İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

İSTATİSTİK

Vol. 11, No. 1-2, January-July 2018, pp. 12–28
ISSN 1300-4077 | 18 | 1-2 | 12 | 28

# BAYESIAN PREDICTION OF PROGRESSIVELY FIRST-FAILURE-CENSORED ORDER STATISTICS BASED ON $k$-RECORD VALUES FROM WEIBULL DISTRIBUTION

Mohammad Vali Ahmadi *
Department of Statistics,
University of Bojnord,
94531-1339, Bojnord, Iran

Mahdi Doostparast
Department of Statistics,
Ferdowsi University of Mashhad,
91775-1159, Mashhad, Iran

***Abstract:*** Prediction on the basis of censored data has an important role in lifetime studies. This paper discusses Bayesian two-sample prediction of progressively first-failure-censored order statistics coming from a future sample based on observed $k$-record values from two-parameter Weibull distribution. Bayesian interval predictions are obtained based on a continuous-discrete joint prior for the unknown two parameters. Moreover, Bayesian point predictors are investigated under symmetric and asymmetric loss functions. Finally, the estimated risks of various point predictors obtained are compared using the Monte Carlo method.

*Key words*: Bayesian prediction; $k$-record; Progressive first-failure censoring; Weibull distribution
*History*: Submitted: 22 April 2017; Revised: 18 December 2017; Accepted: 7 January 2018

## 1. Introduction

Prediction of the future samples based on current sample information (called informative sample) is very interesting topic in Statistics. Here, the informative sample and the future sample are supposed stochastically independent. The aim of this paper is to discuss the predicting progressively first-failure-censored order statistics arising from a future sample based on observed $k$-records. First, we present a brief description to the progressively first-failure-censored samples and the $k$-records.

Censoring is usual in lifetime data due to time and cost restrictions. There are various types of censoring in survival analysis and progressive censoring is one of the most common for consideration. This censoring allows the experimenter to remove units from a life test at various stages during the experiment. For a comprehensive review of theory, methods and applications of the progressive censoring, we refer the reader to Balakrishnan and Aggarwala [7], Balakrishnan [6] and the references contained therein. Also, recently a book due to Balakrishnan and Cramer [8] offers a thorough and updated guide to the theory and methods of progressive censoring along with its practical applications to reliability and survival analyses. Progressive first-failure censoring, introduced by Wu and Kuş [30], is a type of progressive censoring in which $n$ disjoint groups with $s$ identical units within each group ($N = n \times s$) are placed on a life test at time zero. Suppose the random variables $X_1, \ldots, X_N$ denote their corresponding lifetimes. The life test is terminated at the time of $m$-th failure. When the $i$-th unit fails ($i = 1, 2, \ldots, m-1$), randomly selected $R_i$ groups

* Corresponding author. E-mail address: mv.ahmadi@ub.ac.ir

Table 1. The $k$-record times and $k$-record values for the insulating fluid data set when $k = 2$.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $T_{r(k)}$ | 2 | 5 | 6 | 9 | 10 | 11 | 18 | 19 |
| $U_{r(k)}$ | 0.96 | 4.15 | 8.01 | 8.27 | 31.75 | 32.52 | 33.91 | 36.71 |

Table 2. The $k$-record times and $k$-record values for the insulating fluid data set when $k = 3$.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $T_{r(k)}$ | 3 | 4 | 5 | 6 | 7 | 9 | 10 | 11 | 18 | 19 |
| $U_{r(k)}$ | 0.19 | 0.78 | 0.96 | 4.15 | 7.35 | 8.01 | 8.27 | 31.75 | 32.52 | 33.91 |

and the group in which the $i$-th failure is observed are removed from the test. When the $m$-th failure occurs, all of the remaining groups are removed from the test. The $m$ observed failure times denoted by $X^{\mathbf{R}}_{1:m:n:s} < \cdots < X^{\mathbf{R}}_{m:m:n:s}$ are the progressively first-failure-censored order statistics with pre-determined censoring scheme $\mathbf{R} = (R_1, \ldots, R_m)$. Note that: (1) for $s = 1$, the progressive first-failure-censoring scheme is reduced to the case of progressive Type-II censoring, (2) if $R_i = 0$ for $i = 1, 2, \ldots, m$, we have the first-failure censoring, (3) if $s = 1$, $R_i = 0$ for $i = 1, 2, \ldots, m-1$ and hence $R_m = n - m$, this scheme is reduced to the Type-II censoring and (4) if $s = 1$ and $R_i = 0$ for $i = 1, 2, \ldots, m$, this scheme is simplified to the complete sample. According to Wu and Kuş [30], "Although more units are used (only $m$ of $N$ units fail) in the progressive first-failure-censoring plan than in others, it has advantages in terms of reducing test cost and test time".

Let $\{Y_i, i \geq 0\}$ be a sequence of continuous random variables. Then, $Y_j$ will be called an upper record value (or simply record value) if $Y_j > Y_i$ for every $i < j$. A similar definition can be proposed for lower record values. Analogously, an upper $k$-record value (or simply $k$-record value) is defined in terms of the $k$-th largest $Y$ yet seen. Precisely, let $T_{1(k)} = k$, $U_{1(k)} = Y_{1:k}$ and for $r \geq 2$,

$$T_{r(k)} = \min\{j : j > T_{r-1(k)}, Y_j > Y_{T_{r-1(k)}-k+1:T_{r-1(k)}}\},$$

where $Y_{i:n}$ denotes the $i$-th order statistic in a random sample of size $n$. In the literature, $\{T_{r(k)}, r \geq 1\}$ is said to be the $k$-record times sequence. Thus, the sequence of $k$-record values is defined by $U_{r(k)} = Y_{T_{r(k)}-k+1:T_{r(k)}}$ for $r \geq 1$. For the special case $k = 1$, the usual record values are obtained. Dziubdziela and Kopocinski [13] showed that $\{U_{r(k)}, r \geq 1\}$ arising from a sequence of independent and identically distributed (IID) random variables with the common distribution function (DF) $F$ is distributed as the usual record values (i.e. $k = 1$) coming from a sequence of IID random variables with the common DF $1 - (1 - F)^k$. Various applications of $k$-record values can be found in the literature; See, e. g. Arnold *et al.* [5], Kamps [16, 17] and Nevzorov [22]. For example, the $r$-th $k$-record value can be regarded as the life length of a $k$-out-of-$T_{r(k)}$ system. As an illustration, consider the following data set given by Lawless [19, p. 185]:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.96 | 4.15 | 0.19 | 0.78 | 8.01 | 31.75 | 7.35 | 6.50 | 8.27 | 33.91 | 32.52 | 3.16 | 4.85 | 2.78 |
| 4.67 | 1.31 | 12.06 | 36.71 | 72.89 | | | | | | | | | . |

These data contains times to breakdown of an insulating fluid between electrodes recorded at 34 kilovolts. For $k = 2$ and $k = 3$, the $k$-record times and $k$-record values are reported in Tables 1 and 2.

Statistical predictions via a Bayesian approach have been considered in the literature; See, for example, Madi and Raqab [20], Ali Mousa and Jaheen [3], Ali Mousa and Al-Sagheer [4], Raqab *et al.* [23], Kundu and Howlader [18], Ahmadi *et al.* [1, 2], Ghafoori *et al.* [14], Huang and Wu [15], Balakrishnan and Shafay [9], Saeidi *et al.* [24] and Doostparast [11]. Also, Soliman *et al.*

[27] considered point and interval Bayesian predictions on the basis of general progressively Type-II censored data coming from Weibull model under symmetric and asymmetric loss functions. Recently, Shafay *et al.* [25] discussed the problem of predicting the future sequential order statistics based on observed multiply Type-II censored samples of sequential order statistics from one- and two-parameter exponential distributions.

In this paper, we suppose that the parent population follows the Weibull distribution with DF

$$F(x;\alpha,\beta)=1-\exp\left\{-\alpha x^{\beta}\right\},\ \ x>0,\ \alpha>0,\ \beta>0. \tag{1.1}$$

Hence, the density function of the Weibull distribution is

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}\exp\left\{-\alpha x^{\beta}\right\},\ \ x>0,\ \alpha>0,\ \beta>0. \tag{1.2}$$

The Weibull distribution is one of the most popular distributions that is extensively used for modelling the failure data. For more details on applications of the Weibull distribution, see, for example, Murthy *et al.* [21] and references therein.

The rest of this paper is organized as follows. In Sections 2 and 3, Bayesian interval and point predictors for progressively first-failure-censored order statistics arising from a future sample based on observed $k$-record values are derived. To do this, we consider two cases for the shape parameter $\beta$ in the DF (1.1). The performance analysis of the obtained predictors is carried out by conducting a simulation study in Section 4. Finally, we use a real data set from Dunsmore [12] for illustrating the inferential methods developed here.

## 2. Bayesian prediction interval

Let $U_{1(k)}<\cdots<U_{r(k)}$ be the first $r$ $k$-record values from a continuous population with density function and DF $f(\cdot|\boldsymbol{\theta})$ and $F(\cdot|\boldsymbol{\theta})$, respectively, where $\boldsymbol{\theta}$ is a vector of parameters. Following Arnold *et al.* [5], the associated likelihood function (LF) of the observed data $\mathbf{u}=(u_1,\ldots,u_r)$ is reduced to

$$L(\boldsymbol{\theta};\mathbf{u})=k^{r}\left(1-F(u_r|\boldsymbol{\theta})\right)^{k}\prod_{j=1}^{r}\frac{f(u_j|\boldsymbol{\theta})}{1-F(u_j|\boldsymbol{\theta})}.$$

Under the Weibull distribution, the LF becomes

$$L(\alpha,\beta;\mathbf{u})=(k\alpha\beta)^{r}\prod_{j=1}^{r}u_j^{\beta-1}\exp\left\{-k\alpha u_r^{\beta}\right\},\ \ 0<u_1<\cdots<u_r<\infty. \tag{2.1}$$

On the other hand, let $X^{\mathbf{R}}_{1:m:n:s},\ldots,X^{\mathbf{R}}_{m:m:n:s}$ be a progressively first-failure-censored sample from a continuous population with density function and DF $f(\cdot|\boldsymbol{\theta})$ and $F(\cdot|\boldsymbol{\theta})$, respectively. It can be seen that $X^{\mathbf{R}}_{1:m:n:s},\ldots,X^{\mathbf{R}}_{m:m:n:s}$ can be viewed as a progressively type II censored sample from a continuous population with density function and DF $sf(\cdot|\boldsymbol{\theta})\left(1-F(\cdot|\boldsymbol{\theta})\right)^{s-1}$ and $1-\left(1-F(\cdot|\boldsymbol{\theta})\right)^{s}$, respectively. Then, the marginal density function of $X^{\mathbf{R}}_{i:m:n:s}$, $(1\le i\le m)$ is given by (see, for example, Balakrishnan and Aggarwala [7])

$$f_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\boldsymbol{\theta})=c_{i-1}sf(x_i|\boldsymbol{\theta})\left(1-F(x_i|\boldsymbol{\theta})\right)^{s-1}\sum_{j=1}^{i}a_j\left(1-F(x_i|\boldsymbol{\theta})\right)^{s(\gamma_j-1)}, \tag{2.2}$$

where

$$\gamma_j=\sum_{l=j}^{m}(R_l+1)=n-\sum_{l=1}^{j-1}(R_l+1),\quad c_{i-1}=\prod_{j=1}^{i}\gamma_j,$$
$$a_j=\prod_{l=1,\,l\neq j}^{i}\frac{1}{\gamma_l-\gamma_j},\ \ 1<i\le m\ \ \text{and}\ \ a_1=1.$$

Upon substituting Equations (1.2) and (1.1) into Equation (2.2), the marginal density function of $X^{\mathbf{R}}_{i:m:n:s}$ is obtained as

$$f_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\alpha,\beta) = c_{i-1}s\alpha\beta x_i^{\beta-1}\sum_{j=1}^{i} a_j \exp\left\{-s\alpha\gamma_j x_i^{\beta}\right\}, \quad x_i > 0. \tag{2.3}$$

In the following subsections, two cases for the parameter $\beta$ are considered.

## 2.1. $\beta$ is known

Assume a gamma conjugate prior for $\alpha$ as

$$\pi(\alpha) \propto \alpha^{a-1}\exp\{-b\alpha\}, \quad \alpha > 0, \tag{2.4}$$

where $a$ and $b$ are positive hyperparmeters that are chosen, for example, from a prior knowledge of the mean and variance of $\alpha$. From (2.1) and (2.4), the posterior density function of $\alpha$, given $\mathbf{u}$, is obtained as

$$\pi(\alpha|\mathbf{u}) \propto \alpha^{r+a-1}\exp\{-\alpha g_{\beta,b}(u_r)\}, \quad \alpha > 0, \tag{2.5}$$

where $g_{\beta,b}(u_r) = b + ku_r^{\beta}$. It is obvious that $(\alpha|\mathbf{u}) \sim \Gamma(r+a, g_{\beta,b}(u_r))$. By combining the posterior density function in (2.5) with the conditional density function in (2.3), and then integrating out the parameter $\alpha$, the Bayes predictive density function of $X^{\mathbf{R}}_{i:m:n:s}$, given $\mathbf{u}$, is

$$\begin{aligned} f^{\star}_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u}) &= \int_0^{\infty} f_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\alpha)\pi(\alpha|\mathbf{u})\mathrm{d}\alpha \\ &= c_{i-1}s\beta(r+a)(g_{\beta,b}(u_r))^{r+a}x_i^{\beta-1}\sum_{j=1}^{i} a_j\left(g_{\beta,b}(u_r)+s\gamma_j x_i^{\beta}\right)^{-(r+a+1)}. \end{aligned} \tag{2.6}$$

From (2.6), the predictive survival function of $X^{\mathbf{R}}_{i:m:n:s}$ given $\mathbf{u}$ becomes

$$\begin{aligned} P(X^{\mathbf{R}}_{i:m:n:s} > t|\mathbf{u}) &= \int_t^{\infty} f^{\star}_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u})\mathrm{d}x_i \\ &= c_{i-1}\sum_{j=1}^{i}\frac{a_j}{\gamma_j}\left(\frac{g_{\beta,b}(u_r)}{g_{\beta,b}(u_r)+s\gamma_j t^{\beta}}\right)^{r+a}. \end{aligned} \tag{2.7}$$

Suppose that $\psi_i^{\star}(\tau)$ is the $\tau$-th upper quantile of the predictive distribution of $X^{\mathbf{R}}_{i:m:n:s}$, i.e. $\psi_i^{\star}(\tau)$ satisfies $P(X^{\mathbf{R}}_{i:m:n:s} > \psi_i^{\star}(\tau)|\mathbf{u}) = \tau$. Hence, it is clear that the values $\psi_i^{\star}(\tau)$ and $\psi_i^{\star}(1-\tau)$ are the (one-sided) upper and lower $100(1-\tau)\%$ Bayesian prediction bounds for $X^{\mathbf{R}}_{i:m:n:s}$. Also, the two-sided equi-tialed $100(1-\tau)\%$ Bayesian prediction interval for $X^{\mathbf{R}}_{i:m:n:s}$ is obtained as $(\psi_i^{\star}(1-\tau/2), \psi_i^{\star}(\tau/2))$. Although the quantiles of the predictive density function in Equation (2.7) are not expressible in closed form, we can calculate them numerically. In Section 4, we have used the mathematical package of *Mathematica* version 6 in order to obtain these quantiles.

Remark 1. A closed form for the Bayesian prediction interval of $X^{\mathbf{R}}_{1:m:n:s}$ is derived. The value of $X^{\mathbf{R}}_{1:m:n:s}$ will be the first failure time of the future sample of size $N = n \times s$. Putting $i = 1$ in Equations (2.6) and (2.7), we obtain

$$f^{\star}_{X^{\mathbf{R}}_{1:m:n:s}}(x_1|\mathbf{u}) = ns\beta(r+a)(g_{\beta,b}(u_r))^{r+a}x_1^{\beta-1}\left(g_{\beta,b}(u_r)+nsx_1^{\beta}\right)^{-(r+a+1)},$$

and

$$P(X^{\mathbf{R}}_{1:m:n:s} > t|\mathbf{u}) = \left(\frac{g_{\beta,b}(u_r)}{g_{\beta,b}(u_r)+nst^{\beta}}\right)^{r+a}. \tag{2.8}$$

From (2.8), the $\tau$-th upper quantile is obtained as

$$\psi_1^\star(\tau) = \left\{ \frac{g_{\beta,b}(u_r)}{ns} \left( \tau^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}}.$$

Thus, the two-sided equi-tialed $100(1-\tau)\%$ Bayes prediction interval for $X^{\mathbf{R}}_{1:m:n:s}$ is

$$\left( \left\{ \frac{g_{\beta,b}(U_{r(k)})}{ns} \left( \left(1 - \frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}}, \left\{ \frac{g_{\beta,b}(U_{r(k)})}{ns} \left( \left(\frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right) \right\}^{\frac{1}{\beta}} \right). \tag{2.9}$$

Since, the marginal density function of $U_{r(k)}$ under the Weibull distribution is obtained as (see, for example, Arnold *et al.* [5])

$$f_{U_{r(k)}}(u) = \frac{(k\alpha)^r \beta}{\Gamma(r)} u^{r\beta-1} \exp\{-k\alpha u^\beta\}, \ u > 0,$$

where $\Gamma(\cdot)$ denotes the complete gamma function, one can readily derive that $U^\beta_{r(k)} \sim \Gamma(r, k\alpha)$. Hence, the expected width $(W)$ of Bayesian prediction interval in Equation (2.9) becomes

$$\begin{aligned}
E(W) &= \xi E\left( \frac{g_{\beta,b}(U_{r(k)})}{ns} \right)^{\frac{1}{\beta}} \\
&= \frac{\xi (k\alpha)^r \beta}{(ns)^{\frac{1}{\beta}} \Gamma(r)} \int_0^\infty \left(b + k u^\beta\right)^{\frac{1}{\beta}} u^{r\beta-1} \exp(-k\alpha u^\beta) \mathrm{d}u \\
&= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \int_{\alpha b}^\infty y^{\frac{1}{\beta}} (y - \alpha b)^{r-1} \exp(-y) \mathrm{d}y \\
&= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \sum_{j=1}^{r-1} \binom{r-1}{j} (-\alpha b)^{r-1-j} \int_{\alpha b}^\infty y^{\frac{1}{\beta}+j} \exp(-y) \mathrm{d}y \\
&= \frac{\xi \exp(\alpha b)}{(ns\alpha)^{\frac{1}{\beta}} \Gamma(r)} \sum_{j=1}^{r-1} \binom{r-1}{j} (-\alpha b)^{r-1-j} \Gamma\left( \frac{1}{\beta} + j + 1, \alpha b \right),
\end{aligned} \tag{2.10}$$

where

$$\xi = \left( \left(\frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right)^{\frac{1}{\beta}} - \left( \left(1 - \frac{\tau}{2}\right)^{-\frac{1}{r+a}} - 1 \right)^{\frac{1}{\beta}},$$

and $\Gamma(\cdot,\cdot)$ denotes the incomplete gamma function. In the special case when $b = 0$, the expression in (2.10) simply becomes $E(W) = \xi\Gamma\left(1/\beta + r\right) / \left((ns\alpha)^{1/\beta}\Gamma(r)\right)$. From Equation (2.10), it is observed that the expected width of Bayesian prediction interval for $X^{\mathbf{R}}_{1:m:n:s}$ is decreasing with respect to $n$ and $s$ when all others are fixed.

It is easy to show that $W = 1 + ns(X^{\mathbf{R}}_{1:m:n:s})^\beta / g_{\beta,b}(u_r)$ has the Pareto distribution with the density function $f_W(w) = (r+a)w^{-(r+a+1)}$, $w > 1$. Thus, we can find Bayesian prediction intervals for $X^{\mathbf{R}}_{1:m:n:s}$ on the basis of pivotal quantity $W$. One can readily obtain, in this case, the same Bayesian prediction bounds as in Equation (2.9).

### 2.2. $\alpha$ and $\beta$ are unknown

In order to obtain Bayesian prediction interval for $X^{\mathbf{R}}_{i:m:n:s}$, specifying a general joint prior for $\alpha$ and $\beta$ leads us to computational complexities. For sake of brevity in the Bayesian analysis, we use Soland's method (see, Soland [26]). Therefore, it is assumed that the shape parameter $\beta$ is restricted to a finite number of values $\beta_1, \ldots, \beta_M$ with prior probabilities $\xi_1, \ldots \xi_M$, respectively, such that

$0 < \xi_l < 1$ for $l = 1, \ldots, M$ and $\sum_{l=1}^{M} \xi_l = 1$. Moreover, suppose that the conditional distribution of $\alpha$, given $\beta = \beta_l$, is gamma with shape and scale parameters $a_l$ and $b_l$, respectively. Therefore,

$$P(\beta = \beta_l) = \xi_l, \quad l = 1, \ldots, M,$$
$$\pi(\alpha|\beta = \beta_l) = \frac{b_l^{a_l}}{\Gamma(a_l)} \alpha^{a_l - 1} \exp(-b_l \alpha), \quad \alpha > 0,\, a_l > 0,\, b_l > 0.$$

The hyperparameters $a_l$ and $b_l$ $(l = 1, \ldots, M)$ are chosen so as to reflect prior beliefs on $\alpha$ given that $\beta = \beta_l$. The conditional prior density function of $\alpha$ given $\beta = \beta_l$ for $l = 1, \ldots, M$ is completely specified when the hyperparameters $a_l$ and $b_l$ are known. To simplify the problem, we assume that the shape and scale parameters of the conditional distribution $\alpha$ given $\beta = \beta_l$ are $a$ and $b\beta_l$, respectively, where $a$ and $b$ are two positive specified constants. Thus, using the LF (2.1), the conditional posterior density function of $\alpha$ given $\beta = \beta_l$ is

$$\begin{aligned} \pi(\alpha|\beta = \beta_l, \mathbf{u}) &= \frac{L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)}{\int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)\mathrm{d}\alpha} \\ &\propto \alpha^{r+a-1} \exp(-\alpha q_{\beta_l, b}(u_r)), \end{aligned} \tag{2.11}$$

where $q_{\beta_l,b}(u_r) = b\beta_l + k u_r^{\beta_l}$. Consequently, $(\alpha|\beta = \beta_l, \mathbf{u}) \sim \Gamma(r + a, q_{\beta_l,b}(u_r))$ for $l = 1, \ldots, M$. Also, the posterior density function of $\beta = \beta_l$ is obtained as

$$\begin{aligned} P(\beta = \beta_l|\mathbf{u}) &= \frac{\int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)\mathrm{d}\alpha}{\sum_{l=1}^{M} \int_0^\infty L(\alpha, \beta_l; \mathbf{u})\pi(\alpha|\beta = \beta_l)P(\beta = \beta_l)\mathrm{d}\alpha} \\ &= c^\star \xi_l e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l,b}(u_r)} \right)^{r+a}, \end{aligned} \tag{2.12}$$

where $e_{\beta_l}(\mathbf{u}) = \prod_{j=1}^{r} u_j^{\beta_l - 1}$ and $c^\star = \left[ \sum_{l=1}^{M} \xi_l e_{\beta_l}(\mathbf{u}) \left\{ \beta_l / q_{\beta_l,b}(u_r) \right\}^{r+a} \right]^{-1}$ is the normalizing constant.

The Bayes predictive density function of $X^{\mathbf{R}}_{i:m:n:s}$, given $\mathbf{u}$, is

$$f^{\star\star}_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u}) = \sum_{l=1}^{M} \int_0^\infty f_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\alpha, \beta_l)\pi(\alpha|\beta = \beta_l, \mathbf{u})P(\beta = \beta_l|\mathbf{u})\mathrm{d}\alpha. \tag{2.13}$$

Substituting Equations (2.3), (2.11) and (2.12) into Equation (2.13), the Bayes predictive density function of $X^{\mathbf{R}}_{i:m:n:s}$ reads

$$f^{\star\star}_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u}) = c^\star c_{i-1} s(r+a) \sum_{l=1}^{M} \sum_{j=1}^{i} \xi_l a_j e_{\beta_l}(\mathbf{u}) x_i^{\beta_l - 1} \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + s\gamma_j x_i^{\beta_l}} \right)^{r+a+1}, \tag{2.14}$$

and hence, the predictive survival function of $X^{\mathbf{R}}_{i:m:n:s}$ becomes

$$\begin{aligned} P\left(X^{\mathbf{R}}_{i:m:n:s} \geq t|\mathbf{u}\right) &= \int_t^\infty f^{\star\star}_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u})\mathrm{d}x_i \\ &= c^\star c_{i-1} \sum_{l=1}^{M} \sum_{j=1}^{i} \xi_l a_j \gamma_j^{-1} e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + s\gamma_j t^{\beta_l}} \right)^{r+a}. \end{aligned} \tag{2.15}$$

If $\psi_i^{\star\star}(\tau)$ satisfies $P(X^{\mathbf{R}}_{i:m:n:s} > \psi_i^{\star\star}(\tau)|\mathbf{u}) = \tau$, hence, in this case, the values $\psi_i^{\star\star}(\tau)$ and $\psi_i^{\star\star}(1 - \tau)$ are the (one-sided) upper and lower $100(1 - \tau)\%$ Bayesian prediction bounds for $X^{\mathbf{R}}_{i:m:n:s}$.

Also, the two-sided equi-tialed $100(1-\tau)\%$ Bayesian prediction interval for $X^{\mathbf{R}}_{i:m:n:s}$ is given by $(\psi^{\star\star}_i(1-\tau/2), \psi^{\star\star}_i(\tau/2))$. Again, the quantiles of the predictive density function in Equation (2.15) are not expressible in closed form and then we calculate them numerically using the mathematical package of *Mathematica* version 6 in Section 4.

It is important to determine the values for the hyperparameters $(\beta_l, \xi_l)$, $l = 1, \ldots, M$ and $(a, b)$. In fact, considering the values for $(\beta_l, \xi_l)$, $l = 1, \ldots, M$ is fairly straightforward, but for $(a, b)$, the following method is suggested. Suppose that on the basis of a prior knowledge, we known the conditional expectation and variance of $\alpha$ given a special value of $\beta$ (say, $\beta = \beta_t$). Consequently, we have

$$\mu_t = E(\alpha|\beta = \beta_t) = \frac{a}{b\beta_t} \quad \text{and} \quad \sigma_t^2 = Var(\alpha|\beta = \beta_t) = \frac{a}{(b\beta_t)^2}, \tag{2.16}$$

where $\mu_t$ and $\sigma_t^2$ are the expectation and variance of $\alpha$ given $\beta = \beta_t$, respectively. By solving Equation (2.16), the hyperparameters $(a, b)$ are obtained.

Remark 2. When $i = 1$, the expressions in Equations (2.14) and (2.15) are reduced to

$$f^{\star\star}_{X^{\mathbf{R}}_{1:m:n:s}}(x_1|\mathbf{u}) = c^{\star} ns(r+a) \sum_{l=1}^{M} \xi_l e_{\beta_l}(\mathbf{u}) x_1^{\beta_l - 1} \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + nsx_1^{\beta_l}} \right)^{-(r+a+1)},$$

and

$$P\left(X^{\mathbf{R}}_{1:m:n:s} \geq t|\mathbf{u}\right) = c^{\star} \sum_{l=1}^{M} \xi_l e_{\beta_l}(\mathbf{u}) \left( \frac{\beta_l}{q_{\beta_l,b}(u_r) + nst^{\beta_l}} \right)^{r+a}, \tag{2.17}$$

respectively. Thus, the $100(1-\tau)\%$ Bayesian prediction interval $(\psi^{\star\star}_1(1-\tau/2), \psi^{\star\star}_1(\tau/2))$ for $X^{\mathbf{R}}_{1:m:n:s}$ can be easily obtained numerically using Equation (2.17).

## 3. Bayes point prediction

In the Bayesian framework, the choice of a loss function is essential. One of the most popular widely used loss functions is the squared error (SE) loss defined by $L(\hat{\delta}, \delta) = (\hat{\delta} - \delta)^2$, where $\hat{\delta}$ is an estimate of $\delta$. But, the SE loss function is justified only when losses are symmetric in nature. The symmetric nature of this loss function gives equal weight to overestimation as well as underestimation, while in practice, overestimation may be more serious than underestimation of same magnitude or vice versa. Such conditions are very common in engineering, medical and biomedical sciences. In this case, an asymmetric loss function might be more appropriate. A suitable alternative to the SE loss function is a convex but asymmetric loss function, known as the LINear-EXponential (LINEX) loss function, proposed by Varian [29] and defined by

$$L(\hat{\delta}, \delta) = \exp\left\{\vartheta(\hat{\delta} - \delta)\right\} - \vartheta(\hat{\delta} - \delta) - 1, \quad \vartheta \neq 0. \tag{3.1}$$

The constants $\vartheta$ and $\sigma > 0$ involved in (3.1) are the shape and scale parameters. Obviously, the nature of the LINEX loss function changes according to the choice of $\vartheta$. The sign and magnitude of the shape parameter $\vartheta$ represents the direction and degree of symmetry, respectively. ($\vartheta > 0$ means overestimation is more serious than underestimation and $\vartheta < 0$ means the opposite). The LINEX loss converges to the SE loss as $\vartheta \to 0$. It is proved that the Bayes estimates of $\delta$ under the SE and the LINEX loss functions, respectively denoted by $\hat{\delta}_{BS}$ and $\hat{\delta}_{BL}$, are given by

$$\hat{\delta}_{BS} = E(\delta), \tag{3.2}$$

and

$$\hat{\delta}_{BL} = -\frac{1}{\vartheta} \ln E\big(\exp(-\vartheta\delta)\big), \tag{3.3}$$

provided that $E(\delta)$ and $E\big(\exp(-\vartheta\delta)\big)$ exist and are finite.

Now, we obtain the Bayes point predictor for $X^{\mathbf{R}}_{i:m:n:s}$ on the basis of observed $k$-record data. For this purpose, we consider the following two cases.

### 3.1. $\beta$ is known

Using Equations (2.6), (3.2) and (3.3), the Bayes point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ under SE and LINEX loss functions are given, respectively, by

$$\begin{aligned}\hat{X}^{\mathbf{R}}_{i:m:n:s,BS} &= \int_0^\infty x_i f^\star_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u})\mathrm{d}x_i \\ &= c_{i-1}(r+a)\sum_{j=1}^{i}\frac{a_j}{\gamma_j}\left(\frac{s\gamma_j}{g_{\beta,b}(u_r)}\right)^{-\frac{1}{\beta}}\int_0^1 y^{r+a-\frac{1}{\beta}-1}(1-y)^{\frac{1}{\beta}}\mathrm{d}y \\ &= c_{i-1}(r+a)B\left(r+a-\frac{1}{\beta},1+\frac{1}{\beta}\right)\sum_{j=1}^{i}\frac{a_j}{\gamma_j}\left(\frac{s\gamma_j}{g_{\beta,b}(u_r)}\right)^{-\frac{1}{\beta}}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\hat{X}^{\mathbf{R}}_{i:m:n:s,BL} &= -\frac{1}{\vartheta}\ln\left\{\int_0^\infty \exp(-\vartheta x_i) f^\star_{X^{\mathbf{R}}_{i:m:n:s}}(x_i|\mathbf{u})\mathrm{d}x_i\right\} \\ &= -\frac{1}{\vartheta}\ln\left\{c_{i-1}s\beta(r+a)\big(g_{\beta,b}(u_r)\big)^{r+a}\sum_{j=1}^{i}a_j\int_0^\infty \frac{x_i^{\beta-1}\exp(-\vartheta x_i)}{\big(g_{\beta,b}(u_r)+s\gamma_j x_i^\beta\big)^{r+a+1}}\mathrm{d}x_i\right\} \\ &= -\frac{1}{\vartheta}\ln\left\{c_{i-1}s\beta(r+a)\sum_{j=1}^{i}\sum_{l=0}^{\infty}\binom{-r-a-1}{l}\frac{a_j(s\gamma_j)^l\Gamma\big((l+1)\beta\big)}{\big(g_{\beta,b}(u_r)\big)^{l+1}\vartheta^{(l+1)\beta}}\right\}, \end{aligned} \tag{3.5}$$

where $B(\cdot,\cdot)$ denotes the complete beta function.

REMARK 3. In the special case $i=1$, the expressions in Equations (3.4) and (3.5) simply become

$$\hat{X}^{\mathbf{R}}_{1:m:n:s,BS} = (r+a)B\left(r+a-\frac{1}{\beta},1+\frac{1}{\beta}\right)\left(\frac{ns}{g_{\beta,b}(u_r)}\right)^{-\frac{1}{\beta}},$$

and

$$\hat{X}^{\mathbf{R}}_{1:m:n:s,BL} = -\frac{1}{\vartheta}\ln\left\{ns\beta(r+a)\sum_{l=0}^{\infty}\binom{-r-a-1}{l}\frac{(ns)^l\Gamma\big((l+1)\beta\big)}{\big(g_{\beta,b}(u_r)\big)^{l+1}\vartheta^{(l+1)\beta}}\right\},$$

respectively.

### 3.2. $\alpha$ and $\beta$ are unknown

From Equations (2.14), (3.2) and (3.3), it is easy to show that the Bayes point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ under SE and LINEX loss functions are, respectively,

$$\begin{aligned}\hat{X}^{\mathbf{R}}_{i:m:n:s,BS} =& c^\star c_{i-1}s(r+a) \\ &\times\sum_{l=1}^{M}\sum_{j=1}^{i}\xi_l\beta_l^{r+a}a_je_{\beta_l}(\mathbf{u})B\left(r+a-\frac{1}{\beta_l},1+\frac{1}{\beta_l}\right)\frac{(s\gamma_j)^{-\frac{1}{\beta_l}-1}}{\big(g_{\beta_l,b}(u_r)\big)^{r+a-\frac{1}{\beta_l}}}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\hat{X}^{\mathbf{R}}_{i:m:n:s,BL} =& -\frac{1}{\vartheta}\ln\Bigg\{c^\star c_{i-1}s(r+a) \\ &\times\sum_{l=1}^{M}\sum_{j=1}^{i}\sum_{h=0}^{\infty}\xi_l\beta_l^{r+a+1}a_je_{\beta_l}(\mathbf{u})\binom{-r-a-1}{h}\frac{(s\gamma_j)^h\Gamma\big((h+1)\beta_l\big)}{\big(g_{\beta_l,b}(u_r)\big)^{r+a+h+1}\vartheta^{(h+1)\beta_l}}\Bigg\}. \end{aligned} \tag{3.7}$$

Remark 4. For the special case of $i=1$, the expressions in Equations (3.6) and (3.7) are reduced to

$$\hat{X}^{\mathbf{R}}_{1:m:n:s,BS} = c^{\star} ns(r+a)\sum_{l=1}^{M}\xi_l\beta_l^{r+a}e_{\beta_l}(\mathbf{u})B\left(r+a-\frac{1}{\beta_l},1+\frac{1}{\beta_l}\right)\frac{(ns)^{-\frac{1}{\beta_l}-1}}{\left(g_{\beta_l,b}(u_r)\right)^{r+a-\frac{1}{\beta_l}}},$$

and

$$\hat{X}^{\mathbf{R}}_{1:m:n:s,BL} = -\frac{1}{\vartheta}\ln\left\{c^{\star} ns(r+a)\right.$$
$$\left.\times\sum_{l=1}^{M}\sum_{h=0}^{\infty}\xi_l\beta_l^{r+a+1}e_{\beta_l}(\mathbf{u})\binom{-r-a-1}{h}\frac{(ns)^h\Gamma\left((h+1)\beta_l\right)}{\left(g_{\beta_l,b}(u_r)\right)^{r+a+h+1}\vartheta^{(h+1)\beta_l}}\right\},$$

respectively.

## 4. Numerical Results

In this section, the performances of the procedures proposed in the preceding sections are investigated by conducting a simulation study and analysing an illustrative example.

### 4.1. Simulation study

A simulation study was conducted in order to evaluate the performance of the Bayesian prediction intervals and Bayes point predictors for $i$-th order statistic $X^{\mathbf{R}}_{i:m:n:s}$ in a future progressively first-failure-censored sample under the assumption that $\alpha$ and $\beta$ are unknown. Since in predicting first order statistic $X^{\mathbf{R}}_{1:m:n:s}$, the corresponding Bayesian prediction intervals and point predictors depend on the total number of units put on the test $N=n\times s$ but not on $(R_1,\ldots,R_m)$ and $m$, hence, without loss of generality, we constructed the Bayesian prediction intervals and point predictors for $X^{\mathbf{R}}_{1:m:n:s}$ under the assumption that $m=N$ and $(R_1,\ldots,R_m)=(0,\ldots,0)$. Also, for predicting $X^{\mathbf{R}}_{5:m:n:s}$, we considered different choices for progressive censoring scheme $(R_1,\ldots,R_m)$ which include three progressive censoring schemes $(n-m,0_{(m-1)})$, $(0_{(m-1)},n-m)$ and $(0_{(3)},n-m,0_{(m-4)})$, among others. Note that, for example, the notation $(15,0_{(4)})$ stands for a progressive censoring scheme with specification $(R_1,\ldots,R_5)=(15,0,0,0,0)$.

Our simulation was done according to the following steps:

1. Assuming $\beta_t=1$ and $(\mu_t,\sigma_t^2)=(1,0.1)$ and using Equation (2.16), the hyperparameters $(a,b)$ were obtained as $(10,10)$.

2. Following the work of Soman and Misra [28], we assumed that the value of the shape parameter $\beta$ falls in the interval $(0,3)$. Then, we generated $\beta=2.5$ from discrete uniform distribution on $0.5(0.5)3$ and $\alpha=0.27152$ from $\Gamma(a,b\beta)$ prior distribution.

3. A sequence of independent observations from Weibull distribution with parameters $\alpha=0.27152$ and $\beta=2.5$ was generated and the first $r$ $k$-record values in the sequence for $r=5$ and $k=1,3$ were discovered.

4. Considering the observed $k$-record values and using Equations (2.15), (3.6) and (3.7), the Bayesian two-sided equi-tailed prediction intervals as well as the Bayes point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ $(i=1,5)$ under the SE and LINEX loss functions were obtained.

5. Simulating $t=10^4$ independent progressively first-failure-censored samples, the coverage probabilities of the Bayesian prediction intervals for $X^{\mathbf{R}}_{i:m:n:s}$ $(i=1,5)$ were computed. Also, the Estimated Risks (ER) of the Bayes point predictors $X^{\mathbf{R}}_{i:m:n:s}$ under the SE and LINEX loss functions given, respectively, by

$$\mathrm{ER}(\hat{\delta}_{BS})=\frac{1}{t}\sum_{j=1}^{t}\left(\hat{\delta}_{BS}-\delta_j\right)^2,$$

TABLE 3. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ with $i=1$ and $k=1$.

| $N$ | Coverage probability | | Estimated risk | | |
|---|---|---|---|---|---|
| | $\tau=0.1$ | $\tau=0.05$ | SE | LINEX | |
| | | | | $\vartheta=0.7$ | $\vartheta=1.2$ |
| 5 | 0.8956 | 0.9456 | 0.13348 | 0.03170 | 0.09100 |
| 10 | 0.8929 | 0.9514 | 0.07721 | 0.01857 | 0.05378 |
| 15 | 0.8942 | 0.9506 | 0.05550 | 0.01345 | 0.03917 |
| 20 | 0.9054 | 0.9547 | 0.04317 | 0.01050 | 0.03068 |
| 30 | 0.9038 | 0.9498 | 0.03160 | 0.00771 | 0.02257 |
| 50 | 0.8910 | 0.9534 | 0.02145 | 0.00525 | 0.01541 |

TABLE 4. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ with $i=1$ and $k=3$.

| $N$ | Coverage probability | | Estimated risk | | |
|---|---|---|---|---|---|
| | $\tau=0.1$ | $\tau=0.05$ | SE | LINEX | |
| | | | | $\vartheta=0.7$ | $\vartheta=1.2$ |
| 5 | 0.9101 | 0.9581 | 0.13041 | 0.03125 | 0.09020 |
| 10 | 0.9082 | 0.9547 | 0.07815 | 0.01893 | 0.05509 |
| 15 | 0.8924 | 0.9430 | 0.05943 | 0.01443 | 0.04206 |
| 20 | 0.9076 | 0.9608 | 0.04701 | 0.01145 | 0.03348 |
| 30 | 0.8929 | 0.9429 | 0.03396 | 0.00832 | 0.02441 |
| 50 | 0.9058 | 0.9582 | 0.02343 | 0.00576 | 0.01694 |

and

$$\mathrm{ER}(\hat{\delta}_{BL}) = \frac{1}{t}\sum_{j=1}^{t}\left(\exp\left\{\vartheta(\hat{\delta}_{BL}-\delta_j)\right\} - \vartheta(\hat{\delta}_{BL}-\delta_j) - 1\right),$$

were calculated.

The results are tabulated in Tables 3-6. From empirical evidence in Tables 3-6, the following points can be drawn:

- The simulated coverage probabilities of the Bayesian prediction intervals for $X^{\mathbf{R}}_{i:m:n:s}$ are close to the their nominal level.
- The estimated risk of predictors $\hat{X}^{\mathbf{R}}_{1:m:n:s,BS}$ and $\hat{X}^{\mathbf{R}}_{1:m:n:s,BL}$ is decreasing in $N$.
- For fixed $s$ and $n$, as $m$ increases, the estimated risk of predictors $\hat{X}^{\mathbf{R}}_{5:m:n:s,BS}$ and $\hat{X}^{\mathbf{R}}_{5:m:n:s,BL}$ reduce significantly under progressive censoring schemes $(n-m, 0_{(m-1)})$ and $(0_{(3)}, n-m, 0_{(m-4)})$.
- It seems that the estimated risk of $\hat{X}^{\mathbf{R}}_{5:m:n:s,BS}$ and $\hat{X}^{\mathbf{R}}_{5:m:n:s,BL}$ under progressive censoring scheme $(0_{(m-1)}, n-m)$ does not depend on $m$, when all other components are kept fixed.
- For fixed $s$ (or $n$), $m$ and progressive censoring scheme, when $n$ (or $s$) becomes large, the estimated risk of $\hat{X}^{\mathbf{R}}_{5:m:n:s,BS}$ and $\hat{X}^{\mathbf{R}}_{5:m:n:s,BL}$ decreases.
- When $s$, $n$ and $m$ are fixed, the progressive censoring scheme $(0_{(m-1)}, n-m)$ possesses the smallest estimated risk. Also, it seems that when $s$ and $n$ are fixed, the distance between the estimated risk of different progressive censoring schemes decreases with respect to $m$.
- Experimentation with different values of the prior parameters led to the same results, which shows a good stability with respect to the prior setting. The results are available upon readers request to the authors.

TABLE 5. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ with $i=5$ and $k=1$.

| $s$ | $n$ | $m$ | $(R_1,\ldots,R_m)$ | Coverage probability | | Estimated risk | | |
|---|---|---|---|---|---|---|---|---|
| | | | | $\tau=0.1$ | $\tau=0.05$ | SE | LINEX | |
| | | | | | | | $\vartheta=0.7$ | $\vartheta=1.2$ |
| 1 | 20 | 5 | $(15,0_{(4)})$ | 0.9066 | 0.9664 | 0.32170 | 0.07763 | 0.22375 |
| | | | $(0_{(4)},15)$ | 0.8750 | 0.9382 | 0.06308 | 0.01601 | 0.04828 |
| | | | $(0_{(3)},15,0)$ | 0.8952 | 0.9517 | 0.35890 | 0.08169 | 0.22792 |
| | | | $(3_{(5)})$ | 0.8732 | 0.9314 | 0.11942 | 0.02938 | 0.08675 |
| | | | $(0,5_{(3)},0)$ | 0.9037 | 0.9589 | 0.32485 | 0.07538 | 0.21285 |
| | | 15 | $(5,0_{(14)})$ | 0.8723 | 0.9326 | 0.07678 | 0.01937 | 0.05816 |
| | | | $(0_{(14)},5)$ | 0.8785 | 0.9426 | 0.06297 | 0.01600 | 0.04824 |
| | | | $(0_{(3)},5,0_{(11)})$ | 0.8815 | 0.9414 | 0.06828 | 0.01730 | 0.05206 |
| | | | $(1,0_{(2)},1,0_{(2)},\ldots,1,0_{(2)})$ | 0.8762 | 0.9405 | 0.06606 | 0.01676 | 0.05051 |
| | 50 | 15 | $(35,0_{(14)})$ | 0.8718 | 0.9351 | 0.07639 | 0.01926 | 0.05780 |
| | | | $(0_{(14)},35)$ | 0.9241 | 0.9692 | 0.03005 | 0.00775 | 0.02361 |
| | | | $(0_{(3)},35,0_{(11)})$ | 0.8939 | 0.9440 | 0.05537 | 0.01383 | 0.04122 |
| | | | $(0,5,0,5,\ldots,0,5,0)$ | 0.9229 | 0.9651 | 0.03280 | 0.00845 | 0.02570 |
| | | 30 | $(20,0_{(29)})$ | 0.8989 | 0.9545 | 0.04395 | 0.01123 | 0.03400 |
| | | | $(0_{(29)},20)$ | 0.9257 | 0.9693 | 0.02992 | 0.00772 | 0.02352 |
| | | | $(0_{(3)},20,0_{(26)})$ | 0.9157 | 0.9604 | 0.03485 | 0.00893 | 0.02708 |
| | | | $(0,1_{(2)},0,1_{(2)},\ldots,0,1_{(2)})$ | 0.9241 | 0.9687 | 0.03105 | 0.00800 | 0.02434 |
| | | | $(5,0_{(28)},15)$ | 0.9175 | 0.9660 | 0.03283 | 0.00844 | 0.02567 |
| | | | $(2,0_{(2)},2,0_{(2)},\ldots,2,0_{(2)})$ | 0.9218 | 0.9654 | 0.03185 | 0.00820 | 0.02493 |
| 3 | 20 | 5 | $(15,0_{(4)})$ | 0.8786 | 0.9421 | 0.15401 | 0.03794 | 0.11205 |
| | | | $(0_{(4)},15)$ | 0.9199 | 0.9683 | 0.03383 | 0.00872 | 0.02652 |
| | | | $(0_{(3)},15,0)$ | 0.8893 | 0.9363 | 0.16443 | 0.03884 | 0.11144 |
| | | | $(3_{(5)})$ | 0.8851 | 0.9428 | 0.06340 | 0.01598 | 0.04788 |
| | | | $(0,5_{(3)},0)$ | 0.8874 | 0.9404 | 0.15552 | 0.03726 | 0.10781 |
| | | 15 | $(5,0_{(14)})$ | 0.9079 | 0.9601 | 0.04133 | 0.01063 | 0.03227 |
| | | | $(0_{(14)},5)$ | 0.9212 | 0.9704 | 0.03373 | 0.00871 | 0.02653 |
| | | | $(0_{(3)},5,0_{(11)})$ | 0.9163 | 0.9638 | 0.03660 | 0.00942 | 0.02860 |
| | | | $(1,0_{(2)},1,0_{(2)},\ldots,1,0_{(2)})$ | 0.9167 | 0.9649 | 0.03571 | 0.00920 | 0.02798 |
| | 50 | 15 | $(35,0_{(14)})$ | 0.9053 | 0.9606 | 0.04123 | 0.01056 | 0.03196 |
| | | | $(0_{(14)},35)$ | 0.9257 | 0.9791 | 0.01651 | 0.00427 | 0.01301 |
| | | | $(0_{(3)},35,0_{(11)})$ | 0.8985 | 0.9569 | 0.02754 | 0.00699 | 0.02106 |
| | | | $(0,5,0,5,\ldots,0,5,0)$ | 0.9223 | 0.9791 | 0.01771 | 0.00458 | 0.01396 |
| | | 30 | $(20,0_{(29)})$ | 0.9079 | 0.9704 | 0.02350 | 0.00607 | 0.01846 |
| | | | $(0_{(29)},20)$ | 0.9243 | 0.9778 | 0.01668 | 0.00431 | 0.01313 |
| | | | $(0_{(3)},20,0_{(26)})$ | 0.9189 | 0.9768 | 0.01854 | 0.00479 | 0.01458 |
| | | | $(0,1_{(2)},0,1_{(2)},\ldots,0,1_{(2)})$ | 0.9250 | 0.9804 | 0.01654 | 0.00428 | 0.01305 |
| | | | $(5,0_{(28)},15)$ | 0.9237 | 0.9792 | 0.01737 | 0.00450 | 0.01372 |
| | | | $(2,0_{(2)},2,0_{(2)},\ldots,2,0_{(2)})$ | 0.9286 | 0.9815 | 0.01690 | 0.00437 | 0.01333 |

### 4.2. Illustrative example

Dunsmore [12] has given the size of rock crushed by a rock crushing machine. The machine has to be reset if, at any operation, the size of rock being crushed is larger than that has been crushed before. The following data are the sizes dealt with up to the third time that the machine has been reset.

9.3 0.6 24.4 18.1 6.6 9.0 14.3 6.6 13.0 2.4 5.6 33.8.

TABLE 6. The coverage probabilities of the prediction intervals and the estimated risks of the point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ with $i=5$ and $k=3$.

| $s$ | $n$ | $m$ | $(R_1,\ldots,R_m)$ | Coverage probability | | Estimated risk | | |
|---|---|---|---|---|---|---|---|---|
| | | | | $\tau=0.1$ | $\tau=0.05$ | SE | LINEX | |
| | | | | | | | $\vartheta=0.7$ | $\vartheta=1.2$ |
| 1 | 20 | 5 | $(15,0_{(4)})$ | 0.8909 | 0.9620 | 0.30663 | 0.07283 | 0.20736 |
| | | | $(0_{(4)},15)$ | 0.8897 | 0.9477 | 0.04746 | 0.01184 | 0.03530 |
| | | | $(0_{(3)},15,0)$ | 0.8840 | 0.9470 | 0.33897 | 0.07651 | 0.21198 |
| | | | $(3_{(5)})$ | 0.8729 | 0.9371 | 0.10126 | 0.02447 | 0.07129 |
| | | | $(0,5_{(3)},0)$ | 0.8896 | 0.9496 | 0.31201 | 0.07054 | 0.19593 |
| | | 15 | $(5,0_{(14)})$ | 0.8754 | 0.9362 | 0.06167 | 0.01527 | 0.04526 |
| | | | $(0_{(14)},5)$ | 0.8862 | 0.9435 | 0.04757 | 0.01185 | 0.03530 |
| | | | $(0_{(3)},5,0_{(11)})$ | 0.8870 | 0.9435 | 0.05194 | 0.01289 | 0.03829 |
| | | | $(1,0_{(2)},1,0_{(2)},\ldots,1,0_{(2)})$ | 0.8834 | 0.9409 | 0.05067 | 0.01261 | 0.03748 |
| | 50 | 15 | $(35,0_{(14)})$ | 0.8799 | 0.9376 | 0.06048 | 0.01495 | 0.04426 |
| | | | $(0_{(14)},35)$ | 0.9372 | 0.9723 | 0.02050 | 0.00519 | 0.01566 |
| | | | $(0_{(3)},35,0_{(11)})$ | 0.9108 | 0.9515 | 0.04200 | 0.01029 | 0.03029 |
| | | | $(0,5,0,5,\ldots,0,5,0)$ | 0.9302 | 0.9684 | 0.02298 | 0.00581 | 0.01747 |
| | | 30 | $(20,0_{(29)})$ | 0.9141 | 0.9587 | 0.03155 | 0.00793 | 0.02377 |
| | | | $(0_{(29)},20)$ | 0.9396 | 0.9744 | 0.02033 | 0.00515 | 0.01554 |
| | | | $(0_{(3)},20,0_{(26)})$ | 0.9307 | 0.9680 | 0.02417 | 0.00610 | 0.01834 |
| | | | $(0,1_{(2)},0,1_{(2)},\ldots,0,1_{(2)})$ | 0.9376 | 0.9733 | 0.02055 | 0.00521 | 0.01573 |
| | | | $(5,0_{(28)},15)$ | 0.9336 | 0.9718 | 0.02193 | 0.00555 | 0.01675 |
| | | | $(2,0_{(2)},2,0_{(2)},\ldots,2,0_{(2)})$ | 0.9365 | 0.9708 | 0.02152 | 0.00546 | 0.01645 |
| 3 | 20 | 5 | $(15,0_{(4)})$ | 0.8704 | 0.9337 | 0.13125 | 0.03141 | 0.09085 |
| | | | $(0_{(4)},15)$ | 0.9381 | 0.9727 | 0.01889 | 0.00479 | 0.01446 |
| | | | $(0_{(3)},15,0)$ | 0.8928 | 0.9424 | 0.13666 | 0.03185 | 0.09047 |
| | | | $(3_{(5)})$ | 0.9037 | 0.9499 | 0.04094 | 0.01013 | 0.03005 |
| | | | $(0,5_{(3)},0)$ | 0.8861 | 0.9387 | 0.13453 | 0.03138 | 0.08928 |
| | | 15 | $(5,0_{(14)})$ | 0.9294 | 0.9666 | 0.02414 | 0.00610 | 0.01837 |
| | | | $(0_{(14)},5)$ | 0.9405 | 0.9739 | 0.01870 | 0.00474 | 0.01430 |
| | | | $(0_{(3)},5,0_{(11)})$ | 0.9355 | 0.9708 | 0.02121 | 0.00536 | 0.01614 |
| | | | $(1,0_{(2)},1,0_{(2)},\ldots,1,0_{(2)})$ | 0.9375 | 0.9748 | 0.01970 | 0.00499 | 0.01505 |
| | 50 | 15 | $(35,0_{(14)})$ | 0.9140 | 0.9700 | 0.02405 | 0.00606 | 0.01819 |
| | | | $(0_{(14)},35)$ | 0.9374 | 0.9797 | 0.00793 | 0.00204 | 0.00615 |
| | | | $(0_{(3)},35,0_{(11)})$ | 0.9170 | 0.9641 | 0.01628 | 0.00406 | 0.01209 |
| | | | $(0,5,0,5,\ldots,0,5,0)$ | 0.9337 | 0.9775 | 0.00873 | 0.00223 | 0.00674 |
| | | 30 | $(20,0_{(29)})$ | 0.9251 | 0.9777 | 0.01234 | 0.00314 | 0.00949 |
| | | | $(0_{(29)},20)$ | 0.9372 | 0.9855 | 0.00810 | 0.00208 | 0.00626 |
| | | | $(0_{(3)},20,0_{(26)})$ | 0.9313 | 0.9764 | 0.00947 | 0.00241 | 0.00727 |
| | | | $(0,1_{(2)},0,1_{(2)},\ldots,0,1_{(2)})$ | 0.9385 | 0.9798 | 0.00794 | 0.00203 | 0.00615 |
| | | | $(5,0_{(28)},15)$ | 0.9389 | 0.9789 | 0.00855 | 0.00218 | 0.00661 |
| | | | $(2,0_{(2)},2,0_{(2)},\ldots,2,0_{(2)})$ | 0.9379 | 0.9784 | 0.00822 | 0.00211 | 0.00635 |

Dunsmore [12] assumed that the exponential distribution is suitable for these data. Here, we intend to predict the progressively first-failure-censored order statistics from a future sample based on $k$-records extracted from the above data. The observed $k$-record values, when $k=1$ and $k=2$, are shown in Table 7.

TABLE 7. Observed $k$-record values extracted from the data set in Dunsmore [12], when $k=1$ and $k=2$.

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $U_{j(1)}$ | 9.3 | 24.4 | 33.8 | |
| $U_{j(2)}$ | 0.6 | 9.3 | 18.1 | 24.4 |

For this example, the various progressive first-failure-censoring schemes (CS) reported in Table 8 were considered.

TABLE 8. Different choices for progressive first-failure-censoring scheme.

| CS | $s$ | $n$ | $m$ | $(R_1,\ldots,R_m)$ |
|---|---|---|---|---|
| I | 1 | 10 | 5 | $(5,0_{(4)})$ |
| II | 1 | 10 | 8 | $(0_{(4)},2,0_{(3)})$ |
| III | 2 | 10 | 10 | $(0_{(10)})$ |
| IV | 2 | 30 | 10 | $(0_{(5)},4_{(5)})$ |
| V | 3 | 30 | 10 | $(1_{(3)},2_{(4)},3_{(3)})$ |
| VI | 3 | 30 | 20 | $(0_{(3)},2,0_{(3)},1,1,4,0_{(3)},1,0_{(3)},1,0_{(2)})$ |

We recall that in the case where $\beta$ is known, the gamma conjugate prior (2.4) was considered for $\alpha$, so we readily find $E(\alpha)=a/b$ and $Var(\alpha)=a/b^2$. In order to have least informative, informative, and most informative prior densities, we considered three cases for the hyperparameters $(a,b)=(0.5,0.5),(1,1)$ and $(2,2)$, which correspond to the prior densities with common means 1 and different variances 2, 1 and 0.5, respectively. Considering these settings, we constructed 95% Bayesian two-sided equi-tailed prediction intervals as well as Bayes point predictors for future progressively first-failure-censored order statistics under SE and LINEX loss functions. The results are summarized in Tables 9-14.

By empirical evidences from Tables 9-14, we observe that the width of the prediction intervals decreases as the information about the unknown parameter $\alpha$ increases, i.e., as the hyperparameters $(a,b)$ increase. So, the prediction intervals are shortest when $(a,b)=(2,2)$. As we would expect, the prediction intervals are very sensitive to the prior distribution, and hence, the prior distribution must be determined carefully by the researcher. Also, the width of the prediction intervals is increasing with respect to $i$ when other parameters are kept fixed, while it decreases as each of the parameters $m$, $n$ and $s$ increases. Moreover, the results in Tables 9-14 show that the Bayes point predictor relative to the LINEX loss function is sensitive to the value of the shape parameter $\vartheta$. The problem of choosing the value of the parameter $\vartheta$ has been investigated by Calabria and Pulcini [10]. These results establish that for making an optimum decision, the careful consideration must be given to the choice of the loss function and one should not just consider appropriate prior distribution.

**References**

[1] Ahmadi, J., MirMostafaee, S.M.T.K. and Balakrishnan, N. (2011). Bayesian Prediction of order statistics based on $k$-record values from exponential distribution. *Statistics*, 45, 375-387.

[2] Ahmadi, J., MirMostafaee, S.M.T.K. and Balakrishnan, N. (2012). Bayesian Prediction of $k$-record values based on progressively censored data from exponential distribution. *Journal of Statistical Computation and Simulation*, 82, 51-62.

[3] Ali Mousa, M.A.M. and Jaheen, Z.F. (2002). Bayesian prediction for progressively censored data from the Burr model. *Statistical Papers*, 43, 587-593.

[4] Ali Mousa, M.A.M. and Al-Sagheer, S.A. (2005). Bayesian prediction for progressively Type-II censored data from the Rayleigh model. *Communications in Statistics - Theory and Methods*, 34, 2353-2361.

[5] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998). *Records*, John Wiley & Sons, New York.

[6] Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *Test*, 16, 211-256.

[7] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications*, Birkhäuser, Boston.

[8] Balakrishnan, N. and Cramer, E. (2014). *The Art of Progressive Censoring: Applications to Reliability and Quality*, Birkhäuser, New York.

[9] Balakrishnan, N. and Shafay, R.A. (2012). One- and two-sample Bayesian prediction intervals based on Type-II hybrid censored data. *Communications in Statistics - Theory and Methods*, 41, 1511-1531.

[10] Calabria, R. and Pulcini, G. (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. *Communications in Statistics - Theory and Methods*, 25, 585-600.

[11] Doostparast, M. (2009). A note on estimation based on record data. *Metrika*, 69, 69-80.

[12] Dunsmore, I.R. (1983). The future occurrence of records. *Annals of the Institute of Statistical Mathematics*, 35, 267-277.

[13] Dziubdziela, W. and Kopocinski, B. (1976). Limiting properties of the $k$-th record values. *Zastosowania Matematyki*, 15, 187-190.

[14] Ghafoori, S., Habibi Rad, A. and Doostparast, M. (2011). Bayesian two-sample prediction with progressively Type-II censored data for some lifetime models. *Journal of Iranian Statistical Society*, 10, 63-86.

[15] Huang, S.R. and Wu, S.J. (2012). Bayesian estimation and prediction for Weibull model with progressive censoring. *Journal of Statistical Computation and Simulation*, 82, 1607-1620.

[16] Kamps, U. (1995). *A concept of generalized order statistics*, Teubner, Stuttgart.

[17] Kamps, U. (1995). A concept of generalized order statistics. *Journal of Statistical Planning and Inference*, 48, 1-23.

[18] Kundu, D. and Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Computational Statistics and Data Analysis*, 54, 1547-1558.

[19] Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data*, John Wiley & Sons, New York.

[20] Madi, M.T. and Raqab, M.Z. (2004). Bayesian prediction of temperature records using the Pareto model. *Environmetrics*, 15, 701-710.

[21] Murthy, D.N.P., Xie, M. and Jiang, R. (2004). *Weibull Models*, John Wiley & Sons, New York.

[22] Nevzorov, V. (2001). *Records: Mathematical Theory*, Translation of Mathematical Monographs No. 194, American Mathematical Society, Providence, RI.

[23] Raqab, M.Z., Asgharzadeh, A. and Valiollahi, R. (2010). Prediction for Pareto distribution based on progressively Type-II censored samples. *Computational Statistics and Data Analysis*, 54, 1732-1743.

[24] Saeidi, A. R., Akbari, M. G. and Doostparast, M. (2014). Hypotheses testing with the two-parameter Pareto distribution on the basis of records in fuzzy environment. *Kybernetika*, 50, 744-757.

[25] Shafay, R.A., Balakrishnan, N. and Sultan, K.S. (2014). Two-sample Bayesian prediction for sequential order statistics from exponential distribution based on multiply Type-II censored samples. *Journal of Statistical Computation and Simulation*, 83, 526-544.

[26] Soland, R. M. (1969). Bayesian analysis of the Weibull process with unknown scale and shape parameters. *IEEE Transactions on Reliability Analysis*, 18, 181-184.

[27] Soliman, A.A., Al-Hossain, A.Y. and Al-Harbi, M.M. (2011). Predicting observables from Weibull model based on general progressive censored data with asymmetric loss. *Statistical Methodology*, 8, 451-461.

[28] Soman, K.P. and Misra, K.B. (1992). A least square estimation of three parameters of a Weibull distribution. *Microelectronics Reliability*, 32, 303-305.

[29] Varian, H.R. (1975). A Bayesian Approach to Real Estate Assessment, *Studies in Bayesian econometrics and statistics in honor of Leonard J. Savage*, 195-208.

[30] Wu, S.J. and Kuş, C. (2009). On estimation based on progressive first failure censored sampling. *Computational Statistics and Data Analysis*, 53, 3659-3670.

TABLE 9. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=1$ and $(a,b)=(0.5,0.5)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.0249 | 6.41069 | 1.372 | 0.92544 | 0.75168 |
| | 2 | 0.34552 | 19.6229 | 4.802 | 2.54114 | 1.94704 |
| | 5 | 4.63435 | 107.639 | 29.9553 | 9.35748 | 6.64666 |
| II | 1 | 0.0249 | 6.41069 | 1.372 | 0.92544 | 0.75168 |
| | 2 | 0.22772 | 11.4318 | 2.89644 | 1.84024 | 1.47887 |
| | 5 | 1.58548 | 30.2426 | 8.85811 | 4.74453 | 3.70847 |
| | 8 | 6.29625 | 117.997 | 34.0114 | 11.0019 | 7.97905 |
| III | 1 | 0.01245 | 3.20534 | 0.686 | 0.53721 | 0.46272 |
| | 5 | 0.79274 | 15.1213 | 4.42906 | 2.88621 | 2.37226 |
| | 8 | 2.09218 | 32.2619 | 9.80272 | 5.36927 | 4.25876 |
| | 10 | 4.1676 | 67.5167 | 20.0927 | 8.50762 | 6.45626 |
| IV | 1 | 0.00415 | 1.06845 | 0.22867 | 0.20615 | 0.19088 |
| | 5 | 0.22192 | 4.17791 | 1.22814 | 1.02049 | 0.91275 |
| | 8 | 0.50409 | 7.51108 | 2.30287 | 1.78084 | 1.55317 |
| | 10 | 0.93835 | 14.4759 | 4.36087 | 2.9463 | 2.46621 |
| V | 1 | 0.00277 | 0.7123 | 0.15244 | 0.14165 | 0.13369 |
| | 5 | 0.16196 | 3.06649 | 0.90001 | 0.77625 | 0.70615 |
| | 8 | 0.39885 | 6.02192 | 1.84008 | 1.47205 | 1.29991 |
| | 10 | 0.75676 | 11.8414 | 3.55508 | 2.50422 | 2.12069 |
| VI | 1 | 0.00277 | 0.7123 | 0.15244 | 0.14165 | 0.13369 |
| | 5 | 0.15043 | 2.83637 | 0.83342 | 0.7251 | 0.66239 |
| | 10 | 0.46421 | 6.29245 | 1.96764 | 1.57769 | 1.39644 |
| | 20 | 3.16558 | 46.4877 | 14.0635 | 6.96744 | 5.47593 |

TABLE 10. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=1$ and $(a,b)=(1,1)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.0221 | 5.27174 | 1.16 | 0.83062 | 0.68637 |
| | 2 | 0.30986 | 15.9935 | 4.06 | 2.33045 | 1.81293 |
| | 5 | 4.20602 | 86.7622 | 25.3267 | 8.84877 | 6.35123 |
| II | 1 | 0.0221 | 5.27174 | 1.16 | 0.83062 | 0.68637 |
| | 2 | 0.20428 | 9.28556 | 2.44889 | 1.66607 | 1.36148 |
| | 5 | 1.44197 | 24.2321 | 7.48937 | 4.37811 | 3.47413 |
| | 8 | 5.73966 | 94.7871 | 28.756 | 10.4117 | 7.62936 |
| III | 1 | 0.01105 | 2.63587 | 0.58 | 0.47426 | 0.41531 |
| | 5 | 0.72098 | 12.1161 | 3.74468 | 2.61968 | 2.18905 |
| | 8 | 1.91205 | 25.7465 | 8.28802 | 4.95706 | 3.99095 |
| | 10 | 3.80932 | 54.0393 | 16.988 | 7.97251 | 6.12585 |
| IV | 1 | 0.00368 | 0.87862 | 0.19333 | 0.17821 | 0.16705 |
| | 5 | 0.20186 | 3.34608 | 1.03837 | 0.89844 | 0.81653 |
| | 8 | 0.46094 | 5.98711 | 1.94703 | 1.54238 | 1.40795 |
| | 10 | 0.85838 | 11.5689 | 3.68703 | 2.65471 | 2.26997 |
| V | 1 | 0.00246 | 0.58575 | 0.12889 | 0.12176 | 0.11612 |
| | 5 | 0.14731 | 2.45643 | 0.76094 | 0.67906 | 0.62709 |
| | 8 | 0.36463 | 4.80221 | 1.55575 | 1.30537 | 1.17203 |
| | 10 | 0.69209 | 9.4674 | 3.00575 | 2.25652 | 1.9429 |
| VI | 1 | 0.00246 | 0.58575 | 0.12889 | 0.12176 | 0.11612 |
| | 5 | 0.13683 | 2.27176 | 0.70464 | 0.63333 | 0.58717 |
| | 10 | 0.42542 | 5.00336 | 1.6636 | 1.39984 | 1.25991 |
| | 20 | 2.90145 | 37.1436 | 11.8905 | 6.4773 | 5.16029 |

TABLE 11. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=1$ and $(a,b)=(2,2)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.01817 | 3.90678 | 0.895 | 0.69268 | 0.58805 |
| | 2 | 0.2588 | 11.7032 | 3.1325 | 2.00917 | 1.60276 |
| | 5 | 3.57761 | 62.4403 | 19.5408 | 8.02138 | 5.86283 |
| II | 1 | 0.01817 | 3.90678 | 0.895 | 0.69268 | 0.58805 |
| | 2 | 0.1707 | 6.75879 | 1.88944 | 1.40661 | 1.18075 |
| | 5 | 1.23054 | 17.2714 | 5.77843 | 3.80549 | 3.09625 |
| | 8 | 4.91687 | 67.8579 | 22.1868 | 9.44529 | 7.0482 |
| III | 1 | 0.00909 | 1.95339 | 0.4475 | 0.38597 | 0.34634 |
| | 5 | 0.61527 | 8.63572 | 2.88922 | 2.21718 | 1.90274 |
| | 8 | 1.64457 | 18.233 | 6.39463 | 4.30984 | 3.55814 |
| | 10 | 3.27741 | 38.4613 | 13.1071 | 7.10864 | 5.58014 |
| IV | 1 | 0.00303 | 0.65113 | 0.14917 | 0.14098 | 0.13432 |
| | 5 | 0.1723 | 2.38314 | 0.80115 | 0.72687 | 0.67587 |
| | 8 | 0.39679 | 4.23146 | 1.50224 | 1.30613 | 1.18853 |
| | 10 | 0.7395 | 8.21349 | 2.84474 | 2.24823 | 1.96386 |
| V | 1 | 0.00202 | 0.43409 | 0.09944 | 0.09567 | 0.09242 |
| | 5 | 0.12573 | 1.75008 | 0.5871 | 0.54469 | 0.51355 |
| | 8 | 0.31378 | 3.39655 | 1.20035 | 1.06635 | 0.9812 |
| | 10 | 0.59599 | 6.72616 | 2.3191 | 1.88779 | 1.66835 |
| VI | 1 | 0.00202 | 0.43409 | 0.09944 | 0.09567 | 0.09242 |
| | 5 | 0.11679 | 1.61814 | 0.54367 | 0.50698 | 0.4796 |
| | 10 | 0.36758 | 3.52156 | 1.28355 | 1.14413 | 1.05551 |
| | 20 | 2.50791 | 26.3644 | 9.1741 | 5.69514 | 4.64417 |

TABLE 12. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=2$ and $(a,b)=(0.5,0.5)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.02782 | 6.26078 | 1.40857 | 0.97578 | 0.79333 |
| | 2 | 0.39337 | 18.8613 | 4.93 | 2.70565 | 2.07091 |
| | 5 | 5.39272 | 101.395 | 30.7538 | 10.1172 | 7.15811 |
| II | 1 | 0.02782 | 6.26078 | 1.40857 | 0.97578 | 0.79333 |
| | 2 | 0.25941 | 10.9192 | 2.97365 | 1.9594 | 1.57716 |
| | 5 | 1.85207 | 28.1731 | 9.09423 | 5.15094 | 4.03323 |
| | 8 | 7.38705 | 110.458 | 34.918 | 12.0215 | 8.70581 |
| III | 1 | 0.01391 | 3.13039 | 0.70429 | 0.56421 | 0.48789 |
| | 5 | 0.92603 | 14.0866 | 4.54711 | 3.11531 | 2.5755 |
| | 8 | 2.46617 | 29.8309 | 10.064 | 5.87067 | 4.67704 |
| | 10 | 4.91399 | 62.7767 | 20.6283 | 9.34165 | 7.10311 |
| IV | 1 | 0.00464 | 1.04346 | 0.23476 | 0.2146 | 0.19984 |
| | 5 | 0.2593 | 3.88873 | 1.26088 | 1.08251 | 0.97812 |
| | 8 | 0.59478 | 6.92958 | 2.36426 | 1.91033 | 1.68445 |
| | 10 | 1.10809 | 13.4217 | 4.47711 | 3.19019 | 2.69431 |
| V | 1 | 0.00309 | 0.69564 | 0.15651 | 0.14701 | 0.1395 |
| | 5 | 0.18922 | 2.8553 | 0.924 | 0.81966 | 0.7533 |
| | 8 | 0.47043 | 5.56035 | 1.88913 | 1.57309 | 1.40476 |
| | 10 | 0.89323 | 10.9877 | 3.64985 | 2.70272 | 2.31051 |
| VI | 1 | 0.00309 | 0.69564 | 0.15651 | 0.14701 | 0.1395 |
| | 5 | 0.17576 | 2.64032 | 0.85563 | 0.76481 | 0.70585 |
| | 10 | 0.55004 | 5.77832 | 2.02008 | 2.77549 | 1.51336 |
| | 20 | 3.75202 | 43.0861 | 14.4384 | 7.66772 | 6.06231 |

TABLE 13. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=2$ and $(a,b)=(1,1)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.02528 | 5.43457 | 1.245 | 0.89812 | 0.73976 |
| | 2 | 0.36001 | 16.2798 | 4.3575 | 2.52923 | 1.95856 |
| | 5 | 4.97668 | 86.8583 | 27.1825 | 9.67337 | 6.9003 |
| II | 1 | 0.02528 | 5.43457 | 1.245 | 0.89812 | 0.73976 |
| | 2 | 0.23746 | 9.40189 | 2.62833 | 1.81367 | 1.47859 |
| | 5 | 1.71175 | 24.0256 | 8.03815 | 4.83182 | 3.82752 |
| | 8 | 6.83967 | 94.3945 | 30.8632 | 11.4956 | 8.39332 |
| III | 1 | 0.01264 | 2.71728 | 0.6225 | 0.51302 | 0.44906 |
| | 5 | 0.85588 | 12.0128 | 4.01908 | 2.88576 | 2.41591 |
| | 8 | 2.28769 | 25.3633 | 8.89533 | 5.50661 | 4.43806 |
| | 10 | 4.55907 | 53.502 | 18.2328 | 8.86358 | 6.80511 |
| IV | 1 | 0.00421 | 0.90576 | 0.2075 | 0.19229 | 0.18059 |
| | 5 | 0.23968 | 3.31509 | 1.11446 | 0.98082 | 0.89634 |
| | 8 | 0.55196 | 5.88621 | 2.08971 | 1.74538 | 1.55814 |
| | 10 | 1.02869 | 11.4255 | 3.95721 | 2.94847 | 2.52112 |
| V | 1 | 0.00281 | 0.60384 | 0.13833 | 0.13124 | 0.12539 |
| | 5 | 0.17489 | 2.43447 | 0.8167 | 0.73943 | 0.6867 |
| | 8 | 0.43649 | 4.72481 | 1.66975 | 1.43207 | 1.29427 |
| | 10 | 0.82906 | 9.3565 | 3.226 | 2.48915 | 2.15444 |
| VI | 1 | 0.00281 | 0.60384 | 0.13833 | 0.13124 | 0.12539 |
| | 5 | 0.16246 | 2.25093 | 0.75627 | 0.68924 | 0.64261 |
| | 10 | 0.51132 | 4.89871 | 1.7855 | 1.52902 | 1.39472 |
| | 20 | 3.48866 | 36.6745 | 12.7617 | 7.22771 | 5.77448 |

TABLE 14. 95% prediction intervals and point predictors for $X^{\mathbf{R}}_{i:m:n:s}$ when $k=2$ and $(a,b)=(2,2)$.

| CS | $i$ | Prediction interval | | Point predictor | | |
|---|---|---|---|---|---|---|
| | | Lower | Upper | SE | LINEX | |
| | | | | | $\vartheta=0.5$ | $\vartheta=1$ |
| I | 1 | 0.02148 | 4.3145 | 1.016 | 0.7772 | 0.65407 |
| | 2 | 0.30917 | 12.8144 | 3.556 | 2.2448 | 1.7735 |
| | 5 | 4.33122 | 67.5645 | 22.1827 | 8.92216 | 6.45819 |
| II | 1 | 0.02148 | 4.3145 | 1.016 | 0.7772 | 0.65407 |
| | 2 | 0.20399 | 7.37182 | 2.14489 | 1.58301 | 1.31847 |
| | 5 | 1.49337 | 18.5493 | 6.55965 | 4.30843 | 3.48256 |
| | 8 | 5.98523 | 73.1455 | 25.1863 | 10.6014 | 7.85424 |
| III | 1 | 0.01074 | 2.15725 | 0.508 | 0.43551 | 0.3886 |
| | 5 | 0.74668 | 9.27464 | 3.27983 | 2.51942 | 2.15421 |
| | 8 | 2.00809 | 19.4855 | 7.25916 | 4.90774 | 4.03576 |
| | 10 | 4.00307 | 41.273 | 14.8792 | 8.05928 | 6.29483 |
| IV | 1 | 0.00358 | 0.71908 | 0.16933 | 0.15975 | 0.15188 |
| | 5 | 0.20913 | 2.55793 | 0.90947 | 0.8271 | 0.76899 |
| | 8 | 0.48481 | 4.51474 | 1.70533 | 1.49756 | 1.36445 |
| | 10 | 0.90417 | 8.79669 | 3.22933 | 2.56578 | 2.24017 |
| V | 1 | 0.00239 | 0.47939 | 0.11289 | 0.10848 | 0.10465 |
| | 5 | 0.1526 | 1.87894 | 0.66648 | 0.61969 | 0.58445 |
| | 8 | 0.38329 | 3.62615 | 1.36263 | 1.21599 | 1.11969 |
| | 10 | 0.72847 | 7.20768 | 2.63263 | 2.1524 | 1.90114 |
| VI | 1 | 0.00239 | 0.47939 | 0.11289 | 0.10848 | 0.10465 |
| | 5 | 0.14175 | 1.73695 | 0.61717 | 0.57677 | 0.54584 |
| | 10 | 0.45046 | 3.74476 | 1.45708 | 1.3036 | 1.21045 |
| | 20 | 3.07476 | 28.2347 | 10.4144 | 6.49547 | 5.2373 |