Turk. J. Math. Comput. Sci.
18(1)(2026) 175–182
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DOI : 10.47000/tjmcs.1757179

Local Existence and Blow-Up Analysis for a Damped Viscoelastic Kirchhoff-Type Equation with Logarithmic Power Source

Begüm Çalışkan Desova

Department of Information Security Technology, Faculty of Computer and Information Sciences, Yeditepe University, 34755, İstanbul, Türkiye.

Received: 02-08-2025 • Accepted: 07-11-2025

Abstract. We investigate the initial-boundary value problem for the nonlinear viscoelastic Kirchhoff-type wave equation

$$u_{tt} - \left(1 + \|\nabla u(t)\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \alpha u_t = |u|^{p-1}\ln|u|, \quad x \in \Omega, t > 0,$$

with homogeneous Dirichlet boundary conditions. Local existence and uniqueness of weak solutions are established via a Banach fixed-point argument. Moreover, we prove a finite-time blow-up result: if the initial energy is positive and the data satisfy a compatibility condition, the solution's H_0^1-norm becomes unbounded in finite time. This work extends existing results by capturing the combined effects of Kirchhoff-type nonlinearity, viscoelastic damping, and a logarithmic source.

2020 AMS Classification: 35L05, 35B44

Keywords: Viscoelastic wave equation, Kirchhoff-type nonlinearity, logarithmic source term, finite-time blow-up, positive initial energy.

1. Introduction

Nonlinear wave equations of Kirchhoff type originate from the classical work of Kirchhoff [8], who modeled the vibration of elastic strings or membranes with tension depending on the deformation. A prototype is the Kirchhoff equation

$$u_{tt} - (1 + b\|\nabla u\|^2)\Delta u = 0,$$

which generalizes the d'Alembert wave equation by allowing the coefficient of Δu to depend on the strain $\|\nabla u\|^2$.

In this paper, we investigate a Kirchhoff-type wave equation that includes both viscoelastic damping term and a logarithmic nonlinear source:

$$u_{tt} - \left(1 + \|\nabla u(t)\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \alpha u_t = |u|^{p-1}\ln|u|, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

subject to the homogeneous Dirichlet boundary conditions

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.3}$$

Email address: begum.caliskan@yeditepe.edu.tr (B. Çalışkan Desova)

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\alpha > 0$, $p > 1$, and $g(t)$ is a positive, non-increasing relaxation kernel. This model incorporates Kirchhoff-type nonlinearity through the gradient-dependent coefficient of the Laplacian, a viscoelastic damping term with memory, and a logarithmic source, leading to rich dynamical behavior.

The viscoelastic term $\int_0^t g(t-s)\Delta u(s)\,ds$ was introduced by Dafermos [5] to model fading memory, with $g(t)$ typically a positive, decreasing kernel. Such models have been widely studied; for instance, Messaoudi [9] analyzed the problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u|^{p-1}u = 0,$$

and proved finite-time blow-up under suitable conditions. Guesmia [6] extended these results to include delay and damping effects. More recently, Chen and Tian [4] considered a pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta u = u \ln|u|,$$

and established well-posedness results.

Al-Gharabli, Guesmia, and Messaoudi [2] investigated a viscoelastic plate equation

$$u_{tt} + \Delta^2 u + \int_0^t g(t-s)\Delta^2 u(s)\,ds + \alpha u_t = u \ln|u|,$$

and proved decay and stability. Boulaaras et al. [3] analyzed a Kirchhoff-type model with a logarithmic source and a polynomially decaying kernel and established energy decay estimates.

Pişkin and Irkıl [12] examined the problem

$$u_{tt} - (1 + \|\nabla u\|^2)\Delta u + \alpha u_t = |u|^{p-1}\ln|u|,$$

and studied qualitative behavior. Yang et al. [15] analyzed

$$u_{tt} - (1 + \|\nabla u\|^2)\Delta u + \alpha u_t = |u|^{p-1}\ln(1 + |u|),$$

and established decay estimates. Peyravi [11] studied semilinear wave equations with both memory and logarithmic source:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = u \ln(1 + |u|),$$

and demonstrated exponential growth under suitable conditions.

Our contribution is to provide a rigorous analysis of the viscoelastic Kirchhoff-type problem with a logarithmic source. We establish two main results: (1) Local existence and uniqueness of weak solutions in appropriate Sobolev spaces for general initial data in a finite time interval; and (2) Finite-time blow-up (global nonexistence) of solutions for a class of initial data that satisfy a positive energy condition and an initial velocity condition. The local existence result is proved via a fixed-point argument applied to an equivalent integral equation – this approach allows us to handle the combined nonlinear terms in a relatively straightforward manner. The blow-up result is obtained by combining energy estimates with a concavity argument. The main difficulty lies in handling the mixed-sign and non-Lipschitz behavior of the logarithmic term $|u|^{p-1}\ln|u|$, especially near the origin where it may act like damping. However, we show that for suitable initial data, the solution energy becomes concave and vanishes in finite time, leading to blow-up.

It should be noted that the nonlinear logarithmic source term introduces substantial analytical difficulties due to its non-Lipschitz nature and mixed sign, which prevent direct use of standard monotonicity and compactness arguments. To overcome these challenges, we employ a refined combination of energy estimates in the logarithmic regime together with a Banach fixed-point framework that ensures control over the nonlinear growth. This methodology not only guarantees the local solvability of the problem but also enables a rigorous derivation of the finite-time blow-up condition under suitable energy constraints. This study complements and extends previous works on Kirchhoff-type and viscoelastic wave equations with logarithmic sources (see [7–13]), where similar models have been investigated under different boundary conditions and nonlinearities.

The paper is organized as follows. In Section 2, we introduce the precise functional framework and hypotheses on the memory kernel $g(t)$ and the initial data, and recall some preliminary inequalities and facts. Section 3 states the main results: the local existence and uniqueness theorem for weak solutions, and the finite-time blow-up theorem under high initial energy conditions and contains the detailed proofs. Finally, Section 4 presents concludes and discusses related open problems, such as global existence, decay rates and potential well theory.

2. Preliminaries and Assumptions

Before stating the main results, we outline the assumptions and functional setting for our problem, and define what we mean by a weak solution. Throughout this paper, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded open set with a sufficiently smooth boundary $\partial\Omega$. We use standard notation for function spaces: $L^2(\Omega)$ is the space of square-integrable functions on Ω and $H_0^1(\Omega)$ is the Sobolev space of functions that are in $H^1(\Omega)$ and vanish on the boundary $\partial\Omega$. By $|\cdot|$ and $(\cdot,\cdot)$ we denote the $L^2(\Omega)$ norm and inner product, and $|\nabla u|$ denotes the $L^2(\Omega)$ norm of the gradient. We will frequently use the Poincaré inequality, which implies $|u|_{L^2(\Omega)} \leq C_P|\nabla u|_{L^2(\Omega)}$ for all $u \in H_0^1(\Omega)$, where $C_P > 0$ depends on Ω. For $T > 0$, we use the notation $C([0,T]; H_0^1(\Omega))$ for the space of continuous functions $[0,T] \to H_0^1(\Omega)$, etc. For more details on Sobolev spaces, Poincaré inequality, and related functional analytic tools, we refer the reader to [1, 14].

(A1) Assumptions on the memory kernel $g(t)$: We assume $g : [0,\infty) \to \mathbb{R}^+$ is a non-increasing, continuously differentiable function modeling the relaxation modulus of the material. In particular, $g(t)$ satisfies:

- $g(0) > 0$, and $g(t) \geq 0$ for all $t \geq 0$ (positive definite damping kernel).
- $\displaystyle\int_0^\infty g(s), ds = 1 - \ell$ for some $\ell > 0$. (In many cases, one assumes $\int_0^\infty g(s)ds < \infty$, which is physically natural as it implies the total past influence is finite. Here we set it to $1-\ell$ for convenience; $\ell > 0$ can be thought of as a residual stiffness ratio as $t \to \infty$.)

The condition $\int_0^\infty g(s)ds < 1$ ensures a certain dissipativity of the memory term (often used to guarantee that the viscoelastic damping actually dissipates energy and does not completely counteract the elastic term). A typical example is an exponentially decaying kernel $g(t) = Ae^{-\beta t}$ (with $A, \beta > 0$), for which this integral condition holds if $A < \beta$ (so that $1 - \ell = A/\beta < 1$). We do not require $g(t)$ to be exponential; it could be a polynomially decaying kernel as well, as long as the integral is finite. We note also that the assumption g is nonincreasing can be relaxed in some stability studies, but for our analysis it is convenient to assume $g(t)$ decreases, which implies the memory term provides a kind of distributed damping.

Assumptions on initial data: The initial displacement $u_0(x)$ and initial velocity $u_1(x)$ are given functions satisfying

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega).$$

For technical reasons in the local existence proof, we will also assume that the logarithmic source term is well-defined by requiring $u_0(x)$ to be essentially bounded away from zero. However, since the nonlinearity $|u|^{p-1}\ln|u|$ is defined at $u = 0$ by continuity ($|u|^{p-1}\ln|u| \to 0$ as $u \to 0$ for $p > 1$), we do not actually need a strict positivity assumption on u_0. It suffices that u_0 is in H_0^1 so that $|u_0|^{p-1}\ln|u_0|$ belongs to $L^1(\Omega)$ (this holds because $H^1(\Omega)$ functions are bounded in L^q for some $q > 2$ when $n \leq 3$ by Sobolev embedding, and for high n we can use the fact that $|s|^{p-1}|\ln|s||$ grows subcritically near infinity and is $O(|s|^{p-1+\epsilon})$ for any small $\epsilon > 0$ as $s \to \infty$, making it integrable against the H^1 function). In summary, no very restrictive conditions on (u_0, u_1) beyond natural energy finite conditions are needed for local existence.

We define the energy functional $E(t)$ associated to a regular solution $u(t)$ as:

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}(1 + \|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(s)\|^2\,ds - \int_\Omega F(u(t))\,dx,$$

where $F(u)$ is an antiderivative of the source term $f(u) = |u|^{p-1}\ln|u|$. We can compute $F(u)$ explicitly: for $u \neq 0$,

$$\frac{d}{du}F(u) = |u|^{p-1}\ln|u|.$$

So one convenient primitive is

$$F(u) = \frac{|u|^p}{p}\ln|u| - \frac{|u|^p}{p^2},$$

for $u \neq 0$, and $F(0) = 0$. (This choice comes from integrating $|u|^{p-1}\ln|u|$: indeed $\int |u|^{p-1}\ln|u|, du = \frac{|u|^p}{p}\ln|u| - \frac{|u|^p}{p^2} + C$.) Thus, $F(u)$ is a double-well potential that is negative for small u (since $\ln|u|$ is negative for $|u| < 1$) and positive for large u. We will see that $E(t)$ is the natural energy of the system, combining kinetic energy $\frac{1}{2}|u_t|^2$, "elastic" energy $\frac{1}{2}(1 + |\nabla u|^2)|\nabla u|^2$ (coming from the Kirchhoff term), memory energy $\frac{1}{2}\int_0^t g(t-s)|\nabla u(s)|^2 ds$, and the potential energy

$-\int F(u)dx$ from the source. One can formally differentiate $E(t)$ and, using u's equation, show that energy is dissipated by the damping terms: indeed one finds

$$E'(t) = -\frac{\alpha}{2}\|u_t(t)\|^2 - \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\nabla u(s)\|^2\,ds.$$

By integrating the memory term by parts in t, this can be rewritten as

$$E'(t) = -\frac{\alpha}{2}\|u_t(t)\|^2 - \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s)\|^2\,ds - \frac{1}{2}g(0)\|\nabla u(t)\|^2,$$

which for a nonincreasing g implies $E'(t) \le -\frac{\alpha}{2}|u_t|^2 \le 0$. Thus, in the absence of the source term ($f(u) \equiv 0$), the energy $E(t)$ is nonincreasing. The source term $|u|^{p-1}\ln|u|$ can cause $E(t)$ to increase, however, since it injects energy into the system when $F(u)$ decreases. We will make these intuitive remarks rigorous in the proofs section. Now we define weak solutions. We say that $u(t)$ is a weak solution on $[0,T]$ to problem (1.1)-(1.3) if:

1. $u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and $u(0) = u_0$, $u_t(0) = u_1$.
2. For every test function $\varphi \in H_0^1(\Omega)$ and for almost every $t \in [0,T]$, $u(t)$ satisfies the weak form of the equation:

$$(u_{tt}(t),\varphi) + \alpha,(u_t(t),\varphi) + (1+|\nabla u(t)|^2),(\nabla u(t),\nabla\varphi) + \int_0^t g(t-s),(\nabla u(s),\nabla\varphi),ds =$$
$$(|u(t)|^{p-1}\ln|u(t)|,\varphi)\ ,$$

   where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product.
3. u satisfies the energy inequality corresponding to (1), i.e. $E(t)$ is well-defined and $E(t) \le E(0)$ for all t.

This definition encapsulates the idea that u satisfies equation (1.1) in the sense of distributions, and u attains the initial data in the $H_0^1 \times L^2$ sense. The regularity stated, $u \in C^1([0,T]; L^2)$, along with the weak formulation, is sufficient to justify integration by parts in time when testing with φ that do not depend on time, thus giving the weak form above.

The focus in this paper is on proving existence of at least one weak solution for short time, and then showing that under certain conditions no global weak solution can exist. To conclude the preliminaries, we recall an inequality that will be useful for handling the logarithmic term: for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|s|^{p-1}|\ln|s|| \le \epsilon|s|^p + C\epsilon,$$

for any small $\delta > 0$ (taking $\epsilon = \delta$ above). This provides a lower bound for the source term which will be crucial in the blow-up analysis to handle the negative part of the logarithmic nonlinearity.

3. Main Results

We are now ready to state the main results of this work. The first theorem guarantees local existence of a unique weak solution for our problem under the assumptions given, for any finite energy initial data. The second theorem provides conditions under which the obtained local solution cannot be extended to a global one, i.e. it blows up in finite time. Throughout, we assume the hypotheses on $g(t)$ and (u_0, u_1) from Section 2.

Theorem 3.1. *(Local Existence and Uniqueness) Suppose $g(t)$ satisfies the assumptions (A1) above and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Then, there exists a time $T > 0$ (depending on the initial data and g) such that the problem* (1.1)-(1.3) *admits a unique weak solution $u(t)$ on the time interval* $[0,T]$. *Moreover, u belongs to the class $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and u depends continuously on the initial data.*

Proof. We reformulate the problem as a first-order system by introducing the velocity $v(x,t) := u_t(x,t)$. Then, the original equation becomes:

$$\begin{cases} u_t = v, \\ v_t = M(|\nabla u|^2)\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds - f(u), \end{cases}$$

with initial conditions $u(0) = u_0 \in H_0^1(\Omega)$, $v(0) = u_1 \in L^2(\Omega)$, and homogeneous Dirichlet boundary conditions.

Integrating both equations in time yields the Volterra integral form:

$$u(t) = u_0 + \int_0^t v(s)\,ds,$$
$$v(t) = u_1 + \int_0^t \left[M\left(|\nabla u(s)|^2\right)\Delta u(s) - (g * \Delta u)(s) - f(u(s)) \right] ds.$$

We now define the Banach space

$$X = C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$

equipped with the norm

$$\|w\|_X := \sup_{t\in[0,T]} \left(\|w(t)\|_{H^1} + \|w_t(t)\|_{L^2} \right).$$

Let $B_R := \{w \in X : \|w\|_X \le R\}$ be the closed ball of radius R. We define an operator $T : B_R \to X$ as follows: for each $w \in B_R$, we set

$$\tilde{u}(t) = u_0 + \int_0^t \tilde{v}(s)\,ds,$$
$$\tilde{v}(t) = u_1 + \int_0^t \left[M(|\nabla w(s)|^2)\Delta w(s) - (g * \Delta w)(s) - f(w(s)) \right] ds.$$

The pair $(\tilde{u}, \tilde{v}) = T(w)$ is well-defined due to standard parabolic regularity theory.

We now show that T maps B_R into itself. To that end, we estimate:

$$\|\tilde{u}(t)\|_{H^1} \le \|u_0\|_{H^1} + \int_0^t \|\tilde{v}(s)\|_{H^1}\,ds, \quad \|\tilde{v}(t)\|_{L^2} \le \|u_1\| + \int_0^t \|F_w(s)\|\,ds,$$

where

$$F_w(s) := M(|\nabla w(s)|^2)\Delta w(s) - \int_0^s g(s-\tau)\Delta w(\tau)\,d\tau - f(w(s)).$$

Each term in F_w is estimated using the assumptions: $M(s) = 1 + s$ is smooth and Lipschitz on bounded sets, $f(w) = |w|^{p-1}\ln|w|$ is locally Lipschitz on bounded balls in H^1, $\|\Delta w\|_{H^{-1}} \le C\|w\|_{H^1}$, and convolution with $g \in L^1$ preserves integrability.

Hence, for all $w \in B_R$, we obtain:

$$\|\tilde{u}(t)\|_{H^1} + \|\tilde{v}(t)\|_{L^2} \le C\left(\|u_0\|_{H^1} + \|u_1\|_{L^2} + TR^k \right),$$

for some exponent k depending on p. Choosing $T > 0$ small enough ensures that $\|T(w)\|_X \le R$, so $T(B_R) \subseteq B_R$.

We now show T is a contraction on B_R. Let $w_1, w_2 \in B_R$. Then:

$$\|T(w_1) - T(w_2)\|_X \le C_R T \|w_1 - w_2\|_X,$$

where the constant $C_R > 0$ depends on R, and the estimate is obtained as follows:

- The Kirchhoff term satisfies:

$$\|M(|\nabla w_1|^2)\Delta w_1 - M(|\nabla w_2|^2)\Delta w_2\|_{H^{-1}} \le C_R \|w_1 - w_2\|_{H^1}.$$

- The memory term, being linear, satisfies:

$$\left\| \int_0^t g(t-s)\Delta(w_1 - w_2)(s)\,ds \right\|_{H^{-1}} \le C\|w_1 - w_2\|_{C([0,T];H^1)}.$$

- The nonlinear source $f(u) = |u|^{p-1}\ln|u|$ is locally Lipschitz on bounded subsets of H^1, hence:

$$\|f(w_1) - f(w_2)\|_{H^{-1}} \le C_R \|w_1 - w_2\|_{H^1}.$$

Summing up the above estimates and integrating in time yields:

$$\|\tilde{v}_1 - \tilde{v}_2\|_{L^\infty(0,T;L^2)} \le C_R T \|w_1 - w_2\|_X, \quad \|\tilde{u}_1 - \tilde{u}_2\|_{L^\infty(0,T;H^1)} \le C_R T^2 \|w_1 - w_2\|_X.$$

Therefore, T is a contraction:

$$\|T(w_1) - T(w_2)\|_X \le C_R T \|w_1 - w_2\|_X,$$

and choosing T sufficiently small so that $C_R T < 1$, we invoke Banach's fixed point theorem to conclude that T has a unique fixed point $u \in X$. This function solves the integral system, hence the original PDE in weak sense, and depends continuously on the initial data. By Banach's Fixed Point Theorem, T has a unique fixed point $u \in B_R \subset X$. This fixed point satisfies the integral system, hence the original PDE in weak form. Regularity follows from the construction. Continuous dependence on initial data follows by stability of the fixed point with respect to input. □

Theorem 3.2. *(Finite-Time Blow-up) In addition to the assumptions of Theorem 3.1, assume further that the initial data satisfy a positive energy condition and a compatibility condition:*

$$E(0) > 0,$$

where the initial energy $E(0)$ computed from (u_0, u_1) is given by

$$E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\left(1 + \|\nabla u_0\|^2\right)\|\nabla u_0\|^2 - \int_\Omega F(u_0)\,dx.$$

Assume also that the initial velocity $u_1(x)$ has a positive projection on the first eigenfunction of $-\Delta$ on $H_0^1(\Omega)$. In particular, let $\phi_1(x)$ be the L^2-normalized first eigenfunction of $-\Delta$ on Ω (so that $-\Delta\phi_1 = \lambda_1\phi_1$ with $\lambda_1 > 0$, $\phi_1|_{\partial\Omega} = 0$, and $\|\phi_1\|_{L^2} = 1$). We assume

$$\int_\Omega u_1(x)\,\phi_1(x)\,dx > 0,$$

i.e., the initial kinetic energy has a component in the positive direction of the first mode.

Then the weak solution $u(t)$ given by Theorem 3.1 blows up in finite time. More precisely, there exists a finite time $T^* < \infty$ such that

$$\lim_{t \to T^*} \|\nabla u(t)\|_{L^2(\Omega)} = +\infty,$$

and no weak solution exists on any interval $[0, T]$ with $T > T^*$. Equivalently, the solution cannot be extended globally beyond T^*. In fact, the blow-up occurs with $u(t)$ maintaining a certain sign in the direction of ϕ_1, and an upper bound for the blow-up time T^* can be estimated explicitly in terms of the initial data and parameters.

Proof. The proof is by contradiction using a concavity method. Assume, to the contrary, that the weak solution $u(t)$ given by Theorem 3.1 exists globally for all $t \geq 0$, even though the initial data satisfies the high-energy condition $E(0) > 0$ and the compatibility condition $\int_\Omega u_1(x)\phi_1(x)\,dx > 0$.

Step 1: Define the functional

$$F(t) := E(t) + \delta(u(t), \phi_1),$$

where $\delta > 0$ is a small constant to be determined, and $(u(t), \phi_1) = \int_\Omega u(x,t)\phi_1(x)dx$.

Using assumptions $E(0) > 0$ and $(u_0, \phi_1) \geq 0$ (we may choose the sign of ϕ_1 appropriately), we get

$$F(0) = E(0) + \delta(u_0, \phi_1) > 0.$$

Next, compute the derivative:

$$F'(t) = E'(t) + \delta(u_t(t), \phi_1).$$

From the energy identity,

$$E'(t) = -\alpha\|u_t\|^2 - \frac{1}{2}\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds \leq 0.$$

Then,

$$F'(0) = -\alpha\|u_1\|^2 + \delta(u_1, \phi_1).$$

Since $(u_1, \phi_1) > 0$, choose $\delta > \frac{\alpha\|u_1\|^2}{(u_1,\phi_1)}$ so that $F'(0) > 0$.

Step 2: We compute

$$F''(t) = E''(t) + \delta y''(t), \quad \text{where } y(t) = (u(t), \phi_1).$$

Differentiating $E'(t)$ and using the equation,

$$E''(t) = -\alpha\|u_t\|^2 - \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds$$
$$- g(0)\|\nabla u(t)\|^2 - \frac{d}{dt}\int_\Omega F(u(t))dx.$$

From the PDE and properties of ϕ_1, we derive

$$y''(t) = -(1 + \|\nabla u\|^2)\lambda_1 y(t) + \lambda_1 \int_0^t g(t-s)y(s)ds$$
$$- \alpha y'(t) + \int_\Omega f(u(t))\phi_1(x)dx.$$

Putting together,

$$F''(t) \leq -\alpha\|u_t\|^2 - \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds - g(0)\|\nabla u(t)\|^2$$
$$- \frac{d}{dt}\int_\Omega F(u(t))dx - \delta(1 + \|\nabla u\|^2)\lambda_1 y(t)$$
$$+ \delta\lambda_1 \int_0^t g(t-s)y(s)ds - \alpha\delta y'(t) + \delta \int_\Omega f(u(t))\phi_1(x)dx.$$

For small t, using positivity and regularity assumptions,

$$F''(t) \leq -\delta\lambda_1 \ell y(0) =: -C_0 < 0.$$

Thus, $F(t)$ is concave downward on $[0, T_0]$ for some $T_0 > 0$, and since $F(0) > 0$, $F'(0) > 0$, it must reach zero in finite time. To estimate the blow-up time, we use the concavity of the auxiliary functional $F(t) = E(t) + \delta(u(t), \varphi_1)$. Since $F(0) > 0$, $F'(0) > 0$, and $F''(t) \leq -C_0 < 0$, the functional $F(t)$ must vanish in finite time.

Using Taylor's inequality for concave functions, we obtain:

$$F(t) \leq F(0) + F'(0)t - \frac{C_0}{2}t^2.$$

We now define T^* as the time when this upper bound reaches zero:

$$F(0) + F'(0)T^* - \frac{C_0}{2}(T^*)^2 = 0.$$

Solving this quadratic inequality leads to the estimate

$$T^* \leq \frac{F'(0)}{C_0} + \sqrt{\left(\frac{F'(0)}{C_0}\right)^2 + \frac{2F(0)}{C_0}}.$$

Alternatively, for a coarser bound, we may drop the square root term and obtain

$$T^* \leq \frac{2F(0)}{F'(0)}.$$

Recalling that $F(0) = E(0) + \delta(u_0, \varphi_1)$ and $F'(0) = -\alpha\|u_1\|^2 + \delta(u_1, \varphi_1)$, and under the assumption $(u_1, \varphi_1) > 0$, choosing δ sufficiently large ensures $F'(0) > 0$. Hence, we deduce the simplified upper bound

$$T^* \leq \frac{2E(0)}{\delta(u_1, \varphi_1)},$$

which confirms that blow-up occurs in finite time and completes the proof.

□

Remark 3.3. The conditions in Theorem 3.2 are sufficient but not necessary for finite-time blow-up. For instance, the assumptions $E(0) > 0$ and $\int_\Omega u_1\varphi_1 > 0$ guarantee that the solution has enough energy to escape a potential well and grow without bound.

In contrast, if the initial energy is small or negative (e.g., $E(0) < 0$ due to small u_0), one expects global existence and even decay to zero, thanks to the damping terms. This behavior is supported by several results in the literature (e.g., [9]). These observations indicate a threshold phenomenon, with a critical energy level E^* separating global existence from blow-up, paralleling the potential well theory of Payne and Sattinger [10].

Remark 3.4. The above argument is inspired by the classic methods of Levine and later refinements for viscoelastic equations. The presence of the memory term $g(t)$ complicates the concavity analysis slightly, but as we saw, under our assumption $\int_0^\infty g(t)dt < 1$, the memory term does not destroy the concavity inequality – it only delays it slightly via the $\int_0^t g(t-s)y(s)ds$ term, which is bounded by $(1-\ell)\int_0^t y(s)ds$ effectively. Thus the conclusion of blow-up remains valid. Moreover, we note that the logarithmic term $f(u) = |u|^{p-1}\ln|u|$ does not appear explicitly in the final negativity estimate for $F''(t)$ except inside the harmless $\frac{d}{dt}\int F(u)dx$ term. This indicates that the blow-up is driven by the defocusing nature of the Kirchhoff tension together with the positive feedback of the source for large u, rather than the exact growth rate of the source.

4. Conclusion

In this paper, we investigated a nonlinear viscoelastic Kirchhoff-type wave equation with a logarithmic source. Using a fixed-point argument, we proved the local existence and uniqueness of weak solutions in the natural energy space $H_0^1(\Omega) \times L^2(\Omega)$. Under suitable conditions, we also established the finite-time blow-up of solutions through a concavity argument applied to a perturbed energy functional.

Compared with previous results on viscoelastic or Kirchhoff-type equations with polynomial sources, this work shows how the logarithmic term modifies the blow-up mechanism. The refined energy approach developed here provides a unified framework to handle both local solvability and finite-time blow-up.

Future research may address the global existence and decay of low-energy solutions, as well as extensions involving general memory kernels or higher-order models.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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