Hacettepe Journal of Mathematics and Statistics
Volume 46 (5) (2017), 985 – 1014

Ordering and transfer policy and variable deterioration for a warehouse model

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Abstract

This paper represents an inventory model for ordering and transferring policy with random, deterioration and several demand patterns. The number of transfers per order from warehouse to display area and optimal order quantity are determined in this model. The random deterioration makes a realistic scenario when the retailer has several products. The model uses different demand pattern to check the optimum profit for different situation. The main purpose of this model is to maximize average profit per unit time for retailer. The retailer receives products from supplier and store in a warehouse. These items are transferred through multi-delivery policy with equal lot-size. There are four lemmas to establish the global maximum solution analytically. Some numerical examples and graphical representations are given to illustrate the model.

Keywords: Inventory, probabilistic deterioration, variable demand, ordering policies.

Received : 16.01.2015 *Accepted* : 22.11.2015 *Doi* : 10.15672/HJMS.201510414237

1. Introduction

It is common phenomenon/phenomena that demand is increasing with the time increasing. There are many products for which demand rate depends on time. Demand of items may increase or decrease with time. Many mathematical models have been developed to control inventory by considering constant demand rate while in most of the cases, demand of items increase with time. Harris [1] first discovered an EOQ (economic order quantity) for constant demand pattern. Regarding demand as time dependent, many researchers formulated several inventory models. Hsu and Li [2] discussed an inventory model with delivery service strategy for internet shopping and time dependent

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consumer demand. Dye et al. [3] observed an inventory model by including not only cost of lost sales, but also the non-constant purchase cost. They extended their model from a constant demand to any log-concave demand function. Khanra et al. [4] developed an inventory model for time dependent demand with delay-in-payments, but with constant deterioration rate. Sarkar et al. [5] formed an economic production quantity (EPQ) model for both continuous and discrete random demand. They considered certain percent of total product is of imperfect quality which followed a probability distribution. The imperfect quality items are reworked at a fixed cost. The percent of defectiveness in the total product usually increases with an increase in production run-time. Sarkar and Moon [6] extended Sarkar et al.'s [5] model with the effect of inflation. They highlighted imperfect items which are reworked at some fixed costs and considered shortages due to the production of imperfect products. The lifetime of defective items followed a Weibull distribution. Sarkar et al. [7] studied an imperfect production model which produces a single type of items. Their model formulated by time-dependent demand with reliability as a decision variable under effect of inflation and time-value of money.

The loss due to deterioration of items like vegetable, or commodities cannot be ignored. The growth and application of inventory control models regarding deterioration products is the main concerns of researcher. Many previous studies have been done in this field by assuming constant deterioration. But deterioration of item may vary with time. Using present value concept, many researchers stated about the distribution processing for deterioration. Wee and Law [8] presented an inventory model with deterioration, time-value of money, and price-dependent demand. Their model applied the discounted cash flows (DCF) approach for problem analysis. Chu and Chen [9] proposed the inventory carrying cost is in proportion to the cost of deteriorated items. Khanra and Chaudhuri [10] invented an order-level inventory problem on continuous and quadratic function of time-dependent demand. They assumed a constant fraction of the on-hand inventory which deteriorated per unit of time. In their model for infinite and finite time-horizon, the solution of model was discussed analytically. Chern et al. [11] extended previous inventory model by allowing general partial backlogging rate and inflation. They considered inventory lot size models for deteriorating items with fluctuating demand under inflation. Sett et al. [12] developed a two-warehouse inventory model based on the assumption of quadratic demand which is useful for those items whose demand increases very rapidly. Their study discussed about time-dependent deterioration rates. Sarkar et al. [13] formulated an optimal inventory replenishment policy for a deteriorating item with quadratic time-dependent demand and time-dependent partial backlogging. Sarkar [14] constructed an inventory model for finite replenishment rate where demand and deterioration rate both are time-dependent. Sarkar and Sarkar [15] developed an inventory model for time-dependent deterioration rate. Their study discussed about inventory-dependent demand function. They considered three possible cases for demand and inventory. Sarkar and Sarkar [16] extended earlier literature with infinite replenishment rate by including stock-dependent demand, time-varying deterioration, and partial backlogging. Sarkar [17] presented a production-inventory model for deteriorating item in a two-echelon supply chain management (SCM). An algebraical approach with the help of three types of continuous probabilistic deterioration functions are employed to obtain the associated cost. Sarkar and Sarkar [18] extended an economic manufacturing quantity (EMQ) model with deterioration and exponential demand under the effect of inflation and time value of money. The production rate is a dynamic variable (varying with time) in a production system. To reduce the production of improper items, they incorporated development cost, production cost, and material cost which are dependent on reliability parameter with probabilistic deterioration. Sarkar et al. [19] discussed an inventory model for finite production rate and deteriorating items with time-dependent increasing demand. The component cost and selling price both are considered at a continuous rate of time in their model. Sarkar et al. [20] developed an inventory model with two-level trade-credit policy for fixed lifetime products. They highlighted the assumption that suppliers offer full trade-credit to retailers, but retailers offer partial trade-credit to their customers.

Pricing is also an important factor in success of business for any item. In general, when selling price of items decreases, customers are more attracted to that product. Hence, demand rate of an item may consider based on the selling-price dependent. Wee [21] analyzed an inventory model for price-dependent demand of items with variable deterioration and completely backorder. Datta and Paul [22] derived an inventory system where the demand rate was influenced by stock-level and selling-price. They considered a finite period system under multi-replenishment scenario. Goyal and Chang [23] obtained an ordering-transfer inventory model for determining retailer's optimal order quantity and the number of transfers per order from warehouse to display area. They assumed limited display space and stock-level-dependent demand rate. Sarkar et al. [24] discovered an inventory model under the assumption that retailers are allowed a period by supplier to obtain trade-credit for goods bought with some discount rates. They developed retailer's optimal replenishment decision under trade-credit policy with inflation. They assumed several types of deterministic demand patterns with the delay-periods and different discounts rates on purchasing cost. Sarkar et al. [25] developed an imperfect production process for stock-dependent demand. These imperfect items were reworked at some fixed cost for restoring its original quality. In addition, in their model unit production cost is a function of reliability parameter and production rate. Sana [26] investigated an inventory model to obtain retailer's optimal order quantity for similar products with limited display space. In his article, demand of products depends on selling price, salesmen's initiatives and display stock-level where more stocks of one product makes a negative impression of another products. Sarkar [27] assumed an imperfect production process with price and advertising demand pattern under the effect of inflation. To reduce the production of imperfect items, development cost, production cost, and material cost are dependent on reliability in his model. Sarkar [28] deduced an inventory framework in which supplier generally offers a delay-period to the retailer to buy more. In this point of view, retailer's optimal replenishment policy under permissible delay-in-payment are considered with stock-dependent demand with finite replenishment rate and the production of defective items. See Table 1 for contribution of various authors.

In this article, an inventory model for probabilistic deteriorating rate is considered with several demand function as time and price dependent, and finite production rate. The display space is taken to be limited. This model includes the number of transfer per order from the warehouse to display area. The main objective of this paper is to maximize average profit function over finite planning horizon and obtain the optimal order quantity and the number of transfer per order. In Section 2, there are four cases of demand functions. The average profit per unit time for the retailer is maximized in each case. Section 3 derives some numerical examples and sensitivity analysis for each numerical example. Finally, conclusions and future extensions are discussed in Section 4.

2. Mathematical model

Following notation are used to formulate this model.

Decision variables

- t_1 replenishment cycle time in display area (year)
- n integer number of shipments for stocks from warehouse to display area per order

Table 1. Contribution of various authors

Author(s)	Time-dependent demand	Price-dependent demand	Other demands	Probabilistic deterio-ration	Other deterio-rations
Harris [1]			$\surd$		
Hsu and Li [2]	$\surd$				
Dye et al. [3]	$\surd$				$\surd$
Khanra et al. [4]	$\surd$				$\surd$
Sarkar et al. [5]			$\surd$		$\surd$
Sarkar and Moon [6]			$\surd$	$\surd$	
Sarkar et al. [7]	$\surd$				$\surd$
Wee and Law [8]		$\surd$			$\surd$
Chu and Chen [9]	$\surd$			$\surd$	
Khanra and Chaudhuri [10]	$\surd$				$\surd$
Chern et al. [11]			$\surd$		$\surd$
Sett et al. [12]	$\surd$				$\surd$
Sarkar et al. [13]	$\surd$				$\surd$
Sarkar [14]	$\surd$				$\surd$
Sarkar and Sarkar [15]				$\surd$	
Sarkar and Sarkar [16]			$\surd$		$\surd$
Sarkar [17]			$\surd$		$\surd$
Sarkar and Sarkar [18]			$\surd$	$\surd$	
Sarkar et al. [19]	$\surd$				$\surd$
Sarkar et al. [20]					$\surd$
Wee [21]		$\surd$			$\surd$
Datta and Paul [22]		$\surd$			
Goyal and Chang [23]			$\surd$		
Sarkar et al. [24]		$\surd$			
Sarkar et al. [25]		$\surd$			
Sana [26]		$\surd$	$\surd$		
Sarkar [27]		$\surd$			$\surd$
Sarkar [28]			$\surd$		
This model	$\surd$	$\surd$		$\surd$	

p unit selling price of stocks per unit (\$/units)

Parameters

h_1 unit carrying cost per stock in warehouse (\$/units/unit time)
h unit carrying cost per stock in display area, where $h > h_1$ (\$/units/unit time)
c unit purchasing cost (\$/units)
S retailer's ordering cost per order (\$/order)
s fixed cost of stocks per transfer to display area from warehouse (\$/transfer)
T replenishment cycle time in warehouse
Q order quantity placed to the supplier (units)
$I(t)$ inventory level at time t in the display area
R fixed inventory level of stocks in display area for transfering of q items reducing stockout during variable demand
q stock per transfer to display area from warehouse (units/transfer)
D demand function considered as time-dependent, price-dependent, and time-price dependent
θ probabilistic deterioration rate, $0 < \theta < 1$
AP_1 average profit for demand function $D(t,p) = x + x_1 + yt - y_1p + zt^2 - z_1p^2$
AP_2 average profit while demand function is $D(t) = x + yt + zt^2$

AP_3 average profit for demand function $D(p) = x_1 - y_1 p - z_1 p^2$
AP_4 average profit when demand function is $D(t) = x_2 e^{y_2 t}$

This model is based on the following assumptions

1. The model consider a warehouse problem with random deterioration rate θ which follows a uniform distribution.
   The probability density function of deterioration is
   $$f(X) = \left\{ \begin{array}{ll} \frac{1}{b-a}, & \text{if } X \in [a,b] \\ 0, & otherwise \end{array} \right\}$$
   where a and $b(> a)$ are two parameters of this distribution and $0 < a < b < 1$.
   Therefore, $\theta = E[f(X)] = \frac{b+a}{2}$.
2. The retailer orders quantity Q per order from a supplier and stores them into the warehouse. These items are transferred to display area from warehouse in equal lots of q until inventory level in warehouse reaches to zero.
3. The transferring time of stocks from warehouse to display area is taken as negligible.
4. The demand function as follows
   $D(t,p) = x + x_1 + yt - y_1 p + zt^2 - z_1 p^2$, $D(t) = x + yt + zt^2$, $D(p) = x_1 - y_1 p - z_1 p^2$, and $D(t) = x_2 e^{y_2 t}$. x, y, and z are beginning rate, increasing rate, and rate of change for demand in first and second demand function respectively. x_1, y_1, and z_1 are initial rate, decreasing rate, and rate of change for demand in first and third demand function separately. x_2 is constant parameter and y_2 is increasing rate of demand regarding fourth demand function.
5. Lead time is considered as negligible and shortages are not allowed.

Here, an inventory model related with warehouse and display area are considered. Two types of costs (warehouse cost and display area cost) are given. These costs are used to calculate the profit of the model.

Warehouse cost

When the retailer orders Q items from the supplier, it is instantly supplied to the retailer and the retailer stocks all the items in the warehouse. Now, the items Q can be divided into q equal parts i.e., $Q = nq$ and a part is transferred to the display area when the previous part has just been depleted. The process will continue until the inventory at the warehouse reaches at zero level.

Retailer's ordering cost per order is $= S$.

During the time interval $[0,\ t_1]$, total item is

$$[q + 2q + 3q + ... + (n-1)q]t_1 \quad = \quad \frac{n(n-1)qt_1}{2}.$$

Hence, the stock holding cost is $= h_1 t_1 \frac{n(n-1)}{2} q$.

Cost at display area

Initially at time $t = 0$, the inventory level $I(t)$ starts with a maximum inventory say $\bar{I}$ and then it reaches to R at the end of cycle t_1. Figure 1 represents the inventory system.

Case I

In this case, demand rate is considered as a function of price and time. As demand may increase when the selling price diminishes and vice-versa or it may fluctuate with the change of time. The consideration of time and price-dependent demand is useful for deteriorated items, for example, fashionable goods, fruits, and vegetables. This study discussed an inventory model by assuming demand as a quadratic function of time and

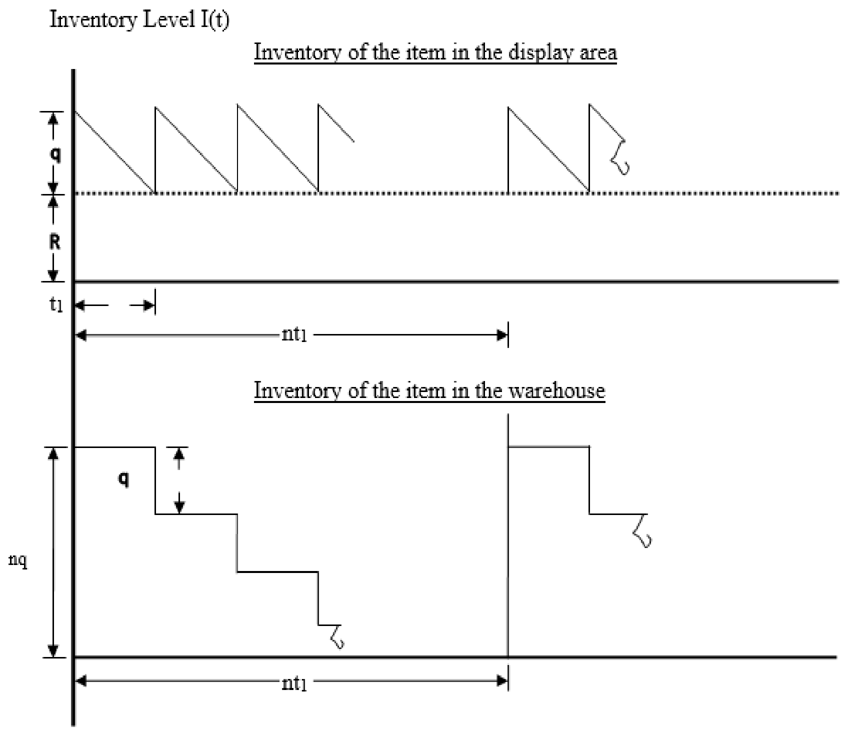

Figure 1

price. i.e.,

$$\begin{aligned} f(t,p) &= D(t,p) = (x + yt + zt^2) + (x_1 - y_1 p - z_1 p^2) \\ &= x + x_1 + yt - y_1 p + zt^2 - z_1 p^2 \end{aligned}$$

The governing differential equation of the inventory system is

$$\begin{aligned} \frac{dI(t)}{dt} + \theta I(t) &= -f(t,p), \quad 0 \le t \le t_1, \ I(t_1) = R \\ &= -(x + x_1 + yt - y_1 p + zt^2 - z_1 p^2) \end{aligned}$$

Using the boundary condition, inventory level *I(t)* as

$$\begin{aligned} I(t) &= \frac{(1 - e^{\theta(t_1 - t)})}{\theta}(y_1 p + z_1 p^2 - x - x_1) + y\Big(\frac{(t_1 e^{\theta(t_1-t)} - t)}{\theta} - \frac{(e^{\theta(t_1-t)} - 1)}{\theta^2}\Big) \\ &+ z\left(\frac{({t_1}^2 e^{\theta(t_1-t)} - t^2)}{\theta} - \frac{(2t_1 e^{\theta(t_1-t)} - 2t)}{\theta^2} + \frac{(2e^{\theta(t_1-t)} - 2)}{\theta^3}\right) + Re^{\theta(t_1-t)} \qquad (2.1) \end{aligned}$$

During $[0,t_1]$, the total costs are as follows:
(i) Fixed cost of stocks per transfer to display area from warehouse is $= s$.

(ii) Holding cost is

$$= h\int_0^{t_1} I(t)dt = \frac{hR(e^{\theta t_1}-1)}{\theta} + \frac{h(y_1p+z_1p^2-x-x_1)}{\theta}\Big(t_1 + \frac{(1-e^{\theta t_1})}{\theta}\Big)$$
$$+hy\Big(\frac{t_1e^{\theta t_1}}{\theta^2} + \frac{(1-e^{\theta t_1})}{\theta^3} - \frac{{t_1}^2}{2\theta}\Big)$$
$$+hz\left(\frac{{t_1}^2e^{\theta t_1}}{\theta^2} - \frac{2t_1e^{\theta t_1}}{\theta^3} + \frac{(2e^{\theta t_1}-2)}{\theta^4} - \frac{{t_1}^3}{\theta^3}\right)$$

(iii) The revenue per cycle is

$$= (p-c)\int_0^{t_1} D(t,p)dt = (p-c)\int_0^{t_1}(x+x_1+yt-y_1p+zt^2-z_1p^2)dt$$
$$= (p-c)\left((x+x_1-y_1p-z_1p^2)t_1 + \frac{y{t_1}^2}{2} + \frac{z{t_1}^3}{3}\right)$$

Equating equation (1) and $I(0)=q+R$, we obtain

$$\begin{aligned} q &= \frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1) + y\Big(\frac{t_1e^{\theta t_1}}{\theta} - \frac{(e^{\theta t_1}-1)}{\theta^2}\Big) \\ &+ z\left(\frac{{t_1}^2e^{\theta t_1}}{\theta} - \frac{2t_1e^{\theta t_1}}{\theta^2} + \frac{(2e^{\theta t_1}-2)}{\theta^3}\right) + Re^{\theta t_1} - R \end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$\begin{aligned} = h_1\left[\frac{n(n-1)}{2}q\right]t_1 &= h_1\Big[\frac{n(n-1)}{2}\Big(\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1) + y\Big(\frac{t_1e^{\theta t_1}}{\theta} \\ &- \frac{(e^{\theta t_1}-1)}{\theta^2}\Big) + z\left(\frac{{t_1}^2e^{\theta t_1}}{\theta} - \frac{2t_1e^{\theta t_1}}{\theta^2} + \frac{(2e^{\theta t_1}-2)}{\theta^3}\right) + Re^{\theta t_1} \\ &- R\Big)\Big]t_1 \end{aligned}$$

Thus, the average profit per unit time is

: $AP_1(n,p,t_1) = \frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where $T=nt_1$)

$$\begin{aligned} &= (p-c)\left((x+x_1-y_1p-z_1p^2) + \frac{yt_1}{2} + \frac{z{t_1}^2}{3}\right) - \Big[h_1\Big(\frac{(n-1)}{2}\Big(\frac{(1-e^{\theta t_1})}{\theta}(y_1p \\ &+ z_1p^2-x-x_1) + y\Big(\frac{t_1e^{\theta t_1}}{\theta} - \frac{(e^{\theta t_1}-1)}{\theta^2}\Big) + z\left(\frac{{t_1}^2e^{\theta t_1}}{\theta} - \frac{2t_1e^{\theta t_1}}{\theta^2} + \frac{(2e^{\theta t_1}-2)}{\theta^3}\right) \\ &+ Re^{\theta t_1} - R\Big)\Big) + \frac{S}{nt_1}\Big] - \frac{s}{t_1} - h\Big[\frac{R(e^{\theta t_1}-1)}{\theta t_1} + \frac{(y_1p+z_1p^2-x-x_1)}{\theta}\Big(\frac{(1-e^{\theta t_1})}{\theta t_1} \\ &+ 1\Big) + y\Big(\frac{e^{\theta t_1}}{\theta^2} + \frac{(1-e^{\theta t_1})}{\theta^3t_1} - \frac{t_1}{2\theta}\Big) + z\left(\frac{t_1e^{\theta t_1}}{\theta^2} - \frac{2e^{\theta t_1}}{\theta^3} + \frac{(2e^{\theta t_1}-2)}{\theta^4t_1} - \frac{{t_1}^2}{\theta^3}\right)\Big] \end{aligned}$$

Now, we have to maximize the total profit function. Thus, we have formulated a lemma to obtain the global optimum solution.

Lemma 1

$AP_1(n^*,p^*,{t_1}^*)$ will have the global maximum solution where n^*, p^*, and ${t_1}^*$ are optimal values of n, p, and t_1 if following conditions are satisfied

$$(i)\ 4S\theta < h_1n^3t_1(1-e^{\theta t_1})(y_1+2z_1p),$$

$$(ii)\ 4z_1p + \frac{h_1}{2}\frac{(1-e^{\theta t_1})}{\theta}(y_1+2z_1p) > 2\Big[cz_1 - y_1 - \frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) +h\left(\frac{1-e^{\theta t_1}}{\theta t_1}+1\right)\Big)\Big],$$

$$(iii)\ h_1 \frac{(e^{\theta t_1}-1)}{\theta}(y_1+2z_1p)M < N\left(\frac{h_1}{2}U+\frac{s}{{t_1}^2}\right),$$

$$(iv)\left(\frac{y}{2}+\frac{2zt_1}{3}\right)+\frac{h}{\theta}\left(\frac{e^{\theta t_1}}{t_1}+\frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right) < \frac{h_1(1-n)}{2}(y_1+2z_1p)e^{\theta t_1}.$$

[See Appendix A for the values of M, N, and U.]

Proof

The necessary condition for optimal solution of $AP_1(n,p,t_1)$ can be calculated by $\frac{\partial AP_1(n,p,t_1)}{\partial n}=0$, $\frac{\partial AP_1(n,p,t_1)}{\partial p}=0$, and $\frac{\partial AP_1(n,p,t_1)}{\partial t_1}=0$.
i.e.,

$$\begin{aligned}\frac{\partial AP_1(n,p,t_1)}{\partial n} &= \frac{S}{n^2{t_1}^2}-\frac{h_1}{2}\Big[\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1)\\ &+ y\left(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\right)\\ &+ z\left(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2}+\frac{2(e^{\theta t_1}-1)}{\theta^3}\right)+Re^{\theta t_1}-R\Big]=0\end{aligned}$$

gives $n=\sqrt{\frac{2S}{h_1t_1f}}$, [See Appendix B for the value of f] .
For the decision variable p,

$$\frac{\partial AP_1(n,p,t_1)}{\partial p} = 0$$

i.e.,

$$\begin{aligned}\frac{\partial AP_1(n,p,t_1)}{\partial p} &= 2z_1p^2-2p\Big[cz_1-y_1-\frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1})+h\Big(\frac{(1-e^{\theta t_1})}{\theta t_1}\\ &+ 1\Big)\Big)\Big]-\Big[x+x_1+\frac{yt_1}{2}+\frac{z{t_1}^2}{3}+cy_1-\frac{y_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1})\\ &+ h\left(\frac{(1-e^{\theta t_1})}{\theta t_1}+1\right)\Big)\Big]=0\end{aligned}$$

Now p^* will be calculated if $\eta(p^*)=0$ where $\frac{\partial AP_1(n,p,t_1)}{\partial p}=\eta(p)$ and $\frac{\partial AP_1(n,p,t_1)}{\partial t_1}=0$ gives i.e.,

$$\begin{aligned}\frac{\partial AP_1(n,p,t_1)}{\partial t_1} &= \left(\frac{y}{2}+\frac{2zt_1}{3}\right)+\frac{h_1(n-1)}{2}(y_1p+z_1p^2-x-x_1)e^{\theta t_1}\\ &- \frac{h_1(n-1)}{2}(yt_1e^{\theta t_1}+z{t_1}^2e^{\theta t_1}+R\theta e^{\theta t_1})+\frac{(S+sn)}{{t_1}^2}\\ &- h\Big[\frac{Re^{\theta t_1}}{t_1}-\frac{(y_1p+z_1p^2-x-x_1)}{\theta}\Big(\frac{e^{\theta t_1}}{t_1}+\frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\Big)\\ &- \frac{(Re^{\theta t_1}-1)}{\theta {t_1}^2}+y\left(\frac{e^{\theta t_1}}{\theta}-\frac{e^{\theta t_1}}{\theta^2t_1}-\frac{(1-e^{\theta t_1})}{\theta^3{t_1}^2}-\frac{1}{2\theta}\right)\\ &- z\Big(\frac{e^{\theta t_1}}{\theta^2}-\frac{t_1e^{\theta t_1}}{\theta}-\frac{2e^{\theta t_1}}{\theta^3t_1}+\frac{2(e^{\theta t_1}-1)}{\theta^4{t_1}^2}+\frac{2t_1}{\theta^3}\Big)\Big]=0\end{aligned}$$

Now ${t_1}^*$ will be calculated if $\xi_1({t_1}^*)=0$ where $\frac{\partial AP_1(n,p,t_1)}{\partial t_1}=\xi_1(t_1)$.
To verify the sufficient conditions for global optimum solution, the second order partial derivatives of $AP_1(n,p,t_1)$ with respect to n, p, and t_1 are as follows:

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3t_1},$$

$$
\begin{aligned}
\frac{\partial^2 AP_1(n,p,t_1)}{\partial {t_1}^2} &= \frac{2z}{3} + \frac{h_1(n-1)}{2}\theta(y_1p + z_1p^2 - x - x_1)e^{\theta t_1} \\
&- \frac{h_1(n-1)}{2}e^{\theta t_1}(y + y\theta t_1 + 2zt_1 + z{t_1}^2\theta + R\theta^2) - \frac{2(S+sn)}{{t_1}^3} \\
&- h\Big[\frac{R\theta e^{\theta t_1}}{t_1} - \frac{2R(e^{\theta t_1}-1)}{\theta {t_1}^3} \\
&+ \frac{(y_1p + z_1p^2 - x - x_1)}{\theta}\Big(\frac{-\theta e^{\theta t_1}}{t_1} + \frac{2e^{\theta t_1}}{{t_1}^2} + \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3}\Big) \\
&+ y\Big(e^{\theta t_1} - \frac{e^{\theta t_1}}{\theta t_1} + \frac{2e^{\theta t_1}}{\theta^2 t_1^2} + \frac{2(1-e^{\theta t_1})}{\theta^3 {t_1}^3}\Big) \\
&+ z\Big(t_1e^{\theta t_1} + \frac{2e^{\theta t_1}}{\theta^2 t_1} - \frac{4e^{\theta t_1}}{\theta^3 {t_1}^2} + \frac{4(e^{\theta t_1}-1)}{\theta^4 {t_1}^3} - \frac{2}{\theta^3}\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 AP_1(n,p,t_1)}{\partial p^2} &= 4z_1p - 2\Big[cz_1 - y_1 - \frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\Big(\frac{(1-e^{\theta t_1})}{\theta t_1} \\
&+ 1\Big)\Big)\Big],
\end{aligned}
$$

$$
\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial t_1} = \frac{h_1}{2}e^{\theta t_1}(y_1p + z_1p^2 - x - x_1 - yt_1 - z{t_1}^2 - R\theta)\frac{s}{{t_1}^2},
$$

$$
\begin{aligned}
\frac{\partial^2 AP_1(n,p,t_1)}{\partial p \partial t_1} &= \left(\frac{y}{2} + \frac{2zt_1}{3}\right) + \frac{h_1(n-1)}{2}(y_1 + 2z_1p)e^{\theta t_1} \\
&+ \frac{h(y_1+2z_1p)}{\theta}\left(\frac{e^{\theta t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right),
\end{aligned}
$$

and

$$
\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 + 2z_1p)\frac{(e^{\theta t_1}-1)}{\theta}\right).
$$

The sufficient condition for global optimum solution for this case is all principal minors are alternating in sign.
i.e., the sufficient condition for the optimum solution of $AP_1(n,t_1)$ are $\frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} < 0$, $\frac{\partial^2 AP_1}{\partial n^2}\frac{\partial^2 AP_1}{\partial p^2} - (\frac{\partial^2 AP_1}{\partial n \partial p})^2 > 0$, and the value of third principal minor i.e., the value of the Hessian matrix $H < 0$.
Now

$$
\frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0
$$

To show the condition of second principal minor, if $\frac{\partial^2 AP_1}{\partial n^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ and $\frac{\partial^2 AP_1}{\partial p^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$, then the condition holds.
Now

$$
\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 + 2z_1p)\frac{(e^{\theta t_1}-1)}{\theta}\right)
$$

which can be written as

$$
\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_1(n,p,t_1)}{\partial n^2} - \xi_1.
$$

where

$$
\xi_1 = \frac{h_1}{2}\frac{(1-e^{\theta t_1})}{\theta}(y_1 + 2z_1p) - \frac{2S}{n^3 t_1}
$$

$\frac{\partial^2 AP_1}{\partial n^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ will hold if $\xi_1 > 0$.
Now $\xi_1 > 0$ exists when

$$4S\theta < h_1 n^3 t_1 (1 - e^{\theta t_1})(y_1 + 2z_1 p)$$

Similarly,

$$\frac{\partial^2 AP_1(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_1(n,p,t_1)}{\partial p^2} - \xi_2$$

where

$$\begin{aligned} \xi_2 &= 4z_1 p - 2\left[cz_1 - y_1 - \frac{z_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{1-e^{\theta t_1}}{\theta t_1} + 1\right)\right)\right] \\ &+ \frac{h_1}{2}\frac{(1-e^{\theta t_1})}{\theta}(y_1 + 2z_1 p) \end{aligned}$$

$\frac{\partial^2 AP_1}{\partial p^2} > \frac{\partial^2 AP_1}{\partial n \partial p}$ will exist if $\xi_2 > 0$.
i.e., if

$$4z_1 p + \frac{h_1}{2}\frac{(1-e^{\theta t_1})}{\theta}(y_1 + 2z_1 p) > 2\left[cz_1 - y_1 - \frac{z_1}{\theta}\left(\frac{h_1(n-1)}{2}(1-e^{\theta t_1}) + h\left(\frac{1-e^{\theta t_1}}{\theta t_1} + 1\right)\right)\right]$$

Similar as above, value of third principal minor i.e., $H < 0$ will hold if

$$h_1 \frac{(e^{\theta t_1} - 1)}{\theta}(y_1 + 2z_1 p)M < N\left(\frac{h_1}{2}U + \frac{s}{{t_1}^2}\right)$$

and

$$\left(\frac{y}{2} + \frac{2zt_1}{3}\right) + \frac{h}{\theta}\left(\frac{e^{\theta t_1}}{t_1} + \frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right) < \frac{h_1(1-n)}{2}(y_1 + 2z_1 p)e^{\theta t_1}$$

Therefore, $AP_1(n^*, p^*, {t_1}^*)$ will have the global maximum (where n^*, p^*, and ${t_1}^*$ are optimal values of n, p, and t_1) if the conditions hold.

Case II
This section provides demand function is time-dependent. As time increases, the demand of each product increases. To show this matter, the demand is considered as quadratic function of time.
For the demand function $f(t) = D(t) = x + yt + zt^2$
The governing differential equation of the inventory system is

$$\begin{aligned} \frac{dI(t)}{dt} + \theta I(t) &= -f(t), \quad 0 \le t \le t_1, \ I(t_1) = R \\ &= -(x + yt + zt^2) \end{aligned}$$

Using the boundary condition, the inventory level *I(t)* as

$$\begin{aligned} I(t) &= Re^{\theta(t_1 - t)} + (1 - e^{\theta(t_1 - t)})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) \\ &+ (t_1 e^{\theta(t_1 - t)} - t)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}\Big(-t^2 {t_1}^2 e^{\theta(t_1 - t)}\Big) \end{aligned} \tag{2.2}$$

During [0,t_1], the total costs are as follows:
(i) Fixed cost of stocks per transfer to display area from warehouse $= s$.

(ii) Holding cost is

$$
\begin{aligned}
= h\int_0^{t_1} I(t)dt \quad &= \quad \frac{-hR(1-e^{\theta t_1})}{\theta} + h\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\left(t_1 + \frac{(1-e^{\theta t_1})}{\theta}\right) \\
&- \quad h\Big(\frac{t_1(1-e^{\theta t_1})}{\theta} + \frac{t_1^2}{2}\Big)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) - \frac{hz}{\theta}\left(\frac{{t_1}^2(1-e^{\theta t_1})}{\theta} + \frac{{t_1}^3}{3}\right)
\end{aligned}
$$

(iii) The revenue per cycle is

$$
\begin{aligned}
= (p-c)\int_0^{t_1} D(t)dt &= (p-c)\int_0^{t_1}(x + yt + zt^2)dt \\
&= (p-c)\left(xt_1 + y\frac{t_1^2}{2} + z\frac{t_1^3}{3}\right)
\end{aligned}
$$

Equating equation (2) and *I(0)=q+R*, one has

$$
\begin{aligned}
q + R \quad &= \quad Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1 e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2 e^{\theta t_1} \\
i.e., \quad q \quad &= \quad Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1 e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2 e^{\theta t_1} - R
\end{aligned}
$$

(iv) Stock holding cost in the warehouse is

$$
\begin{aligned}
&= \quad h_1\left[\frac{n(n-1)}{2}q\right]t_1 \\
&= \quad h_1\Big[\frac{n(n-1)}{2}\Big(Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1 e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) \\
&+ \quad \frac{z}{\theta}{t_1}^2 e^{\theta t_1} - R\Big)\Big]t_1
\end{aligned}
$$

Hence, the average profit per unit time is

: $AP_2(n, t_1) \;=\; \frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where T=nt_1)

$$
\begin{aligned}
&= (p-c)\left(x + y\frac{t_1}{2} + z\frac{t_1^2}{3}\right) - \Big[\frac{S}{nt_1} + h_1\Big(\frac{(n-1)}{2}\Big(Re^{\theta t_1} + (1-e^{\theta(t_1-t)})\Big(\frac{y}{\theta^2} - \frac{2z}{\theta^3} \\
&-\frac{x}{\theta}\Big) + t_1 e^{\theta t_1}\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) + \frac{z}{\theta}{t_1}^2 e^{\theta t_1} - R\Big)\Big)\Big] - \frac{s}{t_1} - h\Big[\frac{-R(1-e^{\theta t_1})}{\theta t_1} + \Big(\frac{y}{\theta^2} - \frac{2z}{\theta^3} \\
&-\frac{x}{\theta}\Big)\Big(1 + \frac{(1-e^{\theta t_1})}{\theta t_1}\Big) - \left(\frac{(1-e^{\theta t_1})}{\theta} + \frac{t_1}{2}\right)\left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) - \frac{z}{\theta}\left(\frac{t_1(1-e^{\theta t_1})}{\theta} + \frac{t_1^2}{3}\right)\Big]
\end{aligned}
$$

which we have to maximize with respect to the decision variables n and t_1. We have made the following lemma to make the global optimum solution for it.

Lemma 2

$AP_2(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) if following conditions are satisfied

(i)

$$
\frac{S}{n^2 t_1^2} + \frac{h}{2}e^{\theta t_1}\left(z{t_1}^2 + \left(y + \frac{2z}{\theta}\right)t_1 + \left(R\theta + \frac{y}{\theta}\right)\right) > \frac{2S}{n^3 t_1} + he^{\theta t_1}\left(\frac{z(\theta t_1 + 1)}{\theta^2} + \frac{e^{-\theta t}\left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)}{2}\right)
$$

and

(ii)

$$
\frac{2(p-c)z}{3} + \frac{2hz}{3\theta} + \frac{S}{n^2 {t_1}^2} + \alpha e^{\theta(t_1 - t)} + (\beta+\gamma)e^{\theta t_1} + \delta > \frac{2}{{t_1}^3}\left(\frac{S}{n} + s\right) + \frac{h_1 Re^{\theta t_1}\theta}{2}[(n-1)\theta - 1]
$$

Proof

From the necessary condition of the optimal solution, $\frac{\partial AP_2(n,t_1)}{\partial n} = 0$.

i.e.,

$$\begin{aligned}\frac{\partial AP_2(n,t_1)}{\partial n} &= \frac{S}{n^2 t_1} - \frac{h_1}{2}\Big(Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} + \frac{2z}{\theta^3} - \frac{x}{\theta}\right) \\ &+ t_1 e^{\theta t_1}\left(\frac{y}{\theta} + \frac{2z}{\theta^2}\right) + \frac{z t_1^2 e^{\theta t_1}}{\theta} - R\Big) = 0\end{aligned}$$

which gives

$$n = \sqrt{\frac{2S}{h_1 t_1 (Re^{\theta t_1} + (1-e^{\theta t_1})\left(\frac{y}{\theta^2} + \frac{2z}{\theta^3} - \frac{x}{\theta}\right) + t_1 e^{\theta t_1}\left(\frac{y}{\theta} + \frac{2z}{\theta^2}\right) + \frac{z t_1^2 e^{\theta t_1}}{\theta} - R)}}$$

For the second decision variable t_1, $\frac{\partial AP_2(n,t_1)}{\partial t_1} = 0$ gives

$$i.e., a_1 {t_1}^6 + a_2 {t_1}^5 + a_3 {t_1}^4 + a_4 {t_1}^3 + a_5 {t_1}^2 + a_6 = 0$$

See Appendix C for the values of $\alpha, \beta, \gamma, \delta, a_1, a_2, a_3, a_4, a_5$, and a_6.

Now ${t_1}^*$ will be obtained if $\xi_2({t_1}^*) = 0$ where $\frac{\partial AP_2(n,t_1)}{\partial t_1} = \xi_2(t_1)$.

To obtain the global maximum, we have to check the sufficient conditions. Thus, we have to calculated the second order partial derivatives of $AP_2(n,t_1)$ with respect to n and t_1 are as follows:

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1},$$

$$\frac{\partial^2 AP_2(n,t_1)}{\partial {t_1}^2} = \frac{\partial^2 \lambda_1}{\partial {t_1}^2} + \frac{\partial^2 \lambda_2}{\partial {t_1}^2}$$

where

$$\begin{aligned}\frac{\partial^2 \lambda_1}{\partial {t_1}^2} &= \frac{2(p-c)z}{3} - \frac{2S}{n {t_1}^3} - \frac{h_1(n-1)}{2}\Big[R\theta^2 e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1 - t)}\theta \\ &+ \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) e^{\theta t_1}\theta(t_1\theta + 2) + \frac{z}{\theta} e^{\theta t_1}(4 t_1 \theta + {t_1}^2 \theta^2 + 2)\Big]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \lambda_2}{\partial {t_1}^2} &= \frac{hR}{\theta}\left[\frac{2(1-e^{\theta t_1})}{{t_1}^3} + \frac{2\theta e^{\theta t_1}}{{t_1}^2} - \frac{\theta^2 e^{\theta t_1}}{t_1}\right] - \frac{2s}{t_1^3} - h\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\Big[\frac{2e^{\theta t_1}}{{t_1}^2} \\ &+ \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3} - \frac{\theta e^{\theta t_1}}{t_1}\Big] - h\left(y - \frac{2z}{\theta}\right) e^{\theta t_1} + \frac{hz}{\theta}\left(-t_1 \theta e^{\theta t_1} - 2 e^{\theta t_1} + \frac{2}{3}\right).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} &= -\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1 - t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) e^{\theta t_1}(\theta t_1 + 1) \\ &+ \frac{z}{\theta} t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]\end{aligned}$$

The sufficient conditions for the optimum solution of $AP_2(n,t_1)$ are $\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} < 0$ and $\frac{\partial^2 AP_2}{\partial n^2}\frac{\partial^2 AP_2}{\partial {t_1}^2} - (\frac{\partial^2 AP_2}{\partial n \partial t_1})^2 > 0$.

Now

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$$

We have to show $\frac{\partial^2 AP_2}{\partial n^2}\frac{\partial^2 AP_2}{\partial {t_1}^2} - (\frac{\partial^2 AP_2}{\partial n \partial t_1})^2 > 0$.

For the proof of this above condition, if $\frac{\partial^2 AP_2}{\partial n^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ and $\frac{\partial^2 AP_2}{\partial {t_1}^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$, then the

conditions hold.
Now

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} = -\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1-t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) e^{\theta t_1}(\theta t_1 + 1) + \frac{z}{\theta} t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]$$

which can be written as

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_2(n,t_1)}{\partial n^2} - \xi_3.$$

where

$$\xi_3 = \frac{S}{n^2 {t_1}^2} + \frac{h_1}{2}\Big[R\theta e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1-t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) e^{\theta t_1}(\theta t_1 + 1) + \frac{z}{\theta} t_1 e^{\theta t_1}(\theta t_1 + 2)\Big] - \frac{2S}{n^3 t_1}$$

$\frac{\partial^2 AP_2}{\partial n^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ will hold if $\xi_3 > 0$.
Now $\xi_3 > 0$ will exist if

$$\frac{S}{n^2 t_1^2} + \frac{h}{2} e^{\theta t_1}\left(z {t_1}^2 + \left(y + \frac{2z}{\theta}\right) t_1 + \left(R\theta + \frac{y}{\theta}\right)\right) > \frac{2S}{n^3 t_1} + h e^{\theta t_1}\left(\frac{z(\theta t_1 + 1)}{\theta^2} + \frac{e^{-\theta t}\left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right)}{2}\right)$$

Similarly,

$$\frac{\partial^2 AP_2(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_2(n,t_1)}{\partial {t_1}^2} - \xi_4$$

where

$$\begin{aligned}
\xi_4 = {} & \frac{2(p-c)z}{3} - \frac{2S}{n {t_1}^3} - \frac{h_1(n-1)}{2}\Big[R\theta^2 e^{\theta t_1} - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1-t)}\theta + \Big(\frac{y}{\theta} \\
& - \frac{2z}{\theta^2}\Big) e^{\theta t_1}\theta(t_1\theta + 2) + \frac{z}{\theta} e^{\theta t_1}(4t_1\theta + {t_1}^2\theta^2 + 2)\Big] - \frac{2s}{t_1^3} + \frac{hR}{\theta}\Big[\frac{2(1-e^{\theta t_1})}{{t_1}^3} \\
& + \frac{2\theta e^{\theta t_1}}{{t_1}^2} - \frac{\theta^2 e^{\theta t_1}}{t_1}\Big] - h\left(\frac{y}{\theta^2} - \frac{2z}{\theta^3} - \frac{x}{\theta}\right)\left[\frac{2e^{\theta t_1}}{{t_1}^2} + \frac{2(1-e^{\theta t_1})}{\theta {t_1}^3} - \frac{\theta e^{\theta t_1}}{t_1}\right] \\
& - h\left(y - \frac{2z}{\theta}\right) e^{\theta t_1} + \frac{hz}{\theta}\left(-t_1\theta e^{\theta t_1} - 2e^{\theta t_1} + \frac{2}{3}\right) + \frac{S}{n^2 {t_1}^2} + \frac{h_1}{2}\Big[R\theta e^{\theta t_1} \\
& - \left(\frac{y}{\theta} - \frac{2z}{\theta^2} - x\right) e^{\theta(t_1-t)} + \left(\frac{y}{\theta} - \frac{2z}{\theta^2}\right) e^{\theta t_1}(\theta t_1 + 1) + \frac{z}{\theta} t_1 e^{\theta t_1}(\theta t_1 + 2)\Big]
\end{aligned}$$

$\frac{\partial^2 AP_2}{\partial {t_1}^2} > \frac{\partial^2 AP_2}{\partial n \partial t_1}$ will exist if $\xi_4 > 0$.
i.e., if

$$\frac{2(p-c)z}{3} + \frac{2hz}{3\theta} + \frac{S}{n^2 {t_1}^2} + \alpha e^{\theta(t_1-t)} + (\beta+\gamma)e^{\theta t_1} + \delta > \frac{2}{{t_1}^3}\left(\frac{S}{n} + s\right) + \frac{h_1 R e^{\theta t_1}\theta}{2}[(n-1)\theta - 1]$$

Therefore, $AP_2(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) if the conditions hold.

Case III

In this section, demand of products is a function of selling price. In general, selling-price decreases means demand of products increases and vice-versa. Customers are more affective to that product whose selling price is low. Therefore, demand can be a function of selling price. Here, demand is taken to be as quadratic function of selling price.

The demand function is $f(p) = D(p) = x_1 - y_1 p - z_1 p^2$
The governing differential equation of the inventory system is

$$\begin{aligned} \frac{dI(t)}{dt} + \theta I(t) &= -f(p), \quad 0 \le t \le t_1, \quad I(t_1) = R \\ &= -(x_1 - y_1 p - z_1 p^2) \end{aligned}$$

Using the boundary condition, the inventory level *I(t)* as

$$I(t) = Re^{\theta(t_1 - t)} + \frac{(x_1 - y_1 p - z_1 p^2)(e^{\theta(t_1 - t)} - 1)}{\theta} \tag{2.3}$$

During $[0,t_1]$, the total costs are as follows:
(i) Fixed cost of stocks per transfer to display area from warehouse $= s$.
(ii) Holding cost is

$$= h\int_0^{t_1} I(t)dt = hR\frac{(e^{\theta t_1} - 1)}{\theta} + \frac{h(x_1 - y_1 p - z_1 p^2)}{\theta}\left(\frac{(e^{\theta}t_1 - 1)}{\theta} - t_1\right)$$

(iii) The revenue per cycle is

$$\begin{aligned} = (p-c)\int_0^{t_1} D(p)dt &= (p-c)\int_0^{t_1} (x_1 - y_1 p - z_1 p^2)dt \\ &= (p-c)(x_1 - y_1 p - z_1 p^2)t_1 \end{aligned}$$

Using equation (3) and *I(0)=q+R*

$$\begin{aligned} q + R &= Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1} - 1)}{\theta} \\ i.e., \quad q &= Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1} - 1)}{\theta} - R \end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$= h_1\left[\frac{n(n-1)}{2}q\right]t_1 = h_1 t_1\frac{n(n-1)}{2}[Re^{\theta t_1} + (x_1 - y_1 p - z_1 p^2)\frac{(e^{\theta t_1} - 1)}{\theta} - R]$$

Hence, the average profit per unit time is
$AP_3(n,p,t_1)=\frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)] (where $T=nt_1$)

$$\begin{aligned} &= (p-c)(x_1 - y_1 p - z_1 p^2) - \Big(\frac{S}{nt_1} + h_1\frac{(n-1)}{2}[Re^{\theta t_1} + (x_1 - y_1 p \\ &- z_1 p^2)\frac{(e^{\theta t_1} - 1)}{\theta} - R]\Big) - \frac{s}{t_1} - h\Big[R\frac{(e^{\theta t_1} - 1)}{\theta t_1} + \frac{(x_1 - y_1 p - z_1 p^2)}{\theta}\Big(\frac{(e^{\theta t_1} - 1)}{\theta t_1} \\ &- 1\Big)\Big] \end{aligned}$$

which we have to maximize with respect to the decision variables n, p, and t_1. To obtain the global optimum solution, we have formulated Lemma 3.

Lemma 3

$AP_3(n^*, p^*, t_1{}^*)$ will have the global maximum (where n^*, p^*, and $t_1{}^*$ are optimal values of n, p, and t_1) if following conditions are satisfied.

$$(i) 4S\theta > n^3 t_1(e^{\theta t_1} - 1)h_1(2z_1 p - y),$$

$$(ii)\frac{2z_1\theta\left(\frac{h}{\theta} + 3p\right)}{(e^{\theta t_1} - 1)} > \left[h_1\left(z_1(n - 1 + p) - \frac{y_1}{2}\right) + \frac{2z_1\theta}{\theta t_1}\right],$$

and

$$(iii)\left[1+\frac{sn}{S}+\frac{nt_1{}^3}{2S}\left(\frac{(n-1)\theta^2e^{\theta t_1}}{2}+h\right)\left(R+\frac{D(p)}{\theta}\right)\right]l_1l_3+\frac{nt_1{}^3}{2S}\Big(l_3l_4{}^2$$
$$+hh_1{}^2\left(\frac{y_1}{2}+z_1p^2\right)^2\Big)<l_5{}^2+\frac{nt_1{}^3h_1}{S}\left(\frac{y_1}{2}+z_1p\right)l_4l_5.$$

[See Appendix D for the values of $D(p), l_1, l_2, l_3, l_4$, and l_5.]

Proof

From the necessary conditions of the optimal solution, $\frac{\partial AP_3(n,p,t_1)}{\partial n}=0$.
i.e.,

$$\frac{\partial AP_3(n,p,t_1)}{\partial n} = \frac{S}{n^2t_1{}^2}-\frac{h_1}{2}\left[Re^{\theta t_1}+D(p)\frac{(e^{\theta t_1}-1)}{\theta}-R\right]=0$$

which gives

$$n=\sqrt{\frac{2S}{h_1t_1{}^2\left[Re^{\theta t_1}+D(p)\frac{(e^{\theta t_1}-1)}{\theta}-R\right]}}$$

For the decision variable p, $\frac{\partial AP_3(n,p,t_1)}{\partial p}=0$ gives

$$i.e., p = \frac{\theta(2y_1-cy_1-x_1-2cz_1)}{2z_1\left[\frac{h_1(n-1)}{2}\frac{(e^{\theta t_1}-1)}{\theta}-h\left(\frac{(e^{\theta t_1}-1)}{\theta t_1}-1\right)\right]}$$

For another decision variable t_1, $\frac{\partial AP_3(n,p,t_1)}{\partial t_1}=0$ gives

$$i.e., \frac{S+n^2s^2}{t_1{}^2} = \left(R+\frac{D(p)}{\theta}\right)\left[\frac{h_1(n-1)}{2}\theta e^{\theta t_1}-h\left(\frac{e^{\theta t_1}}{t_1}-\frac{(e^{\theta t_1}-1)}{\theta t_1{}^2}\right)\right]$$

Now $t_1{}^*$ will be obtained if $\xi_3(t_1{}^*)=0$ where $\frac{\partial AP_3(n,p,t_1)}{\partial t_1}=\xi_3(t_1)$.
To obtain the sufficient conditions, we have to calculate the second order partial derivatives of $AP_3(n,p,t_1)$ with respect to n and t_1 which are as follows:

$$\begin{aligned}
\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} &= \frac{-2S}{n^3t_1},\\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial t_1{}^2} &= -\frac{2(S+n^2s^2)}{t_1{}^3}-\left(R+\frac{D(p)}{\theta}\right)\Big[\frac{h_1(n-1)}{2}\theta^2e^{\theta t_1}\\
&\quad - h\Big(\frac{\theta e^{\theta t_1}}{t_1}-\frac{2e^{\theta t_1}}{t_1{}^2}+\frac{2(e^{\theta t_1}-1)}{\theta t_1{}^3}\Big)\Big],\\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial p^2} &= \left(h_1(n-1)+\frac{2h}{\theta t_1}\right)\frac{(e^{\theta t_1}-1)}{\theta}z_1-2z_1\left(\frac{h}{\theta}+3p\right),\\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial n\partial t_1} &= \frac{2ns^2}{t_1{}^2}-\left(R+\frac{D(p)}{\theta}\right)\frac{h_1}{2}\theta e^{\theta t_1},\\
\frac{\partial^2 AP_3(n,p,t_1)}{\partial p\partial t_1} &= \frac{(y_1+2z_1p)h_1(n-1)\theta e^{\theta t_1}}{2},
\end{aligned}$$

and

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n\partial p} = \frac{h_1}{2}\left((y_1-2z_1p)\frac{(e^{\theta t_1}-1)}{\theta}\right).$$

The sufficient conditions for the optimum solution of $AP_3(n,t_1)$ are $\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} < 0$, $\frac{\partial^2 AP_3}{\partial n^2}\frac{\partial^2 AP_3}{\partial p^2} - (\frac{\partial^2 AP_3}{\partial n \partial p})^2 > 0$, and the value of third principal minor is

$$H = \frac{-2S}{n{t_1}^3}\Bigg[\Bigg(1+\frac{h_1(n-1)n{t_1}^3}{4S}(R\theta + D(p))\theta e^{\theta t_1} + \frac{sn}{S} + \frac{n{t_1}^3 h}{2S}\left(R + \frac{D(p)}{\theta}\right) l_1 l_3$$
$$+ \frac{n{t_1}^3 h}{2S}{h_1}^2\left(\frac{y_1}{2}+z_1 p\right)^2 - {l_5}^2 - \frac{n{t_1}^3 h_1}{S}\left(\frac{y_1}{2}+z_1 p\right) l_4 l_5 + \frac{n{t_1}^3}{2S} l_3 {l_4}^2\Bigg)\Bigg] < 0$$

Now

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$$

We have to show that $\frac{\partial^2 AP_3}{\partial n^2}\frac{\partial^2 AP_3}{\partial p^2} - (\frac{\partial^2 AP_3}{\partial n \partial p})^2 > 0$.

To prove the condition of second principal minor, if $\frac{\partial^2 AP_3}{\partial n^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ and $\frac{\partial^2 AP_3}{\partial p^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$, then the conditions hold.

Now

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{h_1}{2}\left((y_1 - 2z_1 p)\frac{(e^{\theta t_1}-1)}{\theta}\right)$$

which can be written as

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_3(n,p,t_1)}{\partial n^2} - \xi_5.$$

where

$$\xi_5 = -\frac{2S}{n^3 t_1} - \frac{h_1}{2}\left((y_1 - 2z_1 p)\frac{(e^{\theta t_1}-1)}{\theta}\right)$$

$\frac{\partial^2 AP_3}{\partial n^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ will hold if $\xi_5 > 0$.

Now $\xi_5 > 0$ exists when

$$4S\theta > n^3 t_1 (e^{\theta t_1} - 1) h_1 (2z_1 p - y)$$

Similarly,

$$\frac{\partial^2 AP_3(n,p,t_1)}{\partial n \partial p} = \frac{\partial^2 AP_3(n,p,t_1)}{\partial p^2} - \xi_6$$

where

$$\xi_6 = \frac{(e^{\theta t_1}-1)}{\theta}\left[\left(h_1(n-1) + \frac{2h}{\theta t_1}\right) z_1 - \frac{h_1}{2}(y_1 - 2z_1 p)\right] - 2z_1\left(\frac{h}{\theta} + 3p\right)$$

$\frac{\partial^2 AP_3}{\partial p^2} > \frac{\partial^2 AP_3}{\partial n \partial p}$ will exist if $\xi_6 > 0$.

i.e., if

$$\frac{2z_1\theta\left(\frac{h}{\theta}+3p\right)}{(e^{\theta t_1}-1)} > \left[h_1\left(z_1(n-1+p) - \frac{y_1}{2}\right) + \frac{2z_1\theta}{\theta t_1}\right]$$

Similar as above, value of third principal minor i.e., $H < 0$ will be satisfied if

$$\left[1 + \frac{sn}{S} + \frac{n{t_1}^3}{2S}\left(\frac{(n-1)\theta^2 e^{\theta t_1}}{2} + h\right)\left(R + \frac{D(p)}{\theta}\right)\right] l_1 l_3 + \frac{n{t_1}^3}{2S}\Big(l_3 {l_4}^2$$
$$+ h{h_1}^2\left(\frac{y_1}{2} + z_1 p^2\right)^2\Big) < {l_5}^2 + \frac{n{t_1}^3 h_1}{S}\left(\frac{y_1}{2} + z_1 p\right) l_4 l_5$$

Therefore, $AP_3(n^*, p^*, {t_1}^*)$ will have the global maximum (where n^*, p^*, and ${t_1}^*$ are optimal values of n, p, and t_1) if conditions hold.

Case IV

This section describes that demand of products is exponentially time-dependent. For example, electronic goods, fashionable clothes are those products whose demand rate may fluctuate with time. For new products, initially the demand is very high and then it decreases. That situation of demand can be represented by exponential demand pattern. Therefore, it can be observed that demand of products varies exponentially with time.
In this case, the demand function $f(t) = D(t) = x_2e^{y_2t}$
The governing differential equation of the inventory system is

$$\begin{aligned}\frac{dI(t)}{dt} + \theta I(t) &= -f(t), \quad 0 \le t \le t_1, \quad I(t_1) = R\\ &= -x_2e^{y_2t}\end{aligned}$$

Using the boundary condition, the inventory level $I(t)$ as

$$I(t) = Re^{\theta(t_1-t)} + \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1-\theta t} - e^{y_2t}) \tag{2.4}$$

During $[0,t_1]$, the total costs are as follows:
(i) Fixed cost of stocks per transfer to display area from warehouse $= s$.
(ii) Holding cost is

$$= h\int_0^{t_1} I(t)dt = \frac{hR}{\theta}(e^{\theta t_1} - 1) + \frac{hx_2}{(y_2+\theta)}e^{(y_2+\theta)t_1}\frac{(1-e^{-\theta t_1})}{\theta} - \frac{hx_2}{(y_2+\theta)}\frac{(e^{y_2t_1}-1)}{y_2}$$

(iii) The revenue per cycle is

$$\begin{aligned}= (p-c)\int_0^{t_1} D(t)dt &= (p-c)\int_0^{t_1} x_2e^{y_2t}dt\\ &= (p-c)x_2\left[\frac{e^{y_2t_1}}{y_2} - \frac{1}{y_2}\right]\end{aligned}$$

Using Equation (4) and $I(0)=q+R$

$$\begin{aligned}q + R &= \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1}\\ i.e., \quad q &= \frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\end{aligned}$$

(iv) Stock holding cost in the warehouse is

$$= h_1[\frac{n(n-1)}{2}q]t_1 = h_1t_1\frac{n(n-1)}{2}\left[\frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\right]$$

Hence, the average profit per unit time is
$AP_4(n,t_1)$=$\frac{1}{T}$[revenue-(total cost in warehouse)-(total cost in display area)](where T=nt_1)

$$\begin{aligned}&= \frac{(p-c)x_2}{y_2t_1}(e^{y_2t_1} - 1)\\ &- \left[\frac{S}{nt_1} + h_1\frac{(n-1)}{2}\left(\frac{x_2}{(y_2+\theta)}(e^{(y_2+\theta)t_1} - 1) + Re^{\theta t_1} - R\right)\right]\\ &- \frac{s}{t_1} - \frac{h}{t_1}\left[\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1}\frac{(1-e^{-\theta t_1})}{\theta} - \frac{x_2}{(y_2+\theta)}\frac{(e^{y_2t_1}-1)}{y_2} + \frac{R(e^{\theta t_1}-1)}{\theta}\right]\end{aligned}$$

which we have to maximize with respect to the decision variables n and t_1. To obtain the global maximum solution, we have made Lemma 4.

Lemma 4

$AP_4(n^*,{t_1}^*)$ will have the global maximum solution (where n^* and ${t_1}^*$ are optimal values of n and t_1) if following conditions are satisfied

$$(i)e^{\theta t_1}(x_2e^{y_2t_1} + R\theta) > \frac{2S(2-Sn)}{h_1t_1n^3}$$

and

$$(ii)\frac{(c-p)x_2b_1}{y_2} > \frac{2S}{n{t_1}^3} + \frac{h_1(n-1)b_2}{2} + hb_3$$

See Appendix E for the values of b_1, b_2, and b_3.

Proof

From the necessary conditions of optimal solution, one has

$$\frac{\partial AP(n,t_1)}{\partial n} = 0$$

i.e.,

$$\frac{\partial AP_4(n,t_1)}{\partial n} = \frac{S}{n^2t_1} - \frac{h_1}{2}\left(\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right) = 0$$

which implies

$$n = \sqrt{\frac{2S}{h_1t_1\left(\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1} + Re^{\theta t_1} - R\right)}}$$

For the other decision variable t_1,

$$\frac{\partial AP_4(n,t_1)}{\partial t_1} = 0$$

$$\begin{aligned} i.e., \qquad & \frac{(p-c)x_2}{y_2}\left(\frac{e^{y_2t_1}y_2}{t_1} - \frac{(e^{y_2t_1}-1)}{t_1^2}\right) + \frac{\frac{S}{{t_1}^2}}{\sqrt{\frac{2S}{h_1t_1\left(\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1}+Re^{\theta t_1}-R\right)}}} \\ & -h_1\frac{\left[\sqrt{\frac{2S}{h_1t_1\left(\frac{x_2}{(y_2+\theta)}e^{(y_2+\theta)t_1}+Re^{\theta t_1}-R\right)}}-1\right]}{2}(x_2e^{(y_2+\theta)t_1} + R\theta e^{\theta t_1}) \\ & +\frac{s}{{t_1}^2} - h\Big[\frac{x_2}{(y_2+\theta)}\Big(-e^{(y_2+\theta)t_1}\frac{(1-e^{\theta t_1})}{\theta{t_1}^2} + \frac{e^{(y_2+2\theta)t_1}}{t_1} \\ & +\frac{(1-e^{\theta t_1})}{\theta}\frac{e^{(y_2+\theta)t_1}}{t_1}(y_2+\theta)\Big) - e^{y_2t_1} + Re^{\theta t_1}\Big] = 0 \end{aligned}$$

Now ${t_1}^*$ can be obtained if $\xi_4({t_1}^*) = 0$ where $\frac{\partial AP_4(n,t_1)}{\partial t_1} = \xi_4(t_1)$.
From the sufficient conditions, the second order partial derivatives of $AP_4(n,t_1)$ with respect to n and t_1 are as follows:

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} = \frac{-2S}{n^3t_1},$$

$$\begin{aligned} \frac{\partial^2 AP_4(n,t_1)}{\partial {t_1}^2} &= \frac{(p-c)x_2}{y_2}\left[\frac{2(e^{y_2t_1}-1)}{{t_1}^3} - \frac{2e^{y_2t_1}y_2}{{t_1}^2} + \frac{e^{y_2t_1}{y_2}^2}{t_1}\right] - \frac{2S}{n{t_1}^3} \\ &- h_1\frac{(n-1)}{2}[x_2(y_2+\theta)e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta^2] - h\Big[e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} \\ &- y_2e^{y_2t_1})\frac{x_2}{y_2+\theta} + x_2(y_2+\theta)\Big(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2t_1})}{y_2}\Big) \\ &+ 2(e^{-\theta t_1} - e^{y_2t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} + R\theta e^{\theta t_1}\Big], \end{aligned}$$

and

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n\partial t_1} = -\frac{S}{n^2{t_1}^2} - \frac{h_1}{2}[x_2e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta].$$

The sufficient conditions for the optimum solution of $AP_4(n,t_1)$ are $\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} < 0$ and $\frac{\partial^2 AP_4}{\partial n^2}\frac{\partial^2 AP_4}{\partial {t_1}^2} - (\frac{\partial^2 AP_4}{\partial n \partial t_1})^2 > 0$.
Now

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n^2} = \frac{-2S}{n^3 t_1} < 0$$

we have to show $\frac{\partial^2 AP_4}{\partial n^2}\frac{\partial^2 AP_4}{\partial {t_1}^2} - (\frac{\partial^2 AP_4}{\partial n \partial t_1})^2 > 0$.
To justify above condition, if $\frac{\partial^2 AP_4}{\partial n^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ and $\frac{\partial^2 AP_4}{\partial {t_1}^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$, then optimality conditions for second principal minor are satisfied.
Now

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = -\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]$$

which can be written as

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_4(n,t_1)}{\partial n^2} - \xi_7.$$

where

$$\xi_7 = \frac{2S}{n^3 t_1} - \frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]$$

$\frac{\partial^2 AP_4}{\partial n^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ will hold if $\xi_7 > 0$.
Now $\xi_7 > 0$ will exist if

$$e^{\theta t_1}(x_2 e^{y_2 t_1} + R\theta) > \frac{2S(2-Sn)}{h_1 t_1 n^3}$$

Similarly,

$$\frac{\partial^2 AP_4(n,t_1)}{\partial n \partial t_1} = \frac{\partial^2 AP_4(n,t_1)}{\partial {t_1}^2} - \xi_8$$

where

$$\begin{aligned}
\xi_8 &= \frac{(c-p)x_2}{y_2}\left[\frac{2(e^{y_2 t_1}-1)}{{t_1}^3} - \frac{2e^{y_2 t_1}y_2}{{t_1}^2} + \frac{e^{y_2 t_1}{y_2}^2}{t_1}\right] + \frac{2S}{n{t_1}^3} + h_1\frac{(n-1)}{2}[x_2(y_2 \\
&+ \theta)e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta^2] + h\Big[e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} - y_2 e^{y_2 t_1})\frac{x_2}{y_2+\theta} + x_2(y_2 \\
&+ \theta)\Big(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2 t_1})}{y_2}\Big) + 2(e^{-\theta t_1} - e^{y_2 t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} + R\theta e^{\theta t_1}\Big] \\
&- \frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}[x_2 e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta]
\end{aligned}$$

$\frac{\partial^2 AP_4}{\partial {t_1}^2} > \frac{\partial^2 AP_4}{\partial n \partial t_1}$ will exist if $\xi_8 > 0$.
i.e., if

$$\frac{(c-p)x_2 b_1}{y_2} > \frac{2S}{n{t_1}^3} + \frac{h_1(n-1)b_2}{2} + hb_3$$

Therefore, $AP_4(n^*, {t_1}^*)$ will have the global maximum (where n^* and ${t_1}^*$ are optimal values of n and t_1) solution if the conditions hold.

For the demand function $\mathbf{f(p,t) = D(p,t) = (x + x_1 + yt - y_1 p + zt^2 - z_1 p^2)}$

Case A. When t is fixed, n and p are variable

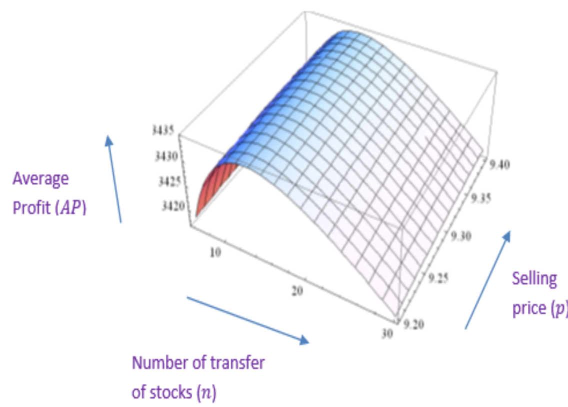

Figure 2. Average Profit (AP) versus number of transfer of stocks (n) and selling price (p) for Example 1

3. Numerical examples

By using numerical data from Goyal and Chang (2009) model, this paper formulates the optimal value of average profit and optimal order quantity.

Example 1. The values for the parameters are taken as follows:
$h = 0.6$ dollar/unit/unit time, $h_1 = 0.3$ dollar/unit/unit time, $c = 1.0$ dollar/unit, $x = 100$, $y = 10$, $z = 0.1$, $x_1 = 800$, $y_1 = 50$, $z_1 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, $s = 10$ dollars/transfer, $S = 100$ dollars/order. Then the optimal solution is $AP_1 = \$3432.74$, $n^* = 10$, $t_1{}^* = 0.33$ year, $p^* = 9.3$ dollars/unit, and optimal order quantity $Q^* = 114.325$ units.

Example 2 The values for the parameters are taken as follows:.
$h = 0.6$ dollar/unit, $h_1 = 0.3$ dollar/unit/unit time, $c = 1.0$ dollar/unit/unit time, $p = 3.0$ dollars/unit, $x = 3000$, $y = 40$, $z = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, $s = 10$ dollars/transfer, $S = 100$ dollars/order. Then the optimal solution is $AP_2 = \$2983.81$, $n^* = 3$, $t_1{}^* = 0.206$ year, and optimal order quantity $Q^* = 1069.69$ units.

Example 3 The values for the parameters are taken as follows:.
$h = 0.6$ dollar/unit/unit time, $h_1 = 0.3$ dollar/unit/unit time, $c = 1.0$ dollar/unit, $x_1 = 700$, $y_1 = 40$, $z_1 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, $s = 10$ dollars/transfer, $S = 100$ dollars/order. Then the optimal solution is $AP_3 = \$2473.03$, $n^* = 3$, and $p^* = 9.1$ dollars/unit, $t_1{}^* = 0.42$ year, and optimal order quantity $Q^* = 426.406$ units.

Example 4 The values for the parameters are taken as follows:.
$h = 0.6$ dollar/unit/unit time, $h_1 = 0.3$ dollar/unit/unit time, $p = 3$ dollars/unit, $c = 1.0$ dollar/unit, $x_2 = 2000$, $y_2 = 0.1$, $R = 1$ unit, $a = 0.1$, $b = 0.2$, $s = 10$ dollars/transfer,

Case B. When p is fixed, n and t are variable

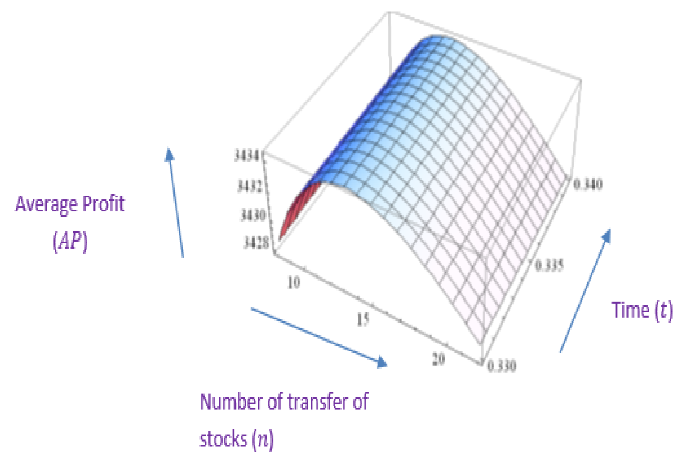

Figure 3. Average Profit (AP) versus number of transfer of stocks (n) and time (t) for Example 1

Case C. When n is fixed, p and t are variable

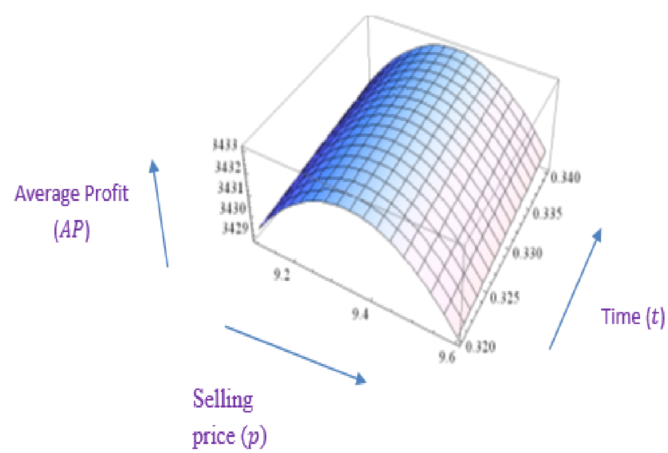

Figure 4. Average Profit (AP) versus selling price (p) and time (t) for Example 1

For the demand function $f(t) = D(t) = (x + yt + zt^2)$

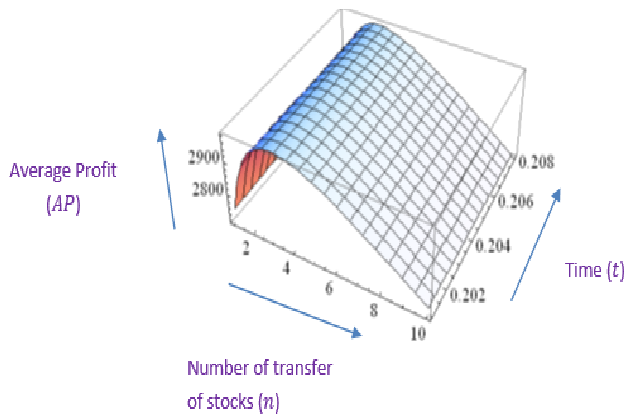

Figure 5. Average Profit (AP) versus number of transfer of stocks (n) and time (t) for Example 2

For the demand function $f(p) = D(p) = (x_1 - y_1 p - z_1 p^2)$

Case A. When *t* is fixed, *n* and *p* are variable

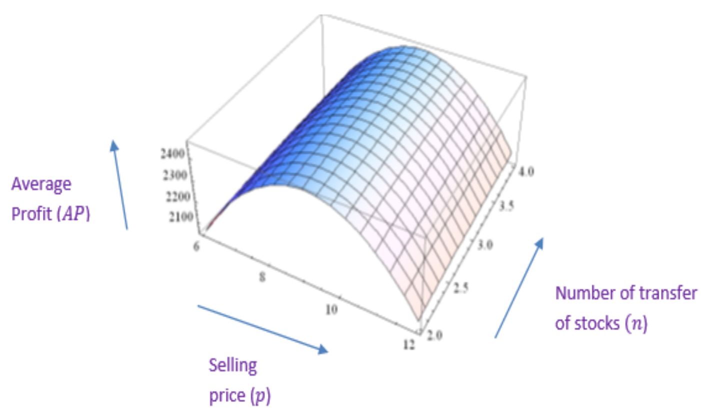

Figure 6. Average Profit (AP) versus selling price (p) and number of transfer of stocks (n) Example 3

$S = 100$ dollars/order. Then the optimal solution is $AP_4 = \$2634.91$, $n^* = 2$, and $t_1{}^* = 0.3$ year, and optimal order quantity $Q^* = 934.702$ units.

Case B. When n is fixed, t and p are variable

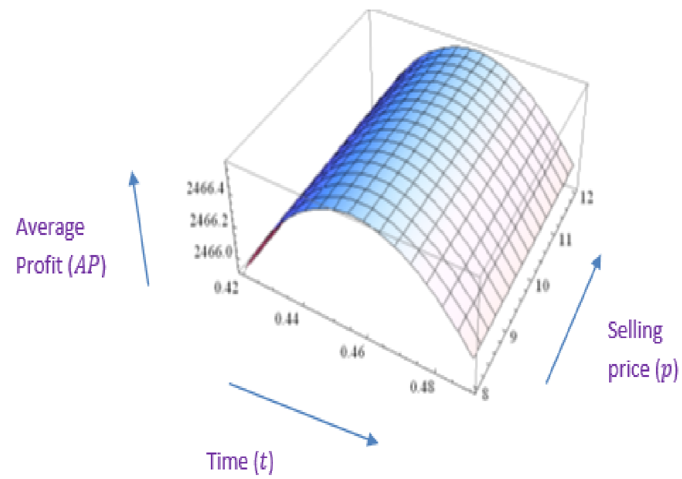

Figure 7. Average Profit (AP) versus time (t) and selling price (p) for Example 3

Case C. When p is fixed, n and t are variable

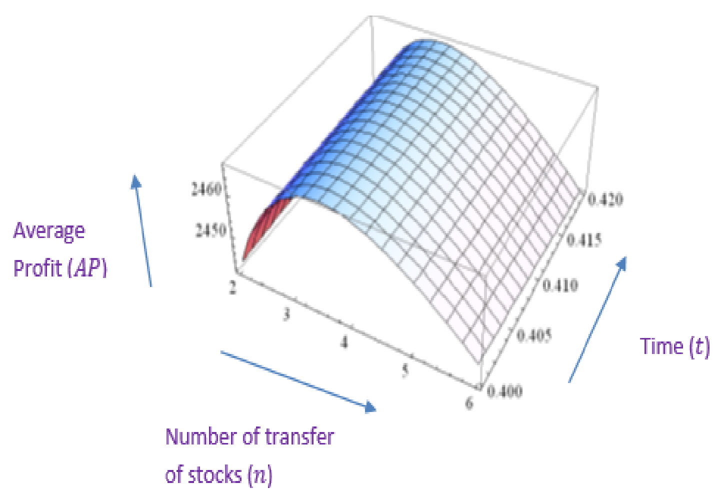

Figure 8. Average Profit (AP) versus number of transfer of stocks (n) and time (t) for Example 3

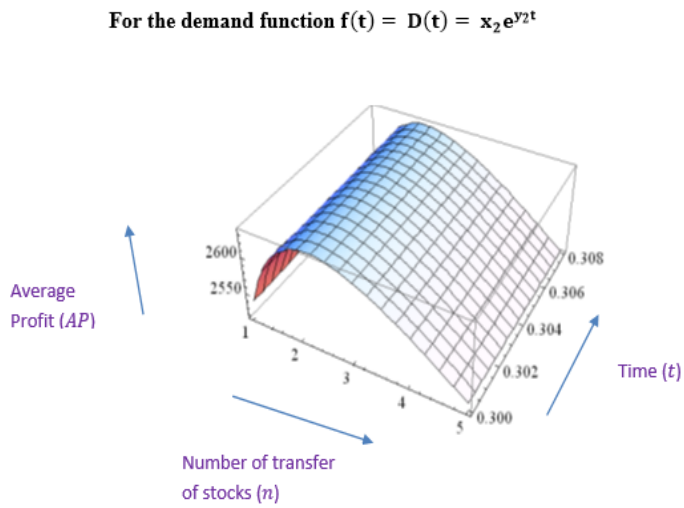

Figure 9. Average Profit (AP) versus number of transfer of stocks (n) and time (t) for Example 4

Sensitivity Analysis

This section provides the sensitivity analysis of each key parameter. The sensitivity analysis of key parameters for several demand functions are given in Table 2, Table 3, Table 4, and Table 5.

Table 2:(a) Sensitivity analysis for Case I

Parameters	Changes(in %)	AP_1
h	-50%	0.83
	-25%	0.35
	$+25\%$	$-$
	$+50\%$	$-$
h_1	-50%	0.36
	-25%	0.16
	$+25\%$	-0.14
	$+50\%$	-0.27
c	-50%	6.28
	-25%	3.11
	$+25\%$	-3.07
	$+50\%$	-6.10
s	-50%	0.51
	-25%	0.23
	$+25\%$	-0.2
	$+50\%$	-0.39

'$-$' refers to infeasible solution.

- As unit carrying cost per stock in display area h increases, average profit AP_1 decreases. But, for $+25\%$ and $+50\%$ increase of this parameter, the model does not allow feasible results. This means, we can decrease holding cost as we need, but we cannot increase it anymore.

- It can be observed that if the parameter h_1 i.e., unit carrying cost per stock in warehouse, increases then the average profit AP_1 gradually decreases. The negative percentage change is greater than positive percentage change for h_1. This is the least sensitive parameter among others.
- If purchasing cost c increases, then the average profit AP_1 decreases. In this case, negative percentage change is greater than the positive percentage change for that parameter. It is the most sensitive parameter among others.
- An increasing value in ordering cost s decreases the average profit AP_1. For that parameter s, positive percentage change is less than the negative percentage change. This is also less sensitive parameter among others.

Table 3:(b) Sensitivity analysis for Case II

Parameters	Changes(in %)	AP_2
h	−50%	−
	−25%	−
	+25%	−0.8
	+50%	−1.47
h_1	−50%	2.36
	−25%	1.02
	+25%	−0.79
	+50%	−1.35
c	−50%	28.56
	−25%	14.28
	+25%	−14.28
	+50%	−28.56
s	−50%	0.95
	−25%	0.43
	+25%	−0.38
	+50%	−0.73

'−' refers to infeasible solution.

- While the parameter unit carrying cost per stock in display area (i.e., h) decreases for −25% and −50%, this model does not give any feasible solution. But for +25% and +50%, this model allows feasible results and in that case average profit AP_2 decreases when unit carrying cost per stock in display area h increases.
- As h_1 i.e., unit carrying cost per stock in warehouse, increases, then the average profit AP_2 decreases gradually. The positive percentage change is less than negative percentage change for h_1.
- For the unit purchasing cost c, negative and positive percentage changes are exactly same. An increasing value in purchasing cost c decreases the average profit AP_2. This is the most sensitive parameter comparing with other parameters.
- When ordering cost s increases, the average profit AP_2 decreases. The negative percentage change is bigger in comparing to positive percentage change for s. This is least sensitive among other parameters.

Table 4:(c) Sensitivity analysis for Case III

Parameters	Changes(in %)	AP_3
h	-50%	-2.96
	-25%	-3.4
	$+25\%$	-4.26
	$+50\%$	-4.70
h_1	-50%	0.06
	-25%	-1.89
	$+25\%$	-5.78
	$+50\%$	-7.72
c	-50%	2.81
	-25%	0.51
	$+25\%$	-7.15
	$+50\%$	-10.47
s	-50%	-3.35
	-25%	-3.6
	$+25\%$	-4.07
	$+50\%$	-4.31

- If unit carrying cost per stock in display area (i.e., h) increases, the average profit AP_3 decreases. It is found that positive and negative percentage changes are almost double for h.
- For h_1, the positive percentage change is greater than the negative percentage change. The result indicates that average profit AP_3 decreases if h_1 increases.
- While purchasing cost c increases from -50% to $+50\%$, average profit AP_3 decreases. The negative percentage change is smaller than positive percentage change for that parameter. This is the most sensitive parameter comparing to others.
- The increasing value of ordering cost s decreases the average profit AP_3. The negative percentage change is not similar with positive percentage change of s.

Table 5:(d) Sensitivity analysis for Case IV

Parameters	Changes(in %)	AP_4
h	-50%	-13.60
	-25%	-14.93
	$+25\%$	-17.60
	$+50\%$	-18.92
h_1	-50%	-4.29
	-25%	-10.28
	$+25\%$	-22.24
	$+50\%$	-28.23
c	-50%	12.62
	-25%	-1.82
	$+25\%$	-30.7
	$+50\%$	-45.14
s	-50%	-15.63
	-25%	-15.94
	$+25\%$	-16.58
	$+50\%$	-16.89

- The negative percentage change and positive percentage change for h is not similar. As unit carrying cost per stock in display area h increases, the average profit AP_4 decreases.
- It can be found that if the parameter h_1 i.e., unit carrying cost per stock in warehouse, increases, then the average profit AP_4 decreases. The negative and positive percentage changes are similar for h_1.
- While purchasing cost c increases, then the average profit AP_4 decreases. From Table 5, it is clear that the negative percentage change is higher than positive percentage change.
- An increasing value in ordering cost s decreases the average profit AP_4. In that case, negative and positive percentage changes are close to each other.

4. Conclusions and future extensions

This model developed an optimal ordering and transfer policy for uniformly distributed deteriorating items. In Goyal and Chang's(2009) [Goyal SK, Chang CT (2009) Optimal ordering and transfer policy for an inventory with stock dependent demand. Euro. J. Oper. Res. 196:177-185] model, demand of products is taken to be only stock-dependent. They does not consider deterioration of products. This model incorporated random deterioration of products. According to Goyal and Chang's (2009) [Goyal SK, Chang CT (2009) Optimal ordering and transfer policy for an inventory with stock dependent demand. Euro. J. Oper. Res. 196:177-185] model, demand of items depends only on display stocks over the whole year. Demand of products may vary with time and price. These two factors i.e., time and price both can affect demand of products. This study derived four different cases of demand functions such as time-price, time, selling price, and exponentially time-dependent demand function which would make this model more realistic to real life situation. This model determined retailer's optimal ordering quantity and maximized average profit for the retailer with finite production rate. The number of transfers from warehouse to display area is determined. The global optimal solutions are obtained analytically. There are four lemmas to find the global optimum solution.Some numerical examples, sensitivity analysis, and graphical representation explained the applicability of this proposed model. This model can be extended in different ways by considering shortages, discounts, and inflation rates.

References

[1] Harris, F.W. *Operations and Cost*, Factory Management Series. A. W. Shaw Co.(1915).

[2] Hsu, C.I. and Li, H.C. *Optimal delivery service strategy for internet shopping with time dependent consumer demand*, Transportation Research Part E: Logistics and Transportation Review 42,473-497,2006.

[3] Dye, C.Y. Chang, H.J. and Teng, J.T. *A deteriorating inventory model with time-varying demand and shortage-dependent partial backlogging*, European Journal of Operational Research 172,417-429,2006.

[4] Khanra, S. Ghosh, S.K. and Chaudhuri, K.S. *An EOQ model for a deteriorating item with time dependent quadratic demand under permissible delay in payment*, Applied Mathematics and Computation 218,1-9,2011.

[5] Sarkar, B. Sana, S.S. and Chaudhuri, K.S. *An Economic production quantity model with stochastic demand in an imperfect production system*, International Journal of Services and Operations Management 9,259-283,2011a.

[6] Sarkar, B. and Moon, I.K. *An EPQ model with inflation in an imperfect production system*, Applied Mathematics and Computation 217,6159-6167,2011.

[7] Sarkar, B. Sana, S.S. and Chaudhuri, K.S. *An imperfect production process for time varying demand with inflation and time value of money - an EMQ model*, Expert Systems with Applications 38,13543-13548,2011b.

[8] Wee, H.M. and Law, S.T. *Replenishment and pricing policy for deteriorating items taking into account the time-value of money*, International Journal of Production Economics 71,213-220,2001.

[9] Chu, P. and Chen, P.S. *A note on inventory replenishment policies for deteriorating items in an exponentially declining market*, Computers and Operations Research 29,1827-1842,2002.

[10] Khanra, S. and Chaudhuri, K.S. *A note on an order-level inventory model for a deteriorating item with time dependent quadratic demand*, Computers and Operations Research 30,1901-1916,2003.

[11] Chern, M.S. Yang, H.L. Teng, J.T. and Papachristos, S. *Partial backlogging inventory lot-size models for deteriorating items with fluctuating demand under inflation*, European Journal of Operational Research 191,127-141,2008.

[12] Sett, B.K. Sarkar, B. and Goswami, A. *A two-warehouse inventory model with increasing demand and time varying deterioration*, Scientia Iranica 19,1969-1977,2012.

[13] Sarkar, T. Ghosh, S.K. and Chaudhuri, K.S. *An optimal inventory replenishment policy for a deteriorating item with time-quadratic demand and time-dependent partial backlogging with shortages in all cycles*, Applied Mathematics and Computation 218,9147-9155,2012.

[14] Sarkar, B. *An EOQ model with delay-in-payments and time-varying deterioration rate*, Mathematical and Computer Modelling 55,367-377,2012b.

[15] Sarkar, B. and Sarkar, S. *Variable deterioration and demand-an inventory model*, Economic Modelling 31,548-556,2013a.

[16] Sarkar, B. and Sarkar, S. *An improved inventory model with partial backlogging, time varying deterioration and stock-dependent demand*, Economic Modelling 30,924-932,2013b.

[17] Sarkar, B. *A production-inventory model with probabilistic deterioration in two-echelon supply chain management*, Applied Mathematical Modelling 37,3138-3151,2013.

[18] Sarkar, M. and Sarkar, B. *An economic manufacturing quantity model with probabilistic deterioration in a production system*, Economic Modelling 31,245-252,2013c.

[19] Sarkar, B. Saren, S. and Wee, H.M. *An EOQ model with time dependent demand, component cost and product selling price for deteriorating items*, Economic Modelling 30,306-310,2013.

[20] Sarkar, B. Saren, S. and Cárdenas-Barrón, L.E. *An inventory model with trade-credit policy and variable deterioration for fixed lifetime products*, Annals of Operations Research 229,677-702,2015.

[21] Wee, H.M. *A replenishment policy for items with a price dependent demand and a varying rate of deterioration*, Production Planning and Control 8,494-499,1997.

[22] Datta, D.K. and Paul, K. *An inventory system with stock-dependent, price-sensitive demand rate*, Production Planning and Control 12,13-20,2001.

[23] Goyal, S.K, and Chang, C.T. *Optimal ordering and transfer policy for an inventory with stock dependent demand*, European Journal of Operational Research 196,177-185,2009.

[24] Sarkar, B. Sana, S.S. and Chaudhuri, K.S. *A finite replenishment model with increasing demand under inflation*, International Journal of Mathematics in Operational Research 2,347-385,2010a.

[25] Sarkar, B. Sana, S.S. and Chaudhuri, K.S. *A stock-dependent inventory model in an imperfect production process*, International Journal of Procurement Management 3,361-378,2010b.

[26] Sana, S.S. *An EOQ model for salesmen's initiatives, stock and price sensitive demand of similar products - A dynamical system*, Applied Mathematics and Computation 218,3277-3288,2011.

[27] Sarkar, B. *An inventory model with reliability in an imperfect production process*, Applied Mathematics and Computation 218,4881-4891,2012c.

[28] Sarkar, B. *An EOQ model with delay in payments and stock dependent demand in the presence of imperfect production*, Applied Mathematics and Computation 218,8295-8308,2012a.

Appendix A

$$
\begin{aligned}
M &= \left(\frac{y}{2}+\frac{2zt_1}{3}\right)+\frac{h_1(n-1)}{2}(y_1+2z_1p)e^{\theta t_1}+\frac{h(y_1+2z_1p)}{\theta}\left(\frac{e^{\theta t_1}}{t_1}+\frac{(1-e^{\theta t_1})}{\theta {t_1}^2}\right)\\
N &= 4z_1p-2\Big[cz_1-y_1-\frac{z_1}{\theta}\Big(\frac{h_1(n-1)}{2}(1-e^{\theta t_1})+h\Big(\frac{(1-e^{\theta t_1})}{\theta t_1}+1\Big)\Big)\Big]\\
U &= (y_1p+z_1p^2-x-x_1-yt_1-z{t_1}^2-R\theta)e^{\theta t_1}
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
f &= \Big[\frac{(1-e^{\theta t_1})}{\theta}(y_1p+z_1p^2-x-x_1)+y\left(\frac{t_1e^{\theta t_1}}{\theta}-\frac{(e^{\theta t_1}-1)}{\theta^2}\right)+z\Big(\frac{{t_1}^2e^{\theta t_1}}{\theta}-\frac{2t_1e^{\theta t_1}}{\theta^2}\\
&+ \frac{2(e^{\theta t_1}-1)}{\theta^3}Big)+Re^{\theta t_1}-R\Big]
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
\alpha &= \frac{h_1}{2}(\theta(n-1)-1)\left(\frac{y}{\theta}-\frac{2}{\theta^2}-x\right)\\
\beta &= \left(y-\frac{2z}{\theta}\right)\left(\frac{h_1}{2\theta}(\theta t_1+1-(n-1)\theta(t_1\theta+2)-h)\right)\\
\gamma &= \frac{z}{\theta}\left[h_1({t_1}^2\theta+2t_1-\frac{(n-1)}{2}(4t_1\theta+\theta^2{t_1}^2+2))-h(t_1\theta+2)\right]\\
\delta &= \frac{h}{2}\left[\frac{2(1-e^{\theta t_1})}{{t_1}^3}+\frac{2\theta e^{\theta t_1}}{{t_1}^2}-\frac{\theta^2e^{\theta t_1}}{t_1}\right]\left[R-\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)\right]\\
a_1 &= \frac{h_1(n-1)}{4}z\theta^2\\
a_2 &= \frac{h_1(n-1)}{2}\Big((y+2z)\frac{\theta^2}{2}+2z\theta\Big)
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= \frac{h_1(n-1)}{2}\left[(y+2z)\frac{3\theta}{2}+3z\right]-\Big[\frac{h_1(n-1)}{2}\Big(\frac{R\theta^3}{2}+\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)e^{-\theta t}\frac{\theta^2}{2}\Big)\\
&+ h\left(y-\frac{2z}{\theta}\right)\frac{\theta}{2}\Big]\\
a_4 &= \Big[\frac{2(p-c)z}{3}-\frac{h_1(n-1)}{2}\Big(R\theta^2+\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)e^{-\theta t}\theta-2\theta\Big(\frac{y}{\theta}+\frac{2z}{\theta}\Big)-\frac{z}{\theta}\Big)\\
&- \frac{hR\theta^2}{2}+h\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)\frac{\theta}{2}-h\left(y-\frac{2z}{\theta}\right)\Big],\\
a_5 &= \Big[(p-c)\frac{y}{2}-\frac{h_1(n-1)}{2}\left[R\theta+\left(\frac{y}{\theta}-\frac{2z}{\theta^2}-x\right)e^{-\theta t}-\left(\frac{y}{\theta}+\frac{2z}{\theta}\right)\right]-\frac{hR\theta}{2}\\
&- \frac{h}{2}\left(\frac{y}{\theta}-\frac{2z}{\theta^2}\right)\Big],\\
a_6 &= \frac{S}{n}+s.
\end{aligned}
$$

Appendix D

$$\begin{aligned}
D(p) &= (x_1 - y_1 p - z_1 p^2). \\
l_1 &= \frac{2(e^{\theta t_1} - 1)}{\theta {t_1}^3} - \frac{2e^{\theta t_1}}{\theta^2} + \frac{\theta e^{\theta t_1}}{t_1}. \\
l_2 &= \left(\frac{e^{\theta t_1}}{t_1} - \frac{e^{\theta t_1} - 1}{\theta {t_1}^2}\right). \\
l_3 &= \left(2cz_1 + h_1(n-1)z_1 - 6z_1 p + \frac{2z_1 h}{\theta} l_2\right). \\
l_4 &= \left(-\frac{S}{n^2 {t_1}^2} - \frac{h_1}{2}(R\theta + D(p))e^{\theta t_1}\right). \\
l_5 &= (y_1 + 2z_1 p)\left(\frac{h_1(1-n)}{2} e^{\theta t_1} + \frac{h}{\theta}\right) l_2.
\end{aligned}$$

Appendix E

$$\begin{aligned}
b_1 &= \left[\frac{2(e^{y_2 t_1} - 1)}{{t_1}^3} - \frac{2e^{y_2 t_1} y_2}{{t_1}^2} + \frac{e^{y_2 t_1} {y_2}^2}{t_1}\right], \\
b_2 &= [x_2(y_2+\theta)e^{(y_2+\theta)t_1} + Re^{\theta t_1}\theta^2], \\
b_3 &= [e^{(y_2+\theta)t_1}(-\theta e^{-\theta t_1} - y_2 e^{y_2 t_1})\frac{x_2}{y_2+\theta} + x_2(y_2+\theta)\Big(\frac{(1-e^{-\theta t_1})}{\theta} + \frac{(1-e^{y_2 t_1})}{y_2}\Big) \\
&+ 2(e^{-\theta t_1} - e^{y_2 t_1})(y_2+\theta)\frac{x_2}{(y_2+\theta)} + R\theta e^{\theta t_1}\Big].
\end{aligned}$$