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Research Article

Upper bounds for the numerical radii of the tensor products of operators

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ABSTRACT. In this paper, we develop new upper bounds for the numerical radii of the tensor products of two operators. These inequalities improve and generalize some earlier related inequalities.

Keywords: Numerical radius, operator norm, tensor product, inequality.

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1. INTRODUCTION

In this paper, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. By $\mathcal{B}(\mathcal{H})$, we denote the algebra of all bounded linear operators acting on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A. The cartesian decomposition of A is $A = \operatorname{Re}(A) + i \operatorname{Im}(A)$, where $\operatorname{Re}(A) = \frac{A+A^*}{2}$ and $\operatorname{Im}(A) = \frac{A-A^*}{2i}$, are known as the real and imaginary parts of A, respectively.

Recall that for $A \in \mathcal{B}(\mathcal{H})$, the the usual operator norm of A is defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H},\ \|x\| = 1\}.$$

The numerical range of A, denoted by $W(A)$, is defined as

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H},\ \|x\| = 1\}$$

and the numerical radius is defined as

$$\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H},\ \|x\| = 1\}.$$

For more material about the numerical radius, we refer the reader to [4, 5, 6, 9]. It is well known that $w(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$ and for all $A \in \mathcal{B}(\mathcal{H})$, the following inequalities

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{1.1}$$

hold, that is, $w(\cdot)$ defines a norm equivalent to $\|\cdot\|$ on $\mathcal{B}(\mathcal{H})$. The first inequality becomes equality if $A^2 = 0$ and the second one turns into equality if A is normal.

In [3], Dragomir proved that for the product of the two operators $A, B \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, we have

$$w^r(B^*A) \leq \frac{1}{2}\| \, |A|^{2r} + |B|^{2r} \, \|. \tag{1.2}$$

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It is known that the tensor product $\mathcal{H} \otimes \mathcal{K}$ of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, which is defined as the completion of the inner product space consisting of elements of the form $\sum_{j=1}^{n} x_i \otimes y_i$, with $x_i \in \mathcal{H}$ and $y_i \in \mathcal{K}$, and $n \geq 1$, is a Hilbert space under the inner product $\langle x \otimes y, u \otimes v\rangle = \langle x, u\rangle \langle y, v\rangle$. The tensor product of the two operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ denoted by $A \otimes B$ is defined on $\mathcal{H} \otimes \mathcal{K}$ as $(A \otimes B)(x \otimes y) = Ax \otimes By$ for $x \otimes y \in \mathcal{H} \otimes \mathcal{K}$. Recall that we can write $A \otimes B$ as the product of $A \otimes \mathcal{I}_{\mathcal{K}}$ and $\mathcal{I}_{\mathcal{H}} \otimes B$, i.e., $A \otimes B = (A \otimes \mathcal{I}_{\mathcal{K}})(\mathcal{I}_{\mathcal{H}} \otimes B)$, where $\mathcal{I}_{\mathcal{K}}$ and $\mathcal{I}_{\mathcal{H}}$ are the identity operators on $\mathcal{H}$ and $\mathcal{K}$, respectively.

For any $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ as it satisfies $\|A \otimes B\| = \|A\| \|B\|$. Therefore, from the inequalities (1.1), it follows that

$$\frac{1}{2}\|A\| \|B\| \leq w(A \otimes B) \leq \|A\| \|B\| . \tag{1.3}$$

Also, $w(A \otimes B)$ satisfies the following inequalities

$$w(A)w(B) \leq w(A \otimes B) \leq 2w(A)w(B). \tag{1.4}$$

In [1], Bhunia et al. proved that

$$\omega^2 (A \otimes B) \leq \left\| |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \right\| \tag{1.5}$$

and

$$\omega^2 (A \otimes B) \leq \frac{1}{4} \left\| |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \right\| + \frac{1}{2} \left\| \mathrm{Re}\left(|A| |A^*| \otimes |B| |B^*|\right) \right\| . \tag{1.6}$$

In this work, we will continue working in this direction and we prove several new numerical radius inequalities for the tensor product of two operators. In particular, we generalize and improve the inequalities (1.5) and (1.6). Some earlier existing inequalities are also refined.

2. Main results

In this section, we present our results. First, we start with the following lemmas that will be used to develop new results in this paper.

Lemma 2.1 ([2]). *If a, b, e are vectors in $\mathcal{H}$ with $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{2}\left(\|a\| \|b\| + |\langle a, b\rangle|\right).$$

Lemma 2.2 ([14, p.20]). *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ such that $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \text{ for all } r \geq 1.$$

Lemma 2.3 ([9]). *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A| x, x\rangle \langle |A^*| y, y\rangle$$

for all $x, y \in \mathcal{H}$.

Lemma 2.4 ([7]). *If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle|^r \leq \frac{2\alpha + 1}{2\alpha + 2} \|a\|^r \|b\|^r + \frac{1}{2\alpha + 2} |\langle a, b\rangle|^r$$

for any $\alpha \geq 0$ and $r \geq 1$.

Lemma 2.5 ([8]). *Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$ and $\zeta \in \mathbb{C} \setminus \{0\}$. Then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{|\zeta|}\left(\max\{1, |1 - \zeta|\} \|a\| \|b\| + |\langle a, b\rangle|\right).$$

Lemma 2.6 ([13]). *If h is a convex function on a real interval J containing the spectrum of the self-adjoint operator A, then for any unit vector $x \in \mathcal{H}$,*

$$h\left(\langle Ax, x\rangle\right) \leq \langle h\left(A\right)x, x\rangle .$$

Lemma 2.7 ([11]). *Let h be a twice differentiable convex function on $[a, b]$ such that $h^{''} \geq \alpha := \min\limits_{x\in[a,b]} h\left(x\right) > 0$. Then*

$$h\left(\frac{a+b}{2}\right) \leq \frac{h\left(a\right)+h\left(b\right)}{2} - \frac{1}{8}\alpha\left(a-b\right)^2 .$$

Lemma 2.8 ([12]). *If $A, B \in \mathcal{B}(\mathcal{H})$ are two normal operators, then*

$$\omega\left(A+B\right) \leq \sqrt{2}\omega\left(|A| + i\,|B|\right).$$

Lemma 2.9 ([10, 12]). *If $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators, then*

$$\omega^2\left(A+iB\right) \leq \left\|A^2+B^2\right\|.$$

Our first result reads as follows.

Theorem 2.1. *Let $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and $\alpha \geq 0$. Then*

$$\omega^{2r}\left(A\otimes B\right) \leq \frac{2\alpha+1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| + \frac{1}{2\alpha+2}\omega^r\left(A^2\otimes B^2\right),$$

for any $r \geq 1$.

Proof. Let $f \in \mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$. Then, by using Lemma 2.4, it follows that

$$\begin{aligned}
&|\langle(A\otimes B)\,f, f\rangle|^{2r} \\
&= \left(|\langle(A\otimes B)\,f, f\rangle|\,|\langle f, (A^*\otimes B^*)\,f\rangle|\right)^r \\
&\leq \frac{2\alpha+1}{2\alpha+2}\left\|(A\otimes B)\,f\right\|^r\left\|(A^*\otimes B^*)\,f\right\|^r + \frac{1}{2\alpha+2}\left|\langle(A\otimes B)\,f, (A^*\otimes B^*)\,f\rangle\right|^r \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left(\left\|(A\otimes B)\,f\right\|^{2r} + \left\|(A^*\otimes B^*)\,f\right\|^{2r}\right) + \frac{1}{2\alpha+2}\left|\left\langle\left(A^2\otimes B^2\right)f, f\right\rangle\right|^r \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{2\alpha+1}{4\alpha+4}\left(\left\langle|A\otimes B|^2 f, f\right\rangle^r + \left\langle|A^*\otimes B^*|^2 f, f\right\rangle^r\right) + \frac{1}{2\alpha+2}\left|\left\langle\left(A^2\otimes B^2\right)f, f\right\rangle\right|^r \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left(\left\langle|A\otimes B|^{2r} f, f\right\rangle + \left\langle|A^*\otimes B^*|^{2r} f, f\right\rangle\right) + \frac{1}{2\alpha+2}\left|\left\langle\left(A^2\otimes B^2\right)f, f\right\rangle\right|^r \\
&\text{(by Lemma 2.2)} \\
&= \frac{2\alpha+1}{4\alpha+4}\left\langle\left(|A\otimes B|^{2r} + |A^*\otimes B^*|^{2r}\right)f, f\right\rangle + \frac{1}{2\alpha+2}\left|\left\langle\left(A^2\otimes B^2\right)f, f\right\rangle\right|^r \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| + \frac{1}{2\alpha+2}\omega^r\left(A^2\otimes B^2\right).
\end{aligned}$$

Taking the supremum over all $f \in \mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$, we get the desired inequality. □

Corollary 2.1. *Let $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and let $\alpha \geq 0$. Then, for any $r \geq 1$, we have*

$$\begin{aligned}
\omega^{2r}\left(A\otimes B\right) &\leq \frac{2\alpha+1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| + \frac{1}{2\alpha+2}\omega^r\left(A^2\otimes B^2\right) \\
&\leq \frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\|.
\end{aligned}$$

Proof. By using the inequality (1.2), we have

$$
\begin{aligned}
&\omega^{2r}\left(A\otimes B\right)\\
\leq&\frac{2\alpha+1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r}+|A^*|^{2r}\otimes|B^*|^{2r}\right\|+\frac{1}{2\alpha+2}\omega^{r}\left(A^2\otimes B^2\right)\\
\leq&\frac{2\alpha+1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r}+|A^*|^{2r}\otimes|B^*|^{2r}\right\|+\frac{1}{4\alpha+4}\left\||A|^{2r}\otimes|B|^{2r}+|A^*|^{2r}\otimes|B^*|^{2r}\right\|\\
=&\frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r}+|A^*|^{2r}\otimes|B^*|^{2r}\right\|.
\end{aligned}
$$

□

Remark 2.1. *By taking $\alpha=0$ and $r=1$ in Corollary 2.1, we get the inequality (1.5). Hence, the inequality in Theorem 2.1 is a generalization and refinement of the inequality (1.5).*

Theorem 2.2. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let $\alpha\geq0$. Then*

$$
\begin{aligned}
\omega^2\left(A\otimes B\right)\leq&\frac{1}{4\left(\alpha+1\right)}\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|+\frac{1}{2\left(\alpha+1\right)}\left\|\mathrm{Re}\left(|A|\,|A^*|\otimes|B|\,|B^*|\right)\right\|\\
&+\frac{\alpha}{2\left(\alpha+1\right)}\left\||A|\otimes|B|+|A^*|\otimes|B^*|\right\|\omega\left(A\otimes B\right).
\end{aligned}
$$

Proof. Let $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\|=1$ and let $\alpha\geq0$. Then

$$
\begin{aligned}
\left|\langle(A\otimes B)\,f,f\rangle\right|^2=&\frac{1}{\alpha+1}\left|\langle(A\otimes B)\,f,f\rangle\right|^2+\frac{\alpha}{\alpha+1}\left|\langle(A\otimes B)\,f,f\rangle\right|^2\\
\leq&\frac{1}{\alpha+1}\langle|A\otimes B|\,f,f\rangle\langle|A^*\otimes B^*|\,f,f\rangle\\
&+\frac{\alpha}{\alpha+1}\langle|A\otimes B|\,f,f\rangle^{\frac{1}{2}}\langle|A^*\otimes B^*|\,f,f\rangle^{\frac{1}{2}}\left|\langle(A\otimes B)\,f,f\rangle\right|\\
&\text{(by Lemma 2.3)}\\
\leq&\frac{1}{4\left(\alpha+1\right)}\langle(|A\otimes B|+|A^*\otimes B^*|)\,f,f\rangle^2\\
&+\frac{\alpha}{2\left(\alpha+1\right)}\langle(|A\otimes B|+|A^*\otimes B^*|)\,f,f\rangle\left|\langle(A\otimes B)\,f,f\rangle\right|\\
\leq&\frac{1}{4\left(\alpha+1\right)}\left\langle(|A\otimes B|+|A^*\otimes B^*|)^2\,f,f\right\rangle\\
&+\frac{\alpha}{2\left(\alpha+1\right)}\langle(|A\otimes B|+|A^*\otimes B^*|)\,f,f\rangle\left|\langle(A\otimes B)\,f,f\rangle\right|\\
=&\frac{1}{4\left(\alpha+1\right)}\left\langle\left(|A\otimes B|^2+|A^*\otimes B^*|^2\right)f,f\right\rangle\\
&+\frac{1}{2\left(\alpha+1\right)}\left\|\mathrm{Re}\left(|A|\,|A^*|\otimes|B|\,|B^*|\right)\right\|\\
&+\frac{\alpha}{2\left(\alpha+1\right)}\langle(|A\otimes B|+|A^*\otimes B^*|)\,f,f\rangle\left|\langle(A\otimes B)\,f,f\rangle\right|\\
=&\frac{1}{4\left(\alpha+1\right)}\left\langle\left(|A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right)f,f\right\rangle\\
&+\frac{1}{2\left(\alpha+1\right)}\langle\mathrm{Re}\left(|A|\,|A^*|\otimes|B|\,|B^*|\right)f,f\rangle\\
&+\frac{\alpha}{2\left(\alpha+1\right)}\langle(|A|\otimes|B|+|A^*|\otimes|B^*|)\,f,f\rangle\left|\langle(A\otimes B)\,f,f\rangle\right|
\end{aligned}
$$

$$\le \frac{1}{4(\alpha+1)} \left\| |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \right\|$$
$$+ \frac{1}{2(\alpha+1)} \langle \mathrm{Re}\left(|A|\,|A^*| \otimes |B|\,|B^*|\right) f, f \rangle$$
$$+ \frac{\alpha}{2(\alpha+1)} \left\| |A| \otimes |B| + |A^*| \otimes |B^*| \right\| \omega\left(A \otimes B\right).$$

Taking the supremum over all $f \in \mathcal{H} \otimes \mathcal{K}$ with $\|f\| = 1$, we get the desired inequality. □

Letting $\alpha = 0$ in Theorem 2.2, we get the following corollary.

Corollary 2.2. *Let $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Then*

$$\omega^2\left(A \otimes B\right) \le \frac{1}{4} \left\| |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \right\| + \frac{1}{2} \left\| \mathrm{Re}\left(|A|\,|A^*| \otimes |B|\,|B^*|\right) \right\|.$$

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^{2r}\left(A \otimes B\right) \le \frac{1}{4}\omega^2\left(|A|^r \otimes |B|^r + i\,|A^*|^r \otimes |B^*|^r\right) + \frac{1}{8} \left\| |A|^{2r} \otimes |B|^{2r} + |A^*|^{2r} \otimes |B^*|^{2r} \right\|$$
$$+ \frac{1}{4}\omega\left(|A^* \otimes B^*|^r\,|A \otimes B|^r\right)$$

for any $r \ge 2$.

Proof. Let $f \in \mathcal{H} \otimes \mathcal{K}$ with $\|f\| = 1$. Then

$$|\langle (A \otimes B) f, f \rangle|^{2r}$$
$$= |\langle (A \otimes B) f, f \rangle|^r \, |\langle (A^* \otimes B^*) f, f \rangle|^r$$
$$\le \|(A \otimes B) f\|^r \, \|(A^* \otimes B^*) f\|^r$$
$$= \left\langle |A \otimes B|^2 f, f \right\rangle^{\frac{r}{2}} \left\langle |A^* \otimes B^*|^2 f, f \right\rangle^{\frac{r}{2}}$$
$$\le \langle |A \otimes B|^r f, f \rangle \, \langle |A^* \otimes B^*|^r f, f \rangle \text{ (by Lemma 2.2)}$$
$$\le \frac{1}{4} \left(\langle |A \otimes B|^r f, f \rangle + \langle |A^* \otimes B^*|^r f, f \rangle \right)^2$$
$$= \frac{1}{4} \left(\langle |A \otimes B|^r f, f \rangle^2 + \langle |A^* \otimes B^*|^r f, f \rangle^2 + 2 \langle |A \otimes B|^r f, f \rangle \langle |A^* \otimes B^*|^r f, f \rangle \right)$$
$$= \frac{1}{4} \left| \langle |A \otimes B|^r f, f \rangle + i \langle |A^* \otimes B^*|^r f, f \rangle \right|^2 + \frac{1}{2} \langle |A \otimes B|^r f, f \rangle \langle f, |A^* \otimes B^*|^r f \rangle$$
$$\le \frac{1}{4} \left| \langle (|A \otimes B|^r + i\,|A^* \otimes B^*|^r) f, f \rangle \right|^2 + \frac{1}{4} \left\| |A \otimes B|^r f \right\| \left\| |A^* \otimes B^*|^r f \right\|$$
$$+ \frac{1}{4} \langle |A \otimes B|^r f, |A^* \otimes B^*|^r f \rangle \text{ (by Lemma 2.1)}$$
$$\le \frac{1}{4} \left| \langle (|A \otimes B|^r + i\,|A \otimes B|^r) f, f \rangle \right|^2 + \frac{1}{8} \left(\left\| |A \otimes B|^r f \right\|^2 + \left\| |A^* \otimes B^*|^r f \right\|^2 \right)$$
$$+ \frac{1}{4} \langle |A^* \otimes B^*|^r \, |A \otimes B|^r f, f \rangle$$

(by the arithmetic-geometric mean inequality)

$$= \frac{1}{4}\left|\left\langle \left(|A\otimes B|^r + i\,|A^*\otimes B^*|^r\right) f, f\right\rangle\right|^2 + \frac{1}{8}\left\langle \left(|A\otimes B|^{2r} + |A^*\otimes B^*|^{2r}\right) f, f\right\rangle$$
$$+ \frac{1}{4}\left\langle |A^*\otimes B^*|^r\,|A\otimes B|^r f, f\right\rangle$$
$$\le \frac{1}{4}\omega^2\left(|A\otimes B|^r + i\,|A^*\otimes B^*|^r\right) + \frac{1}{8}\left\||A\otimes B|^{2r} + |A^*\otimes B^*|^{2r}\right\| + \frac{1}{4}\omega\left(|A^*\otimes B^*|^r\,|A\otimes B|^r\right)$$
$$= \frac{1}{4}\omega^2\left(|A|^r\otimes|B|^r + i\,|A^*|^r\otimes|B^*|^r\right) + \frac{1}{8}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\|$$
$$+ \frac{1}{4}\omega\left(|A^*\otimes B^*|^r\,|A\otimes B|^r\right).$$

Taking the supremum over all $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$, we get the desired inequality. □

Remark 2.2. *The inequality in Theorem 2.3 is a generalization and improvement of the inequality (1.5). Indeed, by using Lemma 2.9 and the inequality (1.2), it can observed that*

$$\begin{aligned}
\omega^{2r}(A\otimes B) &\le \frac{1}{4}\omega^2\left(|A|^r\otimes|B|^r + i\,|A^*|^r\otimes|B^*|^r\right) + \frac{1}{8}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| \\
&\quad + \frac{1}{4}\omega\left(|A^*\otimes B^*|^r\,|A\otimes B|^r\right) \\
&\le \frac{1}{4}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| + \frac{1}{8}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| \\
&\quad + \frac{1}{8}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\| \\
&\le \frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r} + |A^*|^{2r}\otimes|B^*|^{2r}\right\|.
\end{aligned}$$

Theorem 2.4. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let $0\le\alpha\le 1$. Then*

$$\omega^2(A\otimes B) \le \frac{\alpha}{4}\omega^2\left(|A\otimes B| + i\,|A^*\otimes B^*|\right) + \frac{2-\alpha}{8}\left\||A|\otimes|B| + |A^*|\otimes|B^*|\right\|^2.$$

Proof. Let $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$ and let $0\le\alpha\le 1$. Then

$$\left|\left\langle (A\otimes B) f, f\right\rangle\right|^2$$
$$= \alpha\left|\left\langle (A\otimes B) f, f\right\rangle\right|^2 + (1-\alpha)\left|\left\langle (A\otimes B) f, f\right\rangle\right|^2$$
$$\le \alpha\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle + (1-\alpha)\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle$$

(by Lemma 2.3)

$$\le \frac{\alpha}{4}\left(\left\langle |A\otimes B| f, f\right\rangle + \left\langle |A^*\otimes B^*| f, f\right\rangle\right)^2$$
$$+ (1-\alpha)\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle$$
$$= \frac{\alpha}{4}\left(\left\langle |A\otimes B| f, f\right\rangle^2 + \left\langle |A^*\otimes B^*| f, f\right\rangle^2 + 2\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle\right)$$
$$+ (1-\alpha)\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle$$
$$\le \frac{\alpha}{4}\left(\left\langle |A\otimes B| f, f\right\rangle^2 + \left\langle |A^*\otimes B^*| f, f\right\rangle^2 + 2\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle\right)$$
$$+ (1-\alpha)\left\langle |A\otimes B| f, f\right\rangle\left\langle |A^*\otimes B^*| f, f\right\rangle$$

$$\begin{aligned}
&\leq \frac{\alpha}{4}\left|\left\langle\left(|A\otimes B|+i\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle\right|^{2}+\frac{\alpha}{8}\left\langle\left(|A\otimes B|+\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}\\
&+\frac{1-\alpha}{4}\left\langle\left(|A\otimes B|+\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}\\
&=\frac{\alpha}{4}\left|\left\langle\left(|A\otimes B|+i\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle\right|^{2}+\frac{2-\alpha}{8}\left\langle\left(|A\otimes B|+\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}\\
&\leq \frac{\alpha}{4}\omega^{2}\left(|A\otimes B|+i\left|A^{*}\otimes B^{*}\right|\right)+\frac{2-\alpha}{8}\left\||A|\otimes|B|+\left|A^{*}\right|\otimes\left|B^{*}\right|\right\|^{2}.
\end{aligned}$$

Taking the supremum over all $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\|=1$, we get the desired inequality. □

Remark 2.3. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let $0\leq\alpha\leq 1$. We have*

$$\begin{aligned}
\omega^{2}(A\otimes B)&\leq\frac{\alpha}{4}\omega^{2}\left(|A\otimes B|+i\left|A^{*}\otimes B^{*}\right|\right)+\frac{2-\alpha}{8}\left\||A|\otimes|B|+\left|A^{*}\right|\otimes\left|B^{*}\right|\right\|^{2}\\
&=\frac{\alpha}{4}\omega^{2}\left(|A\otimes B|+i\left|A^{*}\otimes B^{*}\right|\right)+\frac{2-\alpha}{8}\omega^{2}\left(|A|\otimes|B|+\left|A^{*}\right|\otimes\left|B^{*}\right|\right)\\
&\leq\frac{\alpha}{4}\omega\left(|A\otimes B|^{2}+\left|A^{*}\otimes B^{*}\right|^{2}\right)+\frac{2-\alpha}{4}\omega^{2}\left(|A|\otimes|B|+i\left|A^{*}\right|\otimes\left|B^{*}\right|\right)\\
&\quad\textit{(by Lemma 2.9 and Lemma 2.8)}\\
&=\frac{\alpha}{4}\omega\left(|A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right)+\frac{2-\alpha}{4}\omega^{2}\left(|A|\otimes|B|+i\left|A^{*}\right|\otimes\left|B^{*}\right|\right)\\
&\leq\frac{\alpha}{4}\omega\left(|A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right)+\frac{2-\alpha}{4}\omega\left(|A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right)\\
&\quad\textit{(by Lemma 2.8)}\\
&=\frac{1}{2}\omega\left(|A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right)\\
&=\frac{1}{2}\left\||A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right\|.
\end{aligned}$$

Theorem 2.5. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let $\zeta\in\mathbb{C}\setminus\{0\}$. Then*

$$\omega^{2}(A\otimes B)\leq\frac{\max\{1,|1-\zeta|\}}{2|\zeta|}\left\||A|^{2}\otimes|B|^{2}+\left|A^{*}\right|^{2}\otimes\left|B^{*}\right|^{2}\right\|+\frac{1}{|\zeta|}\omega\left(A^{2}\otimes B^{2}\right).$$

Proof. Let $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\|=1$. Then, by using Lemma 2.5, we have

$$\begin{aligned}
&|\langle(A\otimes B)f,f\rangle|^{2}\\
&=|\langle(A\otimes B)f,f\rangle||\langle f,(A^{*}\otimes B^{*})f\rangle|\\
&\leq\frac{1}{|\zeta|}\left(\max\{1,|1-\zeta|\}\|(A\otimes B)f\|\|(A^{*}\otimes B^{*})f\|+|\langle(A\otimes B)f,(A^{*}\otimes B^{*})f\rangle|\right)\\
&\leq\frac{1}{|\zeta|}\left(\max\{1,|1-\zeta|\}\|(A\otimes B)f\|\|(A^{*}\otimes B^{*})f\|+\left|\left\langle\left(A^{2}\otimes B^{2}\right)f,f\right\rangle\right|\right)\\
&\leq\frac{1}{|\zeta|}\left(\frac{\max\{1,|1-\zeta|\}}{2}\left(\|(A\otimes B)f\|^{2}+\|(A^{*}\otimes B^{*})f\|^{2}\right)+\left|\left\langle\left(A^{2}\otimes B^{2}\right)f,f\right\rangle\right|\right)\\
&=\frac{\max\{1,|1-\zeta|\}}{2|\zeta|}\left(\langle(A\otimes B)f,(A\otimes B)f\rangle+\langle(A^{*}\otimes B^{*})f,(A^{*}\otimes B^{*})f\rangle\right)\\
&+\frac{1}{|\zeta|}\left|\left\langle\left(A^{2}\otimes B^{2}\right)f,f\right\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\max\{1,|1-\zeta|\}}{2|\zeta|}\left(\left\langle (A\otimes B)^{*}(A\otimes B)f,f\right\rangle + \left\langle (A\otimes B)(A^{*}\otimes B^{*})f,f\right\rangle\right)\\
&+\frac{1}{|\zeta|}\left|\left\langle \left(A^{2}\otimes B^{2}\right)f,f\right\rangle\right|\\
&= \frac{\max\{1,|1-\zeta|\}}{2|\zeta|}\left\langle \left(|A|^{2}\otimes|B|^{2}+|A^{*}|^{2}\otimes|B^{*}|^{2}\right)f,f\right\rangle + \frac{1}{|\zeta|}\left|\left\langle \left(A^{2}\otimes B^{2}\right)f,f\right\rangle\right|\\
&\le \frac{\max\{1,|1-\zeta|\}}{2|\zeta|}\left\||A|^{2}\otimes|B|^{2}+|A^{*}|^{2}\otimes|B^{*}|^{2}\right\| + \frac{1}{|\zeta|}\omega\left(A^{2}\otimes B^{2}\right).
\end{aligned}
$$

Taking the supremum over all $f \in \mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$, we get the desired inequality. □

Considering $\zeta = n \in \mathbb{N}\setminus\{1\}$ in Theorem 2.5, we get the following corollary.

Corollary 2.3. *Let $A\otimes B \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Then*

$$
\omega^{2}(A\otimes B) \le \frac{n-1}{2n}\left\||A|^{2}\otimes|B|^{2}+|A^{*}|^{2}\otimes|B^{*}|^{2}\right\| + \frac{1}{n}\omega\left(A^{2}\otimes B^{2}\right)
$$

for all $n \in \mathbb{N}\setminus\{1\}$.

For $n = 2$ in Corollary 2.3, we get the following inequality.

Corollary 2.4. *Let $A\otimes B \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Then*

$$
\omega^{2}(A\otimes B) \le \frac{1}{4}\left\||A|^{2}\otimes|B|^{2}+|A^{*}|^{2}\otimes|B^{*}|^{2}\right\| + \frac{1}{2}\omega\left(A^{2}\otimes B^{2}\right).
$$

Remark 2.4. *If we take $n\to\infty$ in Corollary 2.3, then we obtain*

$$
\omega^{2}(A\otimes B) \le \frac{1}{2}\left\||A|^{2}\otimes|B|^{2}+|A^{*}|^{2}\otimes|B^{*}|^{2}\right\|.
$$

Theorem 2.6. *Let $A\otimes B \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let h be a twice differentiable nonnegative non-decreasing convex function on $[0,\infty)$ such that $0<\alpha\le h''$. Then*

$$
h(\omega(A)) \le \frac{1}{2}\left\|h(|A\otimes B|)+h(|A^{*}\otimes B^{*}|)\right\| - \inf_{\|f\|=1}\delta(f),
$$

where $\delta(f) = \frac{1}{8}\alpha\left\langle (|A\otimes B| - |A^{*}\otimes B^{*}|)f,f\right\rangle^{2}$.

Proof. Let $f \in \mathcal{H}\otimes\mathcal{K}$ with $\|f\| = 1$. Then, we have

$$
\begin{aligned}
h(|\langle (A\otimes B)f,f\rangle|) &\le h\left(\langle |A\otimes B| f,f\rangle^{\frac{1}{2}}\langle |A^{*}\otimes B^{*}| f,f\rangle^{\frac{1}{2}}\right)\\
&\le h\left(\frac{\langle |A\otimes B| f,f\rangle + \langle |A^{*}\otimes B^{*}| f,f\rangle}{2}\right).\\
&\text{(by the arithmetic - geometric mean inequality)}\\
&\le \frac{h(\langle |A\otimes B| f,f\rangle) + h(\langle |A^{*}\otimes B^{*}| f,f\rangle)}{2}\\
&- \frac{1}{8}\alpha\left(\langle |A\otimes B| f,f\rangle - \langle |A^{*}\otimes B^{*}| f,f\rangle\right)^{2}\\
&\text{(by Lemma 2.7)}\\
&\le \frac{1}{2}\langle (h(|A\otimes B|)+h(|A^{*}\otimes B^{*}|))f,f\rangle - \frac{1}{8}\alpha\langle (|A\otimes B| - |A^{*}\otimes B^{*}|)f,f\rangle^{2}\\
&\text{(by Lemma 2.6).}
\end{aligned}
$$

Thus,

$$\begin{aligned}&h\left(\left|\left\langle\left(A\otimes B\right)f,f\right\rangle\right|\right)\\ \leq&\frac{1}{2}\left\langle\left(h\left(\left|A\otimes B\right|\right)+h\left(\left|A^{*}\otimes B^{*}\right|\right)\right)f,f\right\rangle-\frac{1}{8}\alpha\left\langle\left(\left|A\otimes B\right|-\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}.\end{aligned}$$

Taking the supremum over $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\|=1$ in the above inequality, we get

$$\begin{aligned}h\left(\omega\left(A\otimes B\right)\right)&\leq\frac{1}{2}\omega\left(h\left(\left|A\otimes B\right|\right)+h\left(\left|A^{*}\otimes B^{*}\right|\right)\right)-\inf_{\|f\|=1}\delta\left(f\right)\\ &\leq\frac{1}{2}\left\|h\left(\left|A\otimes B\right|\right)+h\left(\left|A^{*}\otimes B^{*}\right|\right)\right\|-\inf_{\|f\|=1}\delta\left(f\right),\end{aligned}$$

where $\delta\left(f\right)=\frac{1}{8}\alpha\left\langle\left(\left|A\otimes B\right|-\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}$. □

For $h(t)=t^{2}$ in Theorem 2.6, we get $\alpha\leq 2$ and we have the following remark, which is a refinement of the inequality (1.5).

Remark 2.5. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Then*

$$\omega^{2}\left(A\otimes B\right)\leq\frac{1}{2}\left\|\left|A\otimes B\right|^{2}+\left|A^{*}\otimes B^{*}\right|^{2}\right\|-\inf_{\|f\|=1}\delta\left(f\right),$$

where $\delta\left(f\right)=\frac{1}{4}\left\langle\left(\left|A\otimes B\right|-\left|A^{*}\otimes B^{*}\right|\right)f,f\right\rangle^{2}$.

Theorem 2.7. *Let $A\otimes B\in\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ and let $0\leq\alpha\leq 1$. Then*

$$\begin{aligned}\omega^{2}\left(A\otimes B\right)\leq&\frac{\alpha}{2}\left\|\left|A^{*}\right|^{2}\otimes\mathcal{I}_{\mathcal{K}}+\mathcal{I}_{\mathcal{H}}\otimes\left|B\right|^{2}\right\|\omega\left(A\otimes B\right)\\ &+\frac{1-\alpha}{2}\left\|\left|A^{*}\right|^{2}\otimes\mathcal{I}_{\mathcal{K}}+\mathcal{I}_{\mathcal{H}}\otimes\left|B\right|^{2}\right\|^{2}.\end{aligned}$$

Proof. Let $f\in\mathcal{H}\otimes\mathcal{K}$ with $\|f\|=1$ and let $0\leq\alpha\leq 1$. Then

$$\begin{aligned}&\left|\left\langle\left(A\otimes B\right)f,f\right\rangle\right|^{2}\\ &=\left|\left\langle\left(A\otimes\mathcal{I}_{\mathcal{K}}\right)\left(\mathcal{I}_{\mathcal{H}}\otimes B\right)f,f\right\rangle\right|^{2}\\ &=\left|\left\langle\left(\mathcal{I}_{\mathcal{H}}\otimes B\right)f,\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\rangle\right|^{2}\\ &=\alpha\left|\left\langle\left(\mathcal{I}_{\mathcal{H}}\otimes B\right)f,\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\rangle\right|^{2}\\ &+\left(1-\alpha\right)\left|\left\langle\left(\mathcal{I}_{\mathcal{H}}\otimes B\right)f,\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\rangle\right|^{2}\\ &=\alpha\left\|\left(\mathcal{I}_{\mathcal{H}}\otimes\left|B\right|^{2}\right)f\right\|\left\|\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\|\left|\left\langle\left(\mathcal{I}_{\mathcal{H}}\otimes B\right)f,\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\rangle\right|\\ &+\left(1-\alpha\right)\left\|\left(\mathcal{I}_{\mathcal{H}}\otimes\left|B\right|^{2}\right)f\right\|^{2}\left\|\left(A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right)f\right\|^{2}\\ &=\alpha\left\langle\left|\mathcal{I}_{\mathcal{H}}\otimes B\right|^{2}f,f\right\rangle^{\frac{1}{2}}\left\langle\left|A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right|^{2}f,f\right\rangle^{\frac{1}{2}}\left|\left\langle\left(A\otimes B\right)f,f\right\rangle\right|\\ &+\left(1-\alpha\right)\left\langle\left|\mathcal{I}_{\mathcal{H}}\otimes B\right|^{2}f,f\right\rangle\left\langle\left|A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right|^{2}f,f\right\rangle\\ &\leq\frac{\alpha}{2}\left\langle\left(\left|\mathcal{I}_{\mathcal{H}}\otimes B\right|^{2}+\left|A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right|^{2}\right)f,f\right\rangle\left|\left\langle\left(A\otimes B\right)f,f\right\rangle\right|\\ &+\frac{1-\alpha}{2}\left\langle\left(\left|\mathcal{I}_{\mathcal{H}}\otimes B\right|^{2}+\left|A^{*}\otimes\mathcal{I}_{\mathcal{K}}\right|^{2}\right)f,f\right\rangle^{2}\end{aligned}$$

(by the arithmetic-geometric mean inequality)

$$
\begin{aligned}
&\leq \frac{\alpha}{2}\left\langle\left(\left|\mathcal{I}_{\mathcal{H}} \otimes B\right|^{2}+\left|A^{*} \otimes \mathcal{I}_{\mathcal{K}}\right|^{2}\right) f, f\right\rangle|\langle(A \otimes B) f, f\rangle| \\
&+\frac{1-\alpha}{2}\left\langle\left|\mathcal{I}_{\mathcal{H}} \otimes B\right|^{2}+\left|A^{*} \otimes \mathcal{I}_{\mathcal{K}}\right|^{2} f, f\right\rangle \\
&\leq \frac{\alpha}{2}\left\|\left|A^{*}\right|^{2} \otimes \mathcal{I}_{\mathcal{K}}+\mathcal{I}_{\mathcal{H}} \otimes|B|^{2}\right\| \omega(A \otimes B) \\
&+\frac{1-\alpha}{2}\left\|\left|A^{*}\right|^{2} \otimes \mathcal{I}_{\mathcal{K}}+\mathcal{I}_{\mathcal{H}} \otimes|B|^{2}\right\|^{2} .
\end{aligned}
$$

Taking the supremum over all $f \in \mathcal{H} \otimes \mathcal{K}$ with $\|f\|=1$, we get the desired inequality. □

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