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A Study On Various Relations

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Abstract. In 2023, $\Gamma - \mathfrak{I}-$open, pre$-\Gamma - \mathfrak{I}-$open, $\Gamma_\Gamma-$open, and almost $\Gamma - \mathfrak{I}-$open sets were defined and their relationships with each other were searched by Yalaz and Keskin Kaymakçı in [19]. Furthermore, Devika and Thilagavathi introduced an M^*-open set and investigated its relationships with some special sets in topological space in [5]. In this study, we research the relevances of these sets with other some specific sets obtained by the operators Γ and Ψ_Γ in ideal topological spaces.

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1. Introduction and Preliminaries

In general topology, the concept of the ideal has a great importance. This concept was given by Vaidyanathaswamy in [17] and Kuratowski in [9]. Moreover, Kuratowski introduced the idea of the local function in [9]. Vaidyanathaswamy obtained a new topology called $*$-topology via local function in [17]. "New topologies from old via ideals" [7] were studied by Janković and Hamlett in 1990. Additionally, Natkaniec presented the set operator Ψ in [11]. The operators Γ and Ψ_Γ were presented in [1]. Thanks to the operator Γ, various set concepts were presented such as $\Gamma - \mathfrak{I}-$open sets, pre$-\Gamma - \mathfrak{I}-$open sets and almost $\Gamma - \mathfrak{I}-$open sets in [19]. Tunç and Özen Yıldırım introduced an $\mathfrak{I}_\Gamma$-perfect set [15], a Γ-dense-in-itself set [15] and a $\Psi_\Gamma - C$ set [16] via the operators Γ and Ψ_Γ.

The aim of this paper is to investigate the relationships among some special sets obtained by the operators Γ and Ψ_Γ. In this paper, we research these relations in general and in the case $cl(\upsilon) \cap \mathfrak{I} = \{\varnothing\}$. Also, we give the counter-examples.

This paper consists of four sections. In the first section, we present the main definitions and theorems that we will use in the study. In the second section, we explore the relationships $\Gamma - \mathfrak{I}-$open sets, pre$-\Gamma - \mathfrak{I}-$open sets, almost $\Gamma - \mathfrak{I}-$open sets and the other some special sets. Moreover, the counter examples are given some cases. In the third section, we investigate the properties of $\Gamma_\Gamma-$open sets and M^*-open sets in ideal topological spaces. In the final section, we briefly talked about our results.

Throughout this paper, (W, υ), $P(W)$, $int(A)$ and $cl(A)$ represent a topological space, the family of all subsets of W, the interior and the closure of a subset A of W in (W, υ), respectively.

For (W, υ) and $A \subseteq W$, $cl_\theta(A) = \{t \in W : cl(E) \cap A \neq \varnothing$ for each $E \in \upsilon(t)\}$, where $\upsilon(t) = \{E \in \upsilon \mid t \in E\}$, is named the θ-closure of A [18]. A is called θ-closed [18] if $A = cl_\theta(A)$. The θ-interior of A [8], symbolized by $int_\theta(A)$, occures of those elements t of A such that $E \subseteq cl(E) \subseteq A$ for some set E in $\upsilon(t)$. Furthermore, $W \setminus int_\theta(A) = cl_\theta(W \setminus A)$ [3]. If $W \setminus A$ is θ-closed, then A is called θ-open [18] and $\upsilon_\theta = \{A \subseteq W \mid A$ is θ-open$\}$. Besides, υ_θ is a topology on W such that

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$\upsilon_\theta \subseteq \upsilon$. Additionally, the union of all θ-open subsets of A is $int_\theta(A)$ [18]. A is called regular θ-closed [2] (resp. semi θ-open [2], semi-open [10], M^*-open [5]) if $A = cl_\theta(int_\theta(A))$ (resp. $A \subseteq cl_\theta(int_\theta(A))$, $A \subseteq cl(int(A))$, $A \subseteq int(cl(int_\theta(A)))$). A is called θ-semiopen [4] if there exists a θ-open subset U of W such that $U \subseteq A \subseteq cl(U)$.

An ideal $\Im$ [9, 17] on W is a nonempty collection of subsets of W which satisfies

(i) If $A \in \Im$ and $K \subseteq A$, then $K \in \Im$,

(ii) If $A \in \Im$ and $K \in \Im$, then $A \cup K \in \Im$.

An ideal topological space $(W, \upsilon, \Im)$ is a topological space (W, υ) with an ideal $\Im$ on W. Thus, $(W, \upsilon, \Im)$ represents an ideal topological space in this study.

An operator $()^* : P(W) \mapsto P(W)$ is called a local function [9] of A with respect to υ and $\Im$ is defined as follows: for $A \subseteq W$, $A^*(\Im, \upsilon) = \{x \in W \mid E \cap A \notin \Im$ for each $E \in \upsilon(x)\}$ in $(W, \upsilon, \Im)$. We use A^* instead of $A^*(\Im, \upsilon)$. An operator Ψ is defined as $\Psi(A) = W \setminus (W \setminus A)^*$ by using the $()^*$-operator in [11]. $\Gamma(A)(\Im, \upsilon) = \{x \in W \mid A \cap cl(E) \notin \Im$ for every $E \in \upsilon(x)\}$ is called a local closure function [1] of A with respect to $\Im$ and υ in $(W, \upsilon, \Im)$. Shortly, it is symbolized by $\Gamma(A)$ in place of $\Gamma(A)(\Im, \upsilon)$. In [1], an operator $\Psi_\Gamma : P(W) \mapsto \upsilon$ is given by $\Psi_\Gamma(A) = W \setminus \Gamma(W \setminus A)$ and also the topologies are defined on W as follows: $\sigma = \{A \subseteq W : A \subseteq \Psi_\Gamma(A)\}$, $\sigma_0 = \{A \subseteq W : A \subseteq int(cl(\Psi_\Gamma(A)))\}$ such that $\upsilon_\theta \subseteq \sigma \subseteq \sigma_0$. If $A \in \sigma$, A is named σ-open [1] and if $A \in \sigma_0$, A is named σ_0-open [1].

The concept of the $\theta^\Im$-closedness [13] is presented by Noorie and Goyal in $(W, \upsilon, \Im)$. A subset A of W is called $\theta^\Im$-closed [13] if $\Gamma(A) \subseteq A$. Furthermore, a θ-closure of a set with respect to an ideal was defined in [6] where θ-closure of A with respect to an ideal $\Im$ is given as $cl_{\Im_\theta}(A) = A \cup \Gamma(A)(\Im, \upsilon)$.

Lemma 1.1. *In $(W, \upsilon, \Im)$ for $A \subseteq W$,*

(i) *[1] $A^* \subseteq \Gamma(A)$.*

(ii) *[12] A is closed in (W, σ) if and only if $\Gamma(A) \subseteq A$.*

Theorem 1.2 ([1])**.** *Let $A, K \subseteq W$ in $(W, \upsilon, \Im)$. Then, the following properties hold.*

(i) *If $A \subseteq K$, then $\Gamma(A) \subseteq \Gamma(K)$.*

(ii) *If $A \subseteq K$, then $\Psi_\Gamma(A) \subseteq \Psi_\Gamma(K)$.*

(iii) *$\Gamma(A) = cl(\Gamma(A)) \subseteq cl_\theta(A)$ and $\Gamma(A)$ is closed.*

(iv) *$\Psi_\Gamma(A \cap K) = \Psi_\Gamma(A) \cap \Psi_\Gamma(K)$.*

(v) *If $A \in \Im$, then $\Gamma(A) = \emptyset$.*

(vi) *$\Gamma(\emptyset) = \emptyset$.*

(vii) *$W = \Gamma(W)$ if and only if $cl(\upsilon) \cap \Im = \{\emptyset\}$, where $cl(\upsilon) = \{cl(U) : U \in \upsilon\}$.*

Corollary 1.3 ([1])**.** *$U \subseteq \Psi_\Gamma(U)$ for each $U \in \upsilon_\theta$ in $(W, \upsilon, \Im)$.*

Lemma 1.4. *In $(W, \upsilon, \Im)$ for $A \subseteq W$,*

(i) *$\Psi_\Gamma(A) \subseteq \Psi(A)$.*

(ii) *A is closed in (W, σ) if and only if A is $\theta^\Im$-closed.*

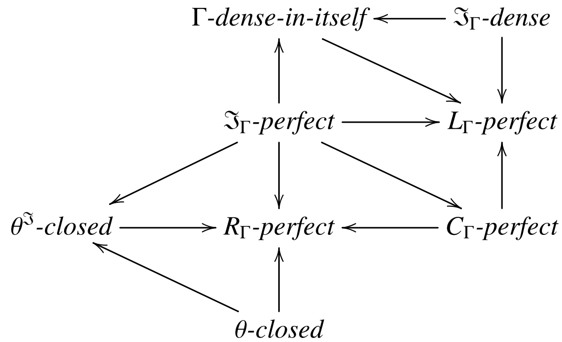
Proof. Let $A \subseteq W$ in $(W, \upsilon, \Im)$.

(i) From the Lemma 1.1 (i), $(W \setminus A)^* \subseteq \Gamma(W \setminus A)$. Then, $\Psi_\Gamma(A) = W \setminus \Gamma(W \setminus A) \subseteq W \setminus (W \setminus A)^* = \Psi(A)$.

(ii) It is obvious from the Lemma 1.1 (ii). □

Definition 1.5 ([15])**.** A set A is called $\Im_\Gamma$-perfect (resp. L_Γ-perfect, R_Γ-perfect, $\Im_\Gamma$-dense, Γ-dense-in-itself) if $A = \Gamma(A)$ (resp. $A \setminus \Gamma(A) \in \Im$, $\Gamma(A) \setminus A \in \Im$, $\Gamma(A) = W$, $A \subseteq \Gamma(A)$) for $A \subseteq W$ in $(W, \upsilon, \Im)$. Moreover, A is called C_Γ-perfect if A is both L_Γ-perfect and R_Γ-perfect.

Remark 1.6 ([15]). In $(W, \upsilon, \Im)$, the below implications hold for $A \subseteq W$:



Definition 1.7 ([16]). A set A is said to be a $\Psi_\Gamma - C$ set if $A \subseteq cl(\Psi_\Gamma(A))$ for $A \subseteq W$ in $(W, \upsilon, \Im)$. The collection of all $\Psi_\Gamma - C$ sets in $(W, \upsilon, \Im)$ is symbolized by $\Psi_\Gamma(W, \upsilon, \Im)$.

Definition 1.8 ([19]). Let $A \subseteq W$ in $(W, \upsilon, \Im)$. A is called
(i) $\Gamma - \Im-$open, if $A \subseteq int(\Gamma(A))$.
(ii) pre$-\Gamma - \Im-$open, if $A \subseteq int(A \cup \Gamma(A))$.
(iii) $\Gamma_\Gamma-$open, if $int_\theta(A) = cl(int(\Gamma(A)))$.
(iv) almost $\Gamma - \Im-$open, if $A \subseteq cl(int(\Gamma(A)))$.
The family of all $\Gamma - \Im-$open (resp. pre$-\Gamma - \Im-$open) sets is denoted by $\Gamma\Im O(W)$ (resp. $\mathcal{P}\Gamma\Im O(W)$).

Theorem 1.9 ([19]). *For $A \subseteq W$ in $(W, \upsilon, \Im)$: if A is $\Gamma - \Im-$open, then*
(i) *A is Γ-dense-in-itself.*
(ii) *A is almost $\Gamma - \Im-$open.*
(iii) *A is pre$-\Gamma - \Im-$open.*

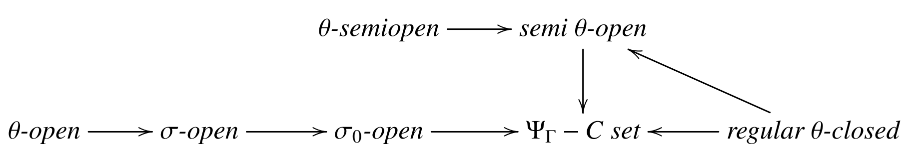
Remark 1.10. In $(W, \upsilon, \Im)$ for $A \subseteq W$, A is pre$-\Gamma - \Im-$open $\Leftrightarrow A \subseteq int(A \cup \Gamma(A)) \Leftrightarrow A \subseteq int(cl_{\Im_\theta}(A))$.

Theorem 1.11 ([16]). *In $(W, \upsilon, \Im)$, $int_\theta(A) \subseteq \Psi_\Gamma(A)$, for each $A \subseteq W$.*

Lemma 1.12. *In (W, υ) for $A \subseteq W$,*
(i) *[1] if A is closed, $int(A) = int_\theta(A)$.*
(ii) *[5] if A is θ-open, A is M^*-open.*
(iii) *[5] if A is M^*-open, A is θ-semiopen.*
(iv) *[18] A is θ-open if and only if $int_\theta(A) = A$.*

Theorem 1.13 ([14]). *In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$, $\Psi_\Gamma(A) \subseteq \Gamma(A)$ for each subset A of W.*

Corollary 1.14 ([1, 2, 16]). *The following diagram is valid in an ideal topological space.*



2. Further Features of $\Gamma - \Im-$open Sets, pre$-\Gamma - \Im-$open Sets and Almost $\Gamma - \Im-$open Sets

Theorem 2.1. *For $H \subseteq W$ in $(W, \upsilon, \Im)$, if H is $\Gamma - \Im-$open, H is L_Γ-perfect.*

Proof. It can be observed from the Theorem 1.9 (i) and the Remark 1.6. □

Remark 2.2. In an ideal topological space, an L_Γ-perfect set may not be $\Gamma - \Im-$open.

Example 2.3. Let $W = \{t, p, l, g\}$, $\Im = \{\varnothing, \{l\}\}$ and $\upsilon = \{\varnothing, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. $H = \{l\}$ is an L_Γ-perfect set but it is not $\Gamma - \Im-$open in $(W, \upsilon, \Im)$.

Theorem 2.4. *In $(W, \upsilon, \Im)$ for $H \subseteq W$,*
(i) if H is $\Im_\Gamma$-dense, H is $\Gamma - \Im-$open.
(ii) if H is both open and Γ-dense-in-itself, H is $\Gamma - \Im-$open.
(iii) if H is both pre$-\Gamma - \Im-$open and Γ-dense-in-itself, H is $\Gamma - \Im-$open.
(iv) if H is both $\Im_\Gamma$-dense and pre$-\Gamma - \Im-$open, H is $\Gamma - \Im-$open.
(v) if H is both $\Im_\Gamma$-perfect and pre$-\Gamma - \Im-$open, H is $\Gamma - \Im-$open.

Proof. In $(W, \upsilon, \Im)$ for $H \subseteq W$,
(i) let H be $\Im_\Gamma$-dense, namely, $\Gamma(H) = W$. It implies that $H \subseteq W = int(W) = int(\Gamma(H))$. As a result, H is $\Gamma - \Im-$open.
(ii) let H be both open and Γ-dense-in-itself. Then, $int(H) = H$ and $H \subseteq \Gamma(H)$. It implies that $H = int(H) \subseteq int(\Gamma(H))$. As a result, H is $\Gamma - \Im-$open.
(iii) let H be both pre$-\Gamma - \Im-$open and Γ-dense-in-itself. Then, $H \subseteq \Gamma(H)$ and $H \subseteq int(H \cup \Gamma(H))$. It implies that $H \subseteq int(H \cup \Gamma(H)) = int(\Gamma(H))$. As a result, H is $\Gamma - \Im-$open.
(iv) It can be observed from the Remark 1.6 and the Theorem 2.4 (iii).
(v) It can be observed from the Remark 1.6 and the Theorem 2.4 (iii). □

Theorem 2.5. *In $(W, \upsilon, \Im)$ for $H \subseteq W$, if H is $\Im_\Gamma$-dense, then H is pre$-\Gamma - \Im-$open.*

Proof. It is clear from the Theorem 2.4 (i) and the Theorem 1.9 (iii). □

Remark 2.6. A $\Gamma - \Im-$open set or a pre$-\Gamma - \Im-$open set may not be $\Im_\Gamma$-dense in an ideal topological space.

Example 2.7. Let $W = \{t, p, l, g\}, \Im = \{\emptyset, \{l\}\}$ and $\upsilon = \{\emptyset, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. $K = \{t\}$ is both $\Gamma - \Im-$open and pre$-\Gamma - \Im-$open, but K is not $\Im_\Gamma$-dense in $(W, \upsilon, \Im)$.

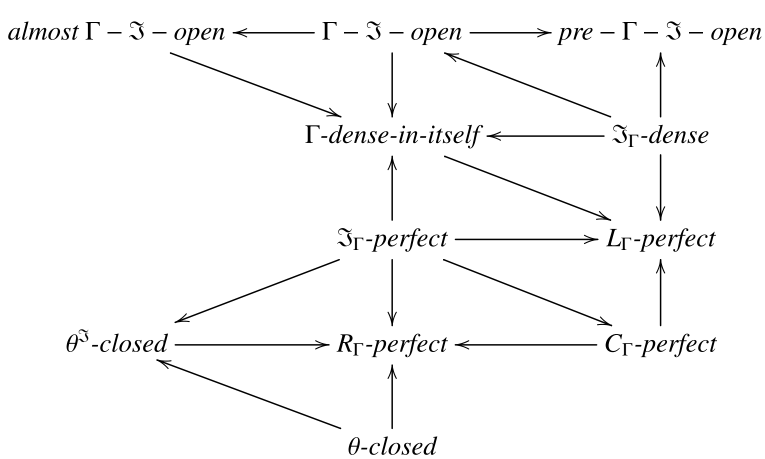
Theorem 2.8. *The following properties hold for $H \subseteq W$ in $(W, \upsilon, \Im)$:*
(i) if H is almost $\Gamma - \Im-$open, then it is Γ-dense-in-itself.
(ii) if H is almost $\Gamma - \Im-$open, then it is L_Γ-perfect.
(iii) if H is almost $\Gamma - \Im-$open and θ-closed, then it is a semi-open set.

Proof. Let $H \subseteq W$ in $(W, \upsilon, \Im)$.
(i) Let H be an almost $\Gamma - \Im-$open set. Then, $H \subseteq cl(int(\Gamma(H))) \subseteq cl(\Gamma(H))$. From the Theorem 1.2 (iii), H is Γ-dense-in-itself.
(ii) It can be observed from the Theorem 2.8 (i) and the Remark 1.6.
(iii) Let H be θ-closed and an almost $\Gamma - \Im-$open set. Then, $H \subseteq cl(int(\Gamma(H))) \subseteq cl(int(cl_\theta(H))) = cl(int(H))$ by the Theorem 1.2 (iii). As a result, H is semi-open. □

Remark 2.9. A Γ-dense-in-itself set or an L_Γ-perfect set may not be almost $\Gamma - \Im-$open in an ideal topological space.

Example 2.10. In the ideal topological space $(\mathbb{R}, \tau_u, \{\emptyset\})$, where $\mathbb{R}$ is the set of all real numbers, $\mathbb{Z}$ is the set of all integers and τ_u is the usual topology on $\mathbb{R}$, the set $\mathbb{Z}$ is both a Γ-dense-in-itself set and an L_Γ-perfect set, but $\mathbb{Z}$ is not almost $\Gamma - \Im-$open.

Remark 2.11. In $(W, \upsilon, \Im)$, the following diagram holds for $H \subseteq W$:



Proof. This follows from the Remark 1.6, Theorem 1.9, Theorem 2.4, Theorem 2.5 and Theorem 2.8. □

Theorem 2.12. *The following properties hold for $H \subseteq W$ in $(W, \upsilon, \Im)$:*
(i) *if H is open and Γ-dense-in-itself, then it is almost $\Gamma - \Im-$open.*
(ii) *if H is θ-open and Γ-dense-in-itself, then it is almost $\Gamma - \Im-$open.*
(iii) *if $\Gamma(H)$ is open and H is Γ-dense-in-itself, then H is almost $\Gamma - \Im-$open.*
(iv) *if H is θ-open and $\Gamma_\Gamma-$open, then it is almost $\Gamma - \Im-$open.*

Proof. Let $H \subseteq W$ in $(W, \upsilon, \Im)$.
(i) It can be observed from the Theorem 2.4 (ii) and the Theorem 1.9 (ii).
(ii) It can be observed from the Theorem 2.12 (i) and $\upsilon_\theta \subseteq \upsilon$.
(iii) Let $\Gamma(H)$ be open and H be Γ-dense-in-itself. Then, $H \subseteq \Gamma(H) = int(\Gamma(H))$ and so $H \subseteq cl(int(\Gamma(H)))$. As a result, H is almost $\Gamma - \Im-$open.
(iv) Let H be θ-open and $\Gamma_\Gamma-$open. Then, $int_\theta(H) = cl(int(\Gamma(H)))$ and $H = int_\theta(H)$ from the Lemma 1.12 (iv) and so $H = cl(int(\Gamma(H)))$. As a result, H is almost $\Gamma - \Im-$open. □

Theorem 2.13. *For $H \subseteq W$ in $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$; if H is σ_0-open, then H is $\Gamma - \Im-$open.*

Proof. For $H \subseteq W$ in $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$; let H be σ_0-open. From the Theorem 1.13, $\Psi_\Gamma(H) \subseteq \Gamma(H)$, then $cl(\Psi_\Gamma(H)) \subseteq cl(\Gamma(H))$. It implies that $cl(\Psi_\Gamma(H)) \subseteq \Gamma(H)$, by the Theorem 1.2 (iii) and so $int(cl(\Psi_\Gamma(H))) \subseteq int(\Gamma(H))$. As H is σ_0-open, $H \subseteq int(\Gamma(H))$. Finally, H is a $\Gamma - \Im-$open set. □

Corollary 2.14. *In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$, $\upsilon_\theta \subseteq \sigma \subseteq \sigma_0 \subseteq \Gamma\Im O(W) \subseteq \mathcal{P}\Gamma\Im O(W)$.*

Proof. In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$: it is obvious from the Corollary 1.14, Theorem 2.13 and Theorem 1.9 (iii). □

Corollary 2.15. *For $H \subseteq W$ in $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$:*
(i) *if H is σ_0-open, it is an almost $\Gamma - \Im-$open set.*
(ii) *if H is σ-open, it is an almost $\Gamma - \Im-$open set.*
(iii) *if H is θ-open, it is an almost $\Gamma - \Im-$open set.*

Proof. For $H \subseteq W$ in $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$:
(i) let H be σ_0-open. From the Theorem 2.13 and Theorem 1.9 (ii), H is an almost $\Gamma - \Im-$open set.
(ii) let H be σ-open. From the Corollary 1.14 and Corollary 2.15 (i), H is an almost $\Gamma - \Im-$open set.
(iii) let H be θ-open. From the Corollary 1.14 and Corollary 2.15 (ii), H is an almost $\Gamma - \Im-$open set. □

Remark 2.16. In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\emptyset\}$, a $\Gamma - \Im-$open set may not σ_0-open. Moreover, an almost $\Gamma - \Im-$open set may not be θ-open (resp. σ-open, σ_0-open).

Example 2.17. $W = \{t, p, l, g\}, \Im = \{\emptyset, \{l\}\}$ and $\upsilon = \{\emptyset, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. Although, $M = \{p\}$ is both $\Gamma - \Im-$open and almost $\Gamma - \Im-$open, it not σ_0-open (resp. σ-open, θ-open) in $(W, \upsilon, \Im)$, where $cl(\upsilon) \cap \Im = \{\emptyset\}$.

Proposition 2.18. $\Gamma\{\emptyset\}O(W) = \mathcal{P}\Gamma\{\emptyset\}O(W)$ *in* $(W, \upsilon, \{\emptyset\})$.

Proof. In $(W, \upsilon, \{\emptyset\})$, the case of $\Gamma\{\emptyset\}O(W) \subseteq \mathcal{P}\Gamma\{\emptyset\}O(W)$ is obvious from the Theorem 1.9 (iii). Let V be a pre$-\Gamma-\Im-$open set in $(W, \upsilon, \{\emptyset\})$. Then, $V \subseteq int(V \cup \Gamma(V)) = int(V \cup cl_\theta(V)) = int(cl_\theta(V)) = int(\Gamma(V))$, that is V is $\Gamma - \Im-$open. Therefore, $\mathcal{P}\Gamma\{\emptyset\}O(W) \subseteq \Gamma\{\emptyset\}O(W)$. As a result, $\mathcal{P}\Gamma\{\emptyset\}O(W) = \Gamma\{\emptyset\}O(W)$. □

Proposition 2.19. *In $(W, \upsilon, P(W))$,*
(i) $\Gamma P(W)O(W) = \{\emptyset\}$
(ii) *the collection of almost $\Gamma - \Im-$open sets is $\{\emptyset\}$.*

Proof. In $(W, \upsilon, P(W))$,
(i) it is obvious that $\emptyset$ is $\Gamma - \Im-$open. Inversely, if H is a $\Gamma - \Im-$open set in $(W, \upsilon, P(W))$, then $H \subseteq int(\Gamma(H)) = int(\emptyset) = \emptyset$. Thus, $H = \emptyset$ and so $\Gamma P(W)O(W) = \{\emptyset\}$.
(ii) $\emptyset$ is an almost $\Gamma - \Im-$open set. Conversely, if H is almost $\Gamma - \Im-$open in $(W, \upsilon, P(W))$, then $H \subseteq cl(int(\Gamma(H))) = cl(int(\emptyset)) = \emptyset$. Therefore, $H = \emptyset$ and so the collection of almost $\Gamma - \Im-$open sets is $\{\emptyset\}$. □

3. On $\Gamma_\Gamma-$open Sets and M^*-open Sets

Proposition 3.1. *In $(W, \upsilon, \Im)$, if H is $\Gamma_\Gamma-$open, then the following properties hold for $H \subseteq W$.*
(i) $cl(int(\Gamma(H))) = cl(int_\theta(\Gamma(H))) \subseteq \Psi_\Gamma(H)$
(ii) $int_\theta(H) \subseteq \Gamma(H)$
(iii) $int(\Gamma(H)) \subseteq \Psi_\Gamma(H)$
(iv) $cl(int(\Gamma(H))) \subseteq H$

Proof. Let $H \subseteq W$ in $(W, \upsilon, \Im)$ and H be a $\Gamma_\Gamma-$open set.
(i) It is known that $int_\theta(H) \subseteq \Psi_\Gamma(H)$ for every $H \subseteq W$ from the Theorem 1.11. Then, by the hypothesis, $cl(int(\Gamma(H))) = int_\theta(H) \subseteq \Psi_\Gamma(H)$. Furthermore, since $\Gamma(H)$ is closed from the Theorem 1.2 (iii), $cl(int(\Gamma(H))) = cl(int_\theta(\Gamma(H)))$ from the Lemma 1.12 (i). As a result, $cl(int(\Gamma(H))) = cl(int_\theta(\Gamma(H))) \subseteq \Psi_\Gamma(H)$.
(ii) By the hypothesis, $int_\theta(H) = cl(int(\Gamma(H))) \subseteq cl(\Gamma(H))$ and thus $int_\theta(H) \subseteq \Gamma(H)$ from the Theorem 1.2 (iii).
(iii) It is observed from the Proposition 3.1 (i).
(iv) It is clear as $int_\theta(H) \subseteq H$ and by the hypothesis. □

Theorem 3.2. *For $H \subseteq W$ in $(W, \upsilon, \Im)$, if $int_\theta(H) \neq \emptyset$ and H is $\Gamma_\Gamma-$open, then there exists an open set G containing x such that $cl(G) \notin \Im$ for each $x \in int_\theta(H)$.*

Proof. Let $int_\theta(H) \neq \emptyset$ and H be a $\Gamma_\Gamma-$open subset of W in $(W, \upsilon, \Im)$. If $x \in int_\theta(H)$, x is also in $\Gamma(H)$ by the Proposition 3.1 (ii). Afterward, there exists $G \in \upsilon(x)$ such that $cl(G) \subseteq H$ and hence $cl(G) \cap H \notin \Im$. Therefore, there exists $G \in \upsilon(x)$ such that $cl(G) \cap H = cl(G) \notin \Im$. □

Theorem 3.3. *The following properties hold for $H \subseteq W$ in $(W, \upsilon, \Im)$:*
(i) *if H is θ-open and $\Gamma_\Gamma-$open, it is almost $\Gamma - \Im-$open.*
(ii) *if H is θ-open and $\Gamma_\Gamma-$open, it is Γ-dense-in-itself.*
(iii) *if H is θ-open and $\Gamma_\Gamma-$open, it is L_Γ-perfect.*
(iv) *if H is $\Gamma - \Im-$open and $\Gamma_\Gamma-$open, it is θ-open.*
(v) *if H is $\Gamma - \Im-$open and $\Gamma_\Gamma-$open, it is σ-open.*
(vi) *if H is $\Gamma - \Im-$open and $\Gamma_\Gamma-$open, it is σ_0-open.*

Proof. Let $H \subseteq W$ in $(W, \upsilon, \Im)$.
(i) Let H be θ-open and $\Gamma_\Gamma-$open. Then, $int_\theta(H) = cl(int(\Gamma(H)))$ and thus from the Lemma 1.12 (iv) $H = cl(int(\Gamma(H)))$. As a result, H is almost $\Gamma - \Im-$open.
(ii) It obvious from the Theorem 3.3 (i) and the Theorem 2.8 (i).
(iii) It obvious from the Theorem 3.3 (ii) and the Remark 1.6.
(iv) Let H be $\Gamma - \Im-$open and $\Gamma_\Gamma-$open. Since $int_\theta(H) = cl(int(\Gamma(H)))$ and $H \subseteq int(\Gamma(H))$, $H \subseteq cl(H) \subseteq cl(int(\Gamma(H))) = int_\theta(H)$. Thus, $H = int_\theta(H)$ and so H is θ-open from the Lemma 1.12 (iv).
(v) The proof can be observed from the Theorem 3.3 (iv) and the Corollary 1.14.
(vi) The proof can be observed from the Theorem 3.3 (v) and the Corollary 1.14. □

Theorem 3.4. *In $(W, \upsilon, \Im)$, for $H \subseteq W$, if $W \setminus H$ is $\Gamma_\Gamma-$open, then H is σ_0-open.*

Proof. Let $H \subseteq W$ in $(W, \upsilon, \Im)$ and $W \setminus H$ be $\Gamma_\Gamma-$open. Then, $cl_\theta(H) = W \setminus int_\theta(W \setminus H) = W \setminus cl(int(\Gamma(W \setminus H))) = int(cl(\Psi_\Gamma(H)))$. Since $H \subseteq cl_\theta(H)$, $H \subseteq int(cl(\Psi_\Gamma(H)))$. As a consequence, H is σ_0-open. □

Remark 3.5. The complement of a σ_0-open set may not be $\Gamma_\Gamma-$open in an ideal topological space.

Example 3.6. $W = \{t, p, l, g\}$, $\Im = \{\emptyset, \{p\}\}$ and $\upsilon = \{\emptyset, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. Altough, $N = \{t, l\}$ is a σ_0-open set, $W \setminus N$ is not $\Gamma_\Gamma-$open in $(W, \upsilon, \Im)$.

Theorem 3.7. *In $(W, \upsilon, \Im)$, for $H \subseteq W$, if H is an M^*-open set, then H is σ_0-open.*

Proof. Let H be subset of W in $(W, \upsilon, \Im)$ and H be an M^*-open set. It is known that $int_\theta(H) \subseteq \Psi_\Gamma(H)$ from the Theorem 1.11. It implies that $cl(int_\theta(H)) \subseteq cl(\Psi_\Gamma(H))$. Then, $H \subseteq int(cl(int_\theta(H))) \subseteq int(cl(\Psi_\Gamma(H)))$. As a consequence, H is σ_0-open. □

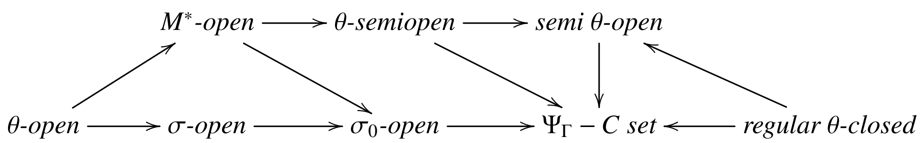
Corollary 3.8. *In $(W, \upsilon, \Im)$, for $H \subseteq W$, if H is an M^*-open set, then*
(i) *H is a $\Psi_\Gamma - C$ set.*
(ii) *H is a semi θ-open set.*

Proof. **(i)** It is obvious from the Theorem 3.7 and the Corollary 1.14.
(ii) It is obvious from the Lemma 1.12 (iii) and the Corollary 1.14. □

Remark 3.9. A σ_0-open set (resp. a semi θ-open set, a $\Psi_\Gamma - C$ set) may not be an M^*-open set in an ideal topological space.

Example 3.10. $W = \{t, p, l, g\}, \Im = \{\varnothing, \{p\}\}$ and $\upsilon = \{\varnothing, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. Although, $N = \{t, l\}$ is both σ_0-open and a $\Psi_\Gamma - C$ set, N is not an M^*-open set in $(W, \upsilon, \Im)$. Furthermore, the set $O = \{t, p, l\}$ is a semi θ-open set, but O is not an M^*-open set in $(W, \upsilon, \Im)$.

Remark 3.11. In $(W, \upsilon, \Im)$, the following implications hold for $H \subseteq W$:



Proof. The proof is obvious from the Lemma 1.12 (ii), the Lemma 1.12 (iii), the Corollary 1.14 and the Theorem 3.7. □

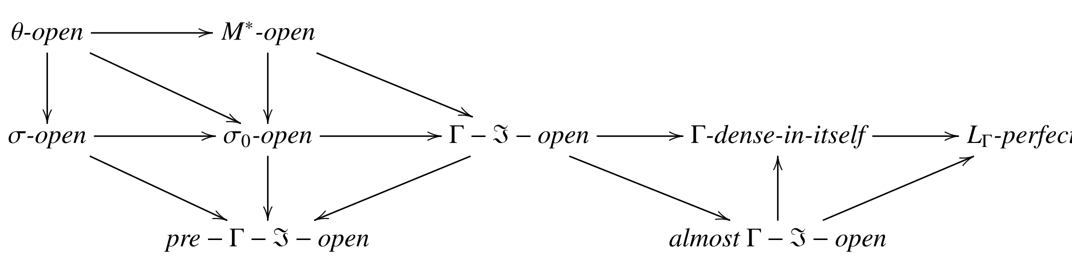
Theorem 3.12. *In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$, for $H \subseteq W$, if H is an M^*-open set, then*
(i) *H is $\Gamma - \Im-$open.*
(ii) *H is pre$-\Gamma - \Im-$open.*
(iii) *H is Γ-dense-in-itself.*
(iv) *H is L_Γ-perfect.*
(v) *H is almost $\Gamma - \Im-$open.*

Proof. In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$, for $H \subseteq W$, let H be an M^*-open set. The proof is obvious
(i) by the Theorem 3.7 and the Theorem 2.13.
(ii) by the Theorem 3.12 (i) and the Theorem 1.9 (iii).
(iii) by the Theorem 3.12 (i) and the Theorem 1.9 (i).
(iv) by the Theorem 3.12 (iii) and the Remark 1.6.
(v) by the Theorem 3.12 (i) and the Theorem 1.9 (ii). □

Remark 3.13. The reverse of the above conditions may not be true in an ideal topological space $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$.

Example 3.14. $W = \{t, p, l, g\}, \Im = \{\varnothing, \{p\}\}$ and $\upsilon = \{\varnothing, \{g\}, \{t, l\}, \{t, l, g\}, W\}$. Altough, $N = \{t, l\}$ is $\Gamma - \Im-$open (resp. pre$-\Gamma - \Im-$open, Γ-dense-in-itself, L_Γ-perfect, almost $\Gamma - \Im-$open), N is not an M^*-open set in $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$.

Remark 3.15. In $(W, \upsilon, \Im)$ where $cl(\upsilon) \cap \Im = \{\varnothing\}$, the following diagram holds for $H \subseteq W$:



Proof. The proof is obvious from the Remark 3.11, the Theorem 3.12 (i), the Corollary 2.14 and the Remark 2.11. □

Proposition 3.16. *In $(W, \upsilon, P(W))$, for $H \subseteq W$, H is $\Gamma_\Gamma-$open if and only if $int_\theta(H) = \varnothing$.*

Proof. Let $H \subseteq W$ in $(W, \upsilon, P(W))$.

$(:\Rightarrow)$ Let H be $\Gamma_\Gamma-$open. Then $int_\theta(H) = cl(int(\Gamma(H))) = cl(int(\emptyset)) = \emptyset$.

$(:\Leftarrow)$ Let $int_\theta(H) = \emptyset$. Then $cl(int(\Gamma(H))) = cl(int(\emptyset)) = \emptyset = int_\theta(H)$ and so H is $\Gamma_\Gamma-$open. □

4. Conclusion

In this study, the relationships between the sets $\Gamma - \Im-$open, pre$-\Gamma - \Im-$open, $\Gamma_\Gamma-$open, almost $\Gamma - \Im-$open, M^*-open, and the some special sets obtained by the operators Γ and Ψ_Γ were investigated in ideal topological spaces. These relations were also searched in the special cases $cl(\upsilon) \cap \Im = \{\emptyset\}$. Moreover, the findings were combined with diagrams to expand the scope of the relationships. Therefore, the results are made more remarkable.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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