753	0.608	0.898	0.865	0.826	0.718	0.549
GMR	0.693	0.603	0.765	1.029	1.034	0.692	0.596	0.751	1.010	1.015	0.689	0.494	0.491	0.638	0.645
GMSR	0.872	0.800	0.712	0.544	0.677	0.871	0.799	0.710	0.541	0.666	0.860	0.782	0.682	0.459	0.444
GMRSR	0.754	0.604	0.587	0.911	1.024	0.754	0.602	0.579	0.894	1.006	0.753	0.562	0.445	0.565	0.639
MO	0.917	0.902	0.890	0.871	0.858	0.916	0.902	0.890	0.871	0.858	0.912	0.898	0.887	0.868	0.855
MR	0.687	0.623	0.856	1.059	1.040	0.685	0.615	0.840	1.040	1.022	0.659	0.474	0.543	0.660	0.649
MSR	0.875	0.815	0.744	0.586	0.569	0.874	0.814	0.743	0.584	0.561	0.869	0.807	0.731	0.539	0.413
MRSR	0.750	0.595	0.609	0.957	1.029	0.750	0.592	0.600	0.940	1.011	0.739	0.534	0.440	0.596	0.642
n=40 p=7															
MSE OLS	0.612	1.225	2.508	13.382	141.652	0.625	1.252	2.562	13.669	144.689	1.047	2.098	4.294	22.908	242.490
FMO	0.793	0.684	0.600	0.542	0.530	0.790	0.680	0.595	0.538	0.527	0.690	0.567	0.492	0.443	0.431
FMR	0.813	0.707	0.675	0.939	1.021	0.815	0.707	0.669	0.923	1.005	0.867	0.752	0.608	0.608	0.683
FMSR	0.800	0.684	0.581	0.506	0.641	0.798	0.681	0.578	0.502	0.633	0.748	0.614	0.502	0.394	0.448
FMRSR	0.809	0.693	0.618	0.835	1.013	0.810	0.693	0.613	0.820	0.996	0.837	0.711	0.562	0.541	0.676
VMO	0.791	0.671	0.572	0.655	0.948	0.788	0.667	0.568	0.647	0.934	0.685	0.542	0.440	0.478	0.646
VMR	0.580	0.755	0.965	1.039	1.026	0.575	0.743	0.950	1.023	1.010	0.484	0.510	0.645	0.700	0.687
VMRSR	0.798	0.678	0.577	0.655	0.949	0.797	0.676	0.573	0.647	0.934	0.746	0.600	0.474	0.460	0.639
VMSR	0.657	0.564	0.682	0.977	1.020	0.655	0.559	0.671	0.962	1.004	0.627	0.457	0.467	0.654	0.684
AMO	0.895	0.856	0.791	0.585	0.723	0.895	0.854	0.789	0.581	0.713	0.869	0.807	0.711	0.455	0.514
AMR	0.684	0.569	0.640	0.913	1.010	0.684	0.565	0.630	0.897	0.993	0.706	0.530	0.468	0.577	0.669
AMSR	0.855	0.778	0.678	0.562	0.860	0.855	0.777	0.676	0.556	0.846	0.837	0.741	0.614	0.420	0.582
AMRSR	0.746	0.604	0.565	0.800	0.991	0.746	0.603	0.559	0.785	0.975	0.754	0.588	0.472	0.514	0.656
HMO	0.926	0.920	0.915	0.906	0.900	0.926	0.920	0.915	0.906	0.900	0.924	0.918	0.913	0.904	0.897
HMR	0.627	0.602	0.825	1.024	1.025	0.625	0.594	0.812	1.007	1.008	0.591	0.445	0.549	0.689	0.686
HMSR	0.877	0.827	0.767	0.627	0.571	0.877	0.827	0.767	0.625	0.565	0.873	0.821	0.756	0.587	0.441
HMRSR	0.761	0.617	0.559	0.821	1.006	0.760	0.615	0.553	0.807	0.990	0.751	0.577	0.451	0.543	0.673
GMO	0.913	0.897	0.879	0.836	0.775	0.912	0.896	0.879	0.835	0.774	0.904	0.887	0.868	0.820	0.752
GMR	0.653	0.575	0.753	1.010	1.023	0.652	0.569	0.740	0.994	1.006	0.638	0.455	0.500	0.677	0.685
GMSR	0.867	0.807	0.735	0.589	0.610	0.867	0.807	0.734	0.586	0.602	0.859	0.794	0.713	0.523	0.433
GMRSR	0.729	0.583	0.585	0.910	1.017	0.729	0.580	0.578	0.895	1.000	0.722	0.531	0.438	0.602	0.681
MO	0.909	0.901	0.892	0.875	0.861	0.909	0.901	0.892	0.875	0.861	0.907	0.898	0.889	0.871	0.857
MR	0.658	0.576	0.775	1.018	1.024	0.657	0.569	0.763	1.001	1.007	0.631	0.446	0.518	0.684	0.686
MSR	0.865	0.811	0.746	0.607	0.583	0.864	0.810	0.745	0.605	0.576	0.861	0.804	0.732	0.558	0.435
MRSR	0.732	0.580	0.591	0.926	1.018	0.732	0.577	0.583	0.911	1.002	0.719	0.520	0.435	0.616	0.682

Appendix 4: MSE of OLS and Relative efficiency of the ridge parameter based on $\hat{K}_{MAO}$

n=10 p=3															
Method	$\sigma = 0.5$					$\sigma = 1$					$\sigma = 5$				
	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999	0.8	0.9	0.95	0.99	0.999
MSE OLS	0.165	0.333	0.686	3.711	39.738	0.168	0.338	0.696	3.768	40.352	0.249	0.502	1.035	5.604	60.002
FMO	0.860	0.756	0.630	0.445	0.410	0.859	0.753	0.627	0.443	0.408	0.810	0.680	0.541	0.365	0.335
FMR	0.868	0.771	0.711	0.953	1.051	0.869	0.772	0.709	0.939	1.037	0.895	0.804	0.696	0.667	0.718
FMSR	0.863	0.756	0.625	0.454	0.617	0.862	0.755	0.623	0.451	0.611	0.838	0.714	0.567	0.369	0.455
FMRSR	0.867	0.762	0.658	0.822	1.039	0.867	0.762	0.656	0.811	1.024	0.880	0.777	0.643	0.581	0.708
VMO	0.707	0.591	0.543	0.676	0.965	0.706	0.588	0.539	0.669	0.952	0.667	0.526	0.447	0.500	0.670
VMR	0.868	0.771	0.711	0.953	1.051	0.869	0.772	0.709	0.939	1.037	0.895	0.804	0.696	0.667	0.718
VMSR	0.788	0.660	0.557	0.643	0.965	0.788	0.659	0.555	0.636	0.952	0.772	0.628	0.497	0.474	0.665
VMRSR	0.867	0.762	0.658	0.822	1.039	0.867	0.762	0.656	0.811	1.024	0.880	0.777	0.643	0.581	0.708
AMO	0.762	0.630	0.529	0.560	0.867	0.761	0.628	0.527	0.555	0.857	0.722	0.572	0.455	0.430	0.614
AMR	0.929	0.858	0.749	0.710	1.005	0.929	0.859	0.749	0.701	0.990	0.937	0.872	0.757	0.539	0.681
AMSR	0.818	0.694	0.573	0.583	0.923	0.817	0.693	0.571	0.577	0.911	0.799	0.661	0.519	0.440	0.640
AMRSR	0.901	0.813	0.691	0.661	0.989	0.902	0.813	0.690	0.653	0.974	0.906	0.818	0.683	0.494	0.672
HMO	0.801	0.675	0.552	0.425	0.420	0.800	0.673	0.549	0.423	0.418	0.751	0.605	0.473	0.346	0.339
HMR	0.910	0.828	0.730	0.875	1.047	0.911	0.828	0.729	0.862	1.033	0.925	0.852	0.737	0.628	0.714
HMSR	0.835	0.717	0.588	0.488	0.715	0.835	0.716	0.586	0.485	0.707	0.812	0.678	0.531	0.386	0.516
HMRSR	0.878	0.777	0.653	0.691	1.009	0.878	0.777	0.652	0.682	0.995	0.881	0.778	0.635	0.506	0.688
GMO	0.782	0.651	0.536	0.481	0.649	0.780	0.649	0.533	0.478	0.643	0.737	0.588	0.461	0.382	0.486
GMR	0.920	0.843	0.736	0.792	1.036	0.920	0.844	0.736	0.781	1.022	0.931	0.862	0.745	0.580	0.705
GMSR	0.827	0.705	0.579	0.538	0.842	0.826	0.704	0.577	0.533	0.832	0.805	0.669	0.525	0.415	0.592
GMRSR	0.896	0.804	0.683	0.706	1.015	0.896	0.804	0.682	0.697	1.001	0.902	0.812	0.676	0.518	0.691
MO	0.774	0.643	0.535	0.538	0.678	0.773	0.641	0.533	0.534	0.672	0.728	0.580	0.461	0.421	0.503
MR	0.918	0.841	0.731	0.740	1.024	0.919	0.841	0.730	0.730	1.009	0.931	0.860	0.741	0.550	0.694
MSR	0.824	0.702	0.578	0.567	0.838	0.823	0.700	0.576	0.562	0.828	0.803	0.666	0.524	0.434	0.591
MRSR	0.896	0.805	0.683	0.679	1.008	0.896	0.805	0.682	0.671	0.994	0.902	0.812	0.676	0.502	0.685
n=40 p=7															
MSE OLS	0.520	1.082	2.279	12.668	46.852	0.528	1.099	2.315	12.868	139.689	0.785	1.634	3.441	19.128	207.649
FMO	0.795	0.669	0.571	0.498	0.485	0.793	0.732	0.569	0.496	0.483	0.732	0.609	0.528	0.472	0.459
FMR	0.805	0.720	0.724	0.920	1.009	0.806	0.792	0.721	0.912	1.001	0.844	0.736	0.676	0.791	0.875
FMSR	0.797	0.670	0.567	0.521	0.664	0.796	0.735	0.565	0.518	0.660	0.766	0.634	0.535	0.480	0.589
FMRSR	0.802	0.694	0.659	0.833	1.000	0.802	0.763	0.656	0.826	0.992	0.822	0.699	0.618	0.716	0.867
VMO	0.588	0.633	0.721	0.878	0.968	0.585	0.691	0.716	0.872	0.961	0.545	0.563	0.632	0.768	0.846
VMR	0.805	0.720	0.724	0.920	1.009	0.806	0.792	0.721	0.912	1.001	0.844	0.736	0.676	0.791	0.875
VMRSR	0.657	0.585	0.608	0.788	0.958	0.656	0.641	0.605	0.782	0.950	0.642	0.548	0.546	0.687	0.834
VMSR	0.802	0.694	0.659	0.833	1.000	0.802	0.763	0.656	0.826	0.992	0.822	0.699	0.618	0.716	0.867
AMO	0.589	0.569	0.615	0.758	0.894	0.587	0.623	0.612	0.754	0.888	0.558	0.522	0.550	0.667	0.786
AMR	0.943	0.876	0.758	0.653	0.950	0.943	0.963	0.758	0.649	0.943	0.945	0.879	0.757	0.580	0.817
AMSR	0.684	0.590	0.581	0.721	0.919	0.683	0.647	0.578	0.717	0.913	0.670	0.561	0.530	0.631	0.801
AMRSR	0.889	0.786	0.664	0.665	0.946	0.889	0.864	0.664	0.661	0.939	0.890	0.785	0.650	0.583	0.816
HMO	0.630	0.545	0.513	0.513	0.525	0.628	0.596	0.511	0.511	0.523	0.590	0.507	0.478	0.476	0.484
HMR	0.914	0.821	0.726	0.828	1.005	0.914	0.903	0.726	0.821	0.997	0.925	0.834	0.710	0.711	0.871
HMSR	0.716	0.602	0.547	0.598	0.773	0.715	0.660	0.545	0.594	0.768	0.694	0.573	0.511	0.536	0.677
HMRSR	0.835	0.716	0.627	0.730	0.972	0.835	0.786	0.626	0.724	0.965	0.838	0.712	0.598	0.632	0.842
GMO	0.599	0.550	0.566	0.664	0.765	0.598	0.602	0.563	0.660	0.760	0.568	0.510	0.516	0.591	0.672
GMR	0.932	0.852	0.728	0.729	0.991	0.933	0.937	0.728	0.724	0.984	0.938	0.860	0.726	0.632	0.858
GMSR	0.696	0.593	0.566	0.675	0.866	0.695	0.650	0.563	0.670	0.860	0.679	0.565	0.522	0.594	0.753
GMRSR	0.880	0.771	0.656	0.707	0.974	0.880	0.848	0.655	0.702	0.966	0.884	0.772	0.639	0.614	0.843
MO	0.583	0.563	0.589	0.687	0.774	0.581	0.616	0.586	0.683	0.769	0.553	0.520	0.534	0.614	0.673
MR	0.945	0.873	0.747	0.705	0.987	0.945	0.960	0.746	0.700	0.979	0.948	0.876	0.741	0.613	0.854
MSR	0.679	0.588	0.572	0.687	0.867	0.678	0.645	0.570	0.682	0.860	0.664	0.560	0.527	0.605	0.751
MRSR	0.891	0.786	0.663	0.695	0.971	0.891	0.864	0.662	0.690	0.964	0.893	0.785	0.645	0.604	0.841

Note: Proposed techniques are bolded.