Konuralp Journal of Mathematics, 6 (1) (2018) 188-193

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Riemannian submersions endowed with a semi-symmetric non-metric connection

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Abstract

We study Riemannian submersions from a Riemannian manifold endowed with a semi-symmetric non-metric connection onto a Riemannian manifold. We give an example and investigate O'Neill's tensor fields, obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold and the fibres by computing curvatures.

Keywords: *Riemannian submersion, semi-symmetric non-metric connection, O'Neill tensors, vertical distribution.*
2010 Mathematics Subject Classification: *Primary 53C43; Secondary 53C20.*

1. Introduction

Semi-symmetric linear connection on a manifold M was introduced by Friedman and Schouten [4] and semi-symmetric metric connection was defined and studied by Hayden [6]. Yano also studied Riemannian manifolds endowed with a semi-symmetric metric connection. In particular, he investigated the following cases; (1) M is conformally flat (2) the covariant derivative of the torsion tensor vanishes. Later Agashe and Chafle [1] introduced a linear connection on a Riemannian manifold which is semi-symmetric but non-metric connection and they studied certain curvatures of such manifolds. Manifolds endowed with a semi-symmetric non-metric connection and their submanifolds have been investigated widely by many authors, see a very recent paper [11] and references therein.

On the other hand Riemannian submersions studied by O'Neill [8] and Gray [5]. In [8], the author obtained curvature relations between the total manifold and base manifold and he showed that this notion is a useful tool to calculate the curvature of certain homogeneous spaces. The theory of Riemannian submersions is an active research area and such maps have many applications in different areas and [2], [3], [9].

In this paper, we consider a Riemannian submersion from a Riemannian manifold endowed with a semi-symmetric non-metric connection onto a Riemannian manifold with Levi-Civita connection.

We first obtain basic formulas and compare with the Riemannian submersions between Riemannian manifolds with Levi-Civita connections. Then we find the derivative of O'Neill's tensor fields. We also investigate curvature relations between a Riemannian manifold with semi-symmetric non-metric connection and a Riemannian manifold with a Levi-Civita connection.

2. Introduction

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In this paper, we consider a Riemannian submersion from a Riemannian manifold endowed with a semi-symmetric non-metric connection onto a Riemannian manifold with Levi-Civita connection.

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3. Preliminaries

Let M be an $m-$dimensional Riemannian manifold with Riemannian metric g. We define a linear connection $\tilde{\nabla}$ on a Riemannian manifold M by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{3.1}$$

where $X,Y \in \chi(M)$ and ∇ is the Riemannian connection on M and η is a 1-form associated with the vector field U on M by

$$\eta(Y) = g(U,Y). \tag{3.2}$$

Using (3.1), the torsion tensor $\tilde{T}$ of M with respect to the connection $\tilde{\nabla}$ is given by

$$\begin{aligned} \tilde{T}(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= \nabla_X Y + \eta(Y)X - \nabla_Y X - \eta(X)Y - [X,Y] \\ &= \eta(Y)X - \eta(X)Y. \end{aligned} \tag{3.3}$$

A linear connection satisfying (3.3) is called a semi-symmetric connection. Moreover, by using (3.1), we obtain

$$(\tilde{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{3.4}$$

for vector fields X,Y,Z on M.
A linear connection $\tilde{\nabla}$ defined by (3.1) satisfies (3.3) and (3.4) and hence we call $\tilde{\nabla}$, a semi-symmetric non-metric connection [1].

Let (M,g) and (B,g') be two Riemannian manifolds of dimension m and n, respectively, with $m>n$. A Riemannian submersion is a smooth map $\psi : M \to B$ which is onto and satisfies the following conditions:
(i) $\psi_{*p} : T_pM \to T_{\pi(p)}B$ is onto for all $p \in M$;
(ii) ψ_* preserves scalar products of vectors normal to fibres.
The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathscr{V}$ the vertical distribution, by $\mathscr{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be basic if X is $\psi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.
We recall that the sections of $\mathscr{V}$, respectively $\mathscr{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to B$ determines two $(1,2)$ tensor field $\mathscr{T}$ and $\mathscr{A}$ on M, by the formulas:

$$\mathscr{T}(E,F) = \mathscr{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{3.5}$$

and

$$\mathscr{A}(E,F) = \mathscr{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{3.6}$$

for any $E,F \in \chi(M)$, where v and h are the vertical and horizontal projections (see, [3]). From (3.5) and (3.6), one can obtain

$$\nabla_U X = \mathscr{T}_U X + h(\nabla_U X); \tag{3.7}$$

$$\nabla_X U = v(\nabla_X U) + \mathscr{A}_X U; \tag{3.8}$$

$$\nabla_X Y = \mathscr{A}_X Y + h(\nabla_X Y), \tag{3.9}$$

for any $X,Y \in \chi^h(M)$, $U \in \chi^v(M)$. Moreover, if X is basic then $h(\nabla_U X) = h(\nabla_X U) = \mathscr{A}_X U$.

We note that for $U,V \in \chi^v(M)$, $\mathscr{T}_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X,Y \in \chi^h(M)$, $\mathscr{A}_X Y = \frac{1}{2}v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathscr{H}$. It is known that $\mathscr{A}$ is alternating on the horizontal distribution: $\mathscr{A}_X Y = -\mathscr{A}_Y X$, for $X,Y \in \chi^h(M)$ and $\mathscr{T}$ is symmetric on the vertical distribution: $\mathscr{T}_U V = \mathscr{T}_V U$, for $U,V \in \chi^v(M)$.

We now recall the following result which will be useful for later.

Lemma 2.1 (see [3],[8]). *If $\psi : M \longrightarrow B$ is a Riemannian submersion and X,Y basic vector fields on M, $\psi-$related to X' and Y' on B, then we have the following properties*

1. *$h[X,Y]$ is a basic vector field and $\psi_* h[X,Y] = [X',Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field $\psi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
3. *$[E,U] \in \Gamma(\mathscr{V})$, $\forall U \in \chi^v(M)$ and $\forall E \in \chi(M)$.*

4. Riemannian Submersions from a Riemannian manifold endowed with a semi-symmetric non-metric connection

In this section, we will give an example for Riemannian submersion and investigate Riemannin submersions defined from a Riemannian manifold with a semi-symmetric non-metric connection onto Riemannian manifold. We obtain O'Neill's tensor fields for this new connection, check the Schouten connection and derive the covariant derivative of O'Neill's tensor fields $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{T}}$.

Example 4.1. *Let $\psi : (\mathbb{R}^4, g) \longrightarrow (\mathbb{R}^2, h)$ be a submersion defined by*

$\psi(x_1, x_2, x_3, x_4) = (y_1, y_2)$,

where

$$y_1 = \sqrt{x_1^2 + x_2^2} \ \text{ and } \ y_2 = \frac{x_3 - x_4}{\sqrt{2}}.$$

Then, the Jacobian matrix of ψ is:

$$\psi_* = \begin{bmatrix} \frac{x_1}{K} & \frac{x_2}{K} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}, \end{bmatrix}$$

where $K = \sqrt{x_1^2 + x_2^2}$. A straight computations yields

$ker\psi_* = span\{Z_1 = -x_2\partial_1 + x_1\partial_2,\ Z_2 = \partial_3 + \partial_4\}$

and

$(ker\psi_*)^{\perp} = span\{H_1 = \frac{x_1}{K}\partial_1 + \frac{x_2}{K}\partial_2, H_2 = \frac{1}{\sqrt{2}}\partial_3 - \frac{1}{\sqrt{2}}\partial_4\}.$

Also by direct computations yields

$$\psi_*(H_1) = \partial v_1 \quad \text{and} \quad \psi_*(H_2) = \partial v_2.$$

Hence, it is easy to see that

$g_{\mathbb{R}^2}(\psi_*(H_i), \psi_*(H_i)) = g_{\mathbb{R}^4}(H_i, H_i), \ \ i = 1, 2.$

Thus ψ is a Riemannian submersion.

Let (M, g) be an Riemannian manifold and $\tilde{\nabla}$ is a symmetric non-metric connection on M. Let also ψ be a Riemannian submersion from M onto a Riemannian manifold N. Then, the tensor $\tilde{\mathscr{T}}$ with type $(1,2)$ on M with respect to $\tilde{\nabla}$ is given by

$\tilde{\mathscr{T}}(E,F) = \tilde{\mathscr{T}}_E F = h\tilde{\nabla}_{vE} vF + v\tilde{\nabla}_{vE} hF, \ \ E, F \in \chi(M).$

By using (3.1) we obtain,

$$\tilde{\mathscr{T}}_E F = \mathscr{T}_E F + \eta(hF)vE. \tag{4.1}$$

Moreover, the tensor $\tilde{\mathscr{A}}$ with type $(1,2)$ on M with respect to $\tilde{\nabla}$ is given by

$\tilde{\mathscr{A}}(E,F) = \tilde{\mathscr{A}}_E F = v\tilde{\nabla}_{hE} hF + h\tilde{\nabla}_{hE} vF, \ \ E, F \in \chi(M).$

In a similar way, using (3.1), we have

$$\tilde{\mathscr{A}}_E F = \mathscr{A}_E F + \eta(vF)hE. \tag{4.2}$$

In the sequel, we show that the tensor fields $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{T}}$ are not skew symmetric with respect to $\tilde{\nabla}$ contrary to the tensor fields $\mathscr{A}$ and $\mathscr{T}$.

Lemma 4.2. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M, g) onto a Riemannian manifold $(B, g^{'})$. Then we have*

(i) $g(\tilde{\mathscr{T}}_E F, G) = g(\tilde{\mathscr{T}}_E G, F) + 2g(\mathscr{T}_E F, G) + \eta(hF)g(vE, vG) - \eta(hG)g(vE, vF).$

(ii) $g(\tilde{\mathscr{A}}_E F, G) = g(\tilde{\mathscr{A}}_E G, F) + 2g(\mathscr{A}_E F, G) + \eta(vF)g(hE, hG) - \eta(vG)g(hE, hF)$

for any $E, F, G \in \chi(M)$.

Proof. (i) For any $E, F, G \in \chi(M)$, by using (4.1), we get

$$\begin{aligned} g(\tilde{\mathscr{T}}_E F, G) &= g(\mathscr{T}_E F, hG) + g(\eta(hF)vE, hG) \\ &\quad + g(\mathscr{T}_E F, vG) + g(\eta(hF)vE, vG) \\ &= g(\mathscr{T}_E F, G) + \eta(hF)g(vE, vG). \end{aligned} \tag{4.3}$$

In a similar way, we have

$$g(\tilde{\mathscr{T}}_E G, F) = g(\mathscr{T}_E G, F) + \eta(hG)g(vE, vF). \tag{4.4}$$

Substracting (4.3) from (4.4), which gives (i). By using the same way, one can easily get (ii). This completes the proof of the lemma. □

We now check symmetry properties of $\tilde{\mathscr{T}}$ and $\tilde{\mathscr{A}}$.

Proposition 4.3. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. Then,*

$$\tilde{\mathscr{T}}_U W = \tilde{\mathscr{T}}_W U, \tag{4.5}$$

$$\tilde{\mathscr{A}}_X Y = \tilde{\mathscr{A}}_Y X + 2\mathscr{A}_X Y, \tag{4.6}$$

for $U,W \in \chi^v(M)$ and $X,Y \in \chi^h(M)$.

Proof. Since $\tilde{\mathscr{T}}_U W = \mathscr{T}_U W$ for $U,W \in \chi^v(M)$ and $\mathscr{T}$ is symmetric on the vertical distribution, we get (4.5). In a similar way, since $\tilde{\mathscr{A}}_X Y = \mathscr{A}_X Y$ for $X,Y \in \chi^h(M)$, and $\mathscr{A}$ is anti-symmetric on the horizontal distribution, we obtain (4.6). □

Combining definition of tensor field $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{T}}$, and using Lemma 4.2 and Proposition 4.3, we have the following equations.

Lemma 4.4. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. Then,*

$$\tilde{\nabla}_V W = \mathscr{T}_V W \ + \hat{\nabla}_V W \tag{4.7}$$

$$\tilde{\nabla}_V X = \mathscr{T}_V X + h\tilde{\nabla}_V X + \eta(X)V \tag{4.8}$$

$$\tilde{\nabla}_X V = \mathscr{A}_X V + v\tilde{\nabla}_X V + \eta(V)X \tag{4.9}$$

$$\tilde{\nabla}_X Y = \mathscr{A}_X Y + h\tilde{\nabla}_X Y \tag{4.10}$$

for $V,W \in \chi^v(M)$ and $X,Y \in \chi^h(M)$, where $\hat{\nabla}_V W = v\tilde{\nabla}_V W$. If X is basic, then $h\tilde{\nabla}_V X = h\tilde{\nabla}_X V = \mathscr{A}_X V$.

Proof. Since ∇ is a Levi-Civita connection, using (3.1), we obtain (4.7). The other assertions can be obtain in a similar way. □

We now obtain Covariant derivatives of the tensor fields $\tilde{\mathscr{T}}$ and $\tilde{\mathscr{A}}$ in terms of $\mathscr{A}$ and $\mathscr{T}$. We first we recall the following definition.

Definition 4.5. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. The covariant derivatives of $\mathscr{A}$ ve $\mathscr{T}$ are given by*

$$(\tilde{\nabla}_E \mathscr{A})_F H = (\tilde{\nabla}_E \mathscr{A})(F,H) = \tilde{\nabla}_E(\mathscr{A}_F H) - \mathscr{A}_{\tilde{\nabla}_E F}(H) - \mathscr{A}_F(\tilde{\nabla}_E H)$$

and

$$(\tilde{\nabla}_E T)_F H = (\tilde{\nabla}_E \mathscr{T})(F,H) = \tilde{\nabla}_E(\mathscr{T}_F H) - \mathscr{T}_{\tilde{\nabla}_E F}(H) - \mathscr{T}_F(\tilde{\nabla}_E H)$$

for any $E,F,G \in \chi(M)$.

Lemma 4.6. *Let $\psi : M \to B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. Then,*

$$(\tilde{\nabla}_V \mathscr{A})W = -\mathscr{A}_{\mathscr{T}_V W} E, \tag{4.11}$$

$$(\tilde{\nabla}_X \mathscr{T})W = -\mathscr{T}_{\mathscr{A}_X Y} E, \tag{4.12}$$

$$(\tilde{\nabla}_X \mathscr{A})W = -\mathscr{A}_{\mathscr{A}_X W + \eta(W)X} E, \tag{4.13}$$

$$(\tilde{\nabla}_V \mathscr{T})Y = -\mathscr{T}_{\mathscr{T}_V Y + \eta(Y)V} E \tag{4.14}$$

for any $X,Y \in \chi^h(M)$ and $V,W \in \chi^v(M)$.

Proof. Let E be an arbitrary vector field on M. Then

$$(\tilde{\nabla}_V \mathscr{A})_W E = (\tilde{\nabla}_V \mathscr{A})(W,E) = \tilde{\nabla}_V(\mathscr{A}_W E) - \mathscr{A}_{\tilde{\nabla}_V W}(E) - \mathscr{A}_W(\tilde{\nabla}_V E).$$

Since $\mathscr{A}$ is horizontal, we see that $\mathscr{A}_W = \mathscr{A}_{hW} = 0$. On the other hand, by using (4.7) we have

$$\mathscr{A}_{\tilde{\nabla}_V W} E = \mathscr{A}_{\mathscr{T}_V W + v\tilde{\nabla}_V W} E = \mathscr{A}_{\mathscr{T}_V W} E + \mathscr{A}_{v\tilde{\nabla}_V W} E = \mathscr{A}_{\mathscr{T}_V W} E.$$

which gives (4.11). One can easily get the others assertions. This completes the proof of the lemma. □

5. Curvature relations with respect to semi-symmetric non-metric connection

In the last section, we are going to obtain curvature relations between base manifold and total manifold by using semi-symmetric non-metric connection $\tilde{\nabla}$. We first note that we denote curvature tensor fields of $\tilde{\nabla}$ by $\tilde{R}$.

Theorem 5.1. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. We denote Riemannian curvatures of M and any fibre $(\psi^{-1}(x),\hat{g}_x)$ by $\tilde{R}$, $\hat{R}$ respectively. Then we have*

$$g(\tilde{R}(U,V)W,F) = g(\hat{R}(U,V)W,F) - g(\mathscr{T}_U F,\mathscr{T}_V W) + g(\mathscr{T}_V F,\mathscr{T}_U W) + \eta(\mathscr{T}_V W)g(U,F) - \eta(\mathscr{T}_U W)g(V,F) \tag{5.1}$$

$$g(\bar{R}(U,V)W,X) = g((\bar{\nabla}_U \mathscr{T})_V W,X) - g((\bar{\nabla}_V \mathscr{T})_U W,X) \tag{5.2}$$

for $U,V,W,F \in \chi^v(M)$ and $X \in \chi^h(M)$.

Proof. We define Riemannian curvature tensor of M with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ by

$$\tilde{R}(U,V)W = \tilde{\nabla}_U \tilde{\nabla}_V W + \tilde{\nabla}_V \tilde{\nabla}_U W - \tilde{\nabla}_{[U,V]} W. \tag{5.3}$$

Then by using (4.7) and a straightforwad computation, we obtain

$$\begin{aligned} \tilde{R}(U,V)W = {} & h\bar{\nabla}_U \mathscr{T}_V W + \mathscr{T}_U \mathscr{T}_V W + \eta(\mathscr{T}_V W)U \\ & + \mathscr{T}_U \hat{\nabla}_V W + \hat{\nabla}_U \hat{\nabla}_V W \\ & - h\bar{\nabla}_V \mathscr{T}_U W - \mathscr{T}_V \mathscr{T}_U W - \eta(\mathscr{T}_U W)V \\ & - \mathscr{T}_V \hat{\nabla}_U W - \hat{\nabla}_V \hat{\nabla}_U W - h\bar{\nabla}_{[U,V]} W - \hat{\nabla}_{[U,V]} W \end{aligned} \tag{5.4}$$

for $U,V,W \in \chi^v(M)$. Taking the inner product in above equation with F, we get (5.1). In a similar way, taking the inner product in (5.4) with X, we get (5.2). □

Theorem 5.2. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. We denote Riemannian curvatures of M and B by $\tilde{R}$, $R^{'}$ respectively. Then we have*

$$g(\tilde{R}(X,Y)Z,H) = g(R^{'}(X,Y)Z,H) - g(\mathscr{A}_X H,\mathscr{A}_Y Z) + g(\mathscr{A}_Y H,\mathscr{A}_X Z) + 2g(\mathscr{A}_Z H,\mathscr{A}_X Y) + \eta(\mathscr{A}_Y Z)g(X,H) - \eta(\mathscr{A}_X Z)g(Y,H) \tag{5.5}$$

$$g(\tilde{R}(X,Y)Z,V) = -2g(\mathscr{T}_V Z,\mathscr{A}_X Y) + g(\tilde{\nabla}_X \mathscr{A}_Y Z,V) - g(\mathscr{A}_Y \tilde{\nabla}_X Z,V) - g(\tilde{\nabla}_Y \mathscr{A}_X Z,V) + g(\mathscr{A}_X \tilde{\nabla}_Y Z,V) - \eta(Z)g([X,Y],V) \tag{5.6}$$

for any $X,Y,Z,H \in \chi^h(M)$ and $V,W \in \chi^v(M)$.

Proof. Since the two equations are tensor equations, we can assume that X,Y and Z are basic vector fields whose brakets are vertical. Then $[X,Y] = 2\mathscr{A}_X Y$. Using Riemannian curvature tensor with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ and using (1.16) we obtain

$$\begin{aligned} \tilde{R}(X,Y)Z = {} & \nabla^{'}_X \nabla^{'}_Y Z - \nabla^{'}_Y \nabla^{'}_X Z + \mathscr{A}_X \mathscr{A}_Y Z - \mathscr{A}_Y \mathscr{A}_X Z - 2\mathscr{A}_Z \mathscr{A}_X Y \\ & - 2\mathscr{T}_{\mathscr{A}_X Y} Z + v\tilde{\nabla}_X \mathscr{A}_Y Z - v\tilde{\nabla}_Y \mathscr{A}_X Z + \mathscr{A}_X \nabla^{'}_Y Z - \mathscr{A}_Y \nabla^{'}_X Z \\ & + \eta(\mathscr{A}_Y Z)X - \eta(\mathscr{A}_X Z)Y - \eta(Z)[X,Y]. \end{aligned} \tag{5.7}$$

Taking the inner product in (5.7) with H, we obtain (5.5). In a similar way, taking the inner product in (5.7) with X, we obtain

$$\begin{aligned} g(\tilde{R}(X,Y)Z,V) = {} & g(R^{'}(X,Y)Z,V) - 2g(\mathscr{T}_{\mathscr{A}_X Y} Z,V) + g(\tilde{\nabla}_X \mathscr{A}_Y Z,V) \\ & - g(\tilde{\nabla}_Y \mathscr{A}_X Z,V) + g(\mathscr{A}_X \nabla^{'}_Y Z,V) - g(\mathscr{A}_Y \nabla^{'}_X Z,V) \\ & - \eta(Z)g([X,Y],V). \end{aligned} \tag{5.8}$$

Since $[X,Y]$ is vertical we have $\tilde{\nabla}_X Y = \tilde{\nabla}_Y X$, we get (5.6). □

Theorem 5.3. *Let $\psi : M \longrightarrow B$ be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold $(B,g^{'})$. We denote Riemannian curvatures of M, B and any fibre $(\psi^{-1}(x),\hat{g}_x)$ by $\tilde{R}$, $R^{'}$ and $\hat{R}$ respectively. Let X and Y are ortonormal horizontal vector fields and U and V be ortonormal vertical vector fields. Then*

$$\tilde{K}(U,V) = \hat{K}(U,V) + ||\mathscr{T}_U V||^2 - g(\mathscr{T}_U U,\mathscr{T}_V V) + \eta(\mathscr{T}_V V) - \eta(\mathscr{T}_U V)g(U,V) \tag{5.9}$$

$$\tilde{K}(X,Y) = K^{'}(X^{'},Y^{'}) \circ \pi - 3||\mathscr{A}_X Y||^2 - \eta(\mathscr{A}_X Y)g(X,Y) \tag{5.10}$$

Proof. Substituting $W = V$ and $F = U$ in (5.1), we have

$$\tilde{K}(U,V) = g(\tilde{R}(U,V)V,U) = g(\hat{R}(U,V)V,U) - g(\mathscr{T}_U U, \mathscr{T}_V V) + g(\mathscr{T}_V U, \mathscr{T}_U V) + \eta(\mathscr{T}_V V)g(U,U) - \eta(\mathscr{T}_U V)g(V,U).$$

Since $\mathscr{T}$ is a symmetric on the vertical distribution, we get (5.9). In a similar way, substituting $Z = Y$ and $H = X$ in (5.5), we obtain

$$\tilde{K}(X,Y) = g(\tilde{R}(X,Y)Y,X) = g(R^{'}(X,Y)Y,X) - g(\mathscr{A}_X X, \mathscr{A}_Y Y) + g(\mathscr{A}_Y X, \mathscr{A}_X Y) + 2g(\mathscr{A}_Y X, \mathscr{A}_X Y) + \eta(\mathscr{A}_Y Y)g(X,X) - \eta(\mathscr{A}_X Y)g(Y,X).$$

Since $\mathscr{A}_X X = 0$, $\mathscr{A}_Y Y = 0$ and $\mathscr{A}$ is an anti-symmetric on the horizontal distribution, we get (5.10). This completes the proof of the theorem. □

Acknowledgement

The author is grateful to the referee for his/her valuable comments and suggestions.

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