Hacettepe Journal of Mathematics and Statistics
Volume 46 (4) (2017), 579 – 592

On the Padovan p-numbers

Ömür DEVECİ * and Erdal KARADUMAN†

Abstract

In this paper, we define the Padovan p-numbers and then we obtain their miscellaneous properties such as the generating matrix, the Binet formula, the generating function, the exponential representation, the combinatorial representations, the sums and permanental representation. Also, we study the Padovan p-numbers modulo m. Furthermore, we define Padovan p-orbit of a finite group and then, we obtain the length of the Padovan p-orbits of the quaternion group Q_{2^n}, $(n \geq 3)$.

Keywords: The Padovan p-numbers, Matrix, Sequence, Group, Length.

2000 AMS Classification: 11B50, 05A15, 11C20, 20D60, 20F05.

Received : 16.06.2015 *Accepted* : 11.10.2016 *Doi* : 10.15672/HJMS.20174622760

1. Introduction and Preliminaries

The well-known the Fibonacci sequence is defined by initial values $F_0 = 0$, $F_1 = 1$ and recurrence relation

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

The Padovan sequence is the sequence of integers $P(n)$ defined by initial values $P(0) = P(1) = P(2) = 1$ and recurrence relation

$$P(n) = P(n-2) + P(n-3) \text{ for } n \geq 3.$$

The Padovan sequence is

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \ldots.$$

*Department of Mathematics, Faculty of Science and Letters, Kafkas University, 36100 Kars, TURKEY
Email : odeveci36@hotmail.com
†Department of Mathematics, Faculty of Science, Atatürk University , 25240 Erzurum, TURKEY
Email : eduman@atauni.edu.tr

The Padovan numbers are generated by a matrix Q,

$$Q=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

The powers of Q give

$$Q^n=\begin{bmatrix} P(n-5) & P(n-3) & P(n-4) \\ P(n-4) & P(n-2) & P(n-3) \\ P(n-3) & P(n-1) & P(n-2) \end{bmatrix}.$$

For more information on this sequence, see [13].
Kalman [14] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [14], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by some authors; see for example, [9, 10, 11, 12, 15, 16, 19, 20, 21, 22, 24, 25, 27, 28, 29, 31]. In Section 2, we define the Padovan p-numbers. Then we obtain the generating matrix, the Binet formula, the generating function, the exponential representation, the combinatorial representations, the sums and permanental representation of the Padovan p-numbers. The study of the linear recurrence sequences in algebraic structures began with the earlier work of Wall [30] where the ordinary Fibonacci sequences in cyclic groups were investigated. Recently, many authors have studied some special linear recurrence sequences in algebraic structures; see for example, [1, 3, 5, 6, 7, 8, 17, 23, 26]. In Section 3, we study the Padovan p-numbers modulo m. Also in this section, we give the definition of Padovan p-sequences in groups generated by two or more elements. Then we examine these sequences in finite groups. Furthermore, we obtain the periods of the Padovan p-sequences of the quaternion group Q_{2^n}, $(n \geq 3)$ as the applications of obtained results in Section 3.

2. The Padovan p-Numbers

Now we define the Padovan p-numbers by the following homogeneous linear recurrence relation for any given $p\,(p = 2, 3, 4, \ldots)$ and $n \geq 1$

(2.1) $\quad Pap\,(n+p+2) = Pap\,(n+p) + Pap\,(n)$

with initial conditions $Pap\,(1) = Pap\,(2) = \cdots = Pap\,(p) = 0$, $Pap\,(p+1) = 1$ and $Pap\,(p+2) = 0$.
When $p = 2$ in (2.1), we obtain $Pa2\,(2n+1) = F_n$ for $n \geq 1$.
By equation (2.1), we have

$$\begin{bmatrix} Pap\,(n+p+2) \\ Pap\,(n+p+1) \\ \vdots \\ Pap\,(n+2) \\ Pap\,(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Pap\,(n+p+1) \\ Pap\,(n+p) \\ \vdots \\ Pap\,(n+1) \\ Pap\,(n) \end{bmatrix}$$

for the sequence of the Padovan p-numbers. Letting

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The matrix M is said to be a Padovan p-matrix. Also, we obtain that

$$M^n = \begin{bmatrix} Pap(n+p+1) & Pap(n+p+2) & Pap(n+1) & Pap(n+2) & \cdots & Pap(n+p) \\ Pap(n+p) & Pap(n+p+1) & Pap(n) & Pap(n+1) & \cdots & Pap(n+p-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Pap(n+1) & Pap(n+2) & Pap(n-p+1) & Pap(n-p+2) & \cdots & Pap(n) \\ Pap(n) & Pap(n+1) & Pap(n-p) & Pap(n-p+1) & \cdots & Pap(n-1) \end{bmatrix} \tag{2.2}$$

for $n \geq 1$, which can be proved by mathematical induction. We easily derive that

$$\det M = (-1)^{p+1}. \tag{2.3}$$

By equation (2.3), we have $\det M^n = (-1)^{np+n}$. Now, we can give a formula for Padovan p-numbers ($n \geq 1$) by using this determinantal representation.

2.1. Lemma. *The characteristic equation of the Padovan p-numbers $x^{p+2} - x^p - 1 = 0$ does not have multiple roots.*

Proof. Let α be a root of $f\,(x) = 0$ where $f\,(x) = x^{p+2} - x^p - 1$ so that $\alpha \notin \{0, 1\}$. If possible, α is a multiple root in which case $f\,(\alpha) = f^{'}\,(\alpha) = 0$. Now $f^{'}\,(\alpha) = 0$ and $\alpha \neq 0$ give $\alpha^2 = \frac{p}{p+2}$ while $f\,(\alpha) = 0$ shows $\alpha^p\left(\alpha^2 - 1\right) - 1 = 0$ so that $\left(\frac{p}{p+2}\right)^{\frac{p}{2}} \cdot \left(\frac{-2}{p+2}\right) = 1$, an impossibility since the left hand side is less than 1 for $p \geq 2$. This contradiction proves the Lemma. □

Let $f\,(u)$ be the characteristic polynomial of the Padovan p-matrix M, then $f\,(u) = u^{p+2} - u^p - 1$. If $u_1, u_2, \ldots, u_{p+2}$ are eigenvalues of the matrix M, then by Lemma 2.1, they are distinct. Let V_p be a $(p+2) \times (p+2)$ Vandermonde matrix such that

$$V_p = \begin{bmatrix} u_1^{p+1} & u_2^{p+1} & \cdots & u_{p+2}^{p+1} \\ u_1^p & u_2^p & \cdots & u_{p+2}^p \\ \vdots & \vdots & \cdots & \vdots \\ u_1 & u_2 & \cdots & u_{p+2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{2.4}$$

Let $V_p^{(i,j)}$ be a $(p+2)\times(p+2)$ matrix obtained from V_p by replacing the j^{th} column of V_p by A_p^i, where, A_p^i is a $(p+2)\times 1$ matrix as follows:

$$A_p^i = \begin{bmatrix} u_1^{n+p+2-i} \\ u_2^{n+p+2-i} \\ \vdots \\ u_{p+2}^{n+p+2-i} \end{bmatrix}.$$

Now we consider the Binet formula for the Padovan p-numbers. We give the following Theorem.

2.2. Theorem. *Let* $M^n = [m_{ij}]$ *. Then,* $m_{ij} = \frac{\det\left(V_p^{(i,j)}\right)}{\det(V_p)}$.

Proof. Since the eigenvalues of the matrix M are distinct, the matrix M is diagonalizable. Let $D = (u_1, u_2, \ldots, u_{p+2})$, then it is easy to see that $MV_p = V_pD$. Since V_p is invertible, $(V_p)^{-1} MV_p = D$. Thus, the matrix M is similar to D. So we get $M^nV_p = V_pD^n$ for $n \geq 1$. Then we have the following linear system of equations for $n \geq 1$:

$$\begin{aligned} &m_{i1}u_1^{p+1} + m_{i2}u_1^{p} + \cdots + m_{ip+2} = u_1^{n+p+2-i} \\ &m_{i1}u_2^{p+1} + m_{i2}u_2^{p} + \cdots + m_{ip+2} = u_2^{n+p+2-i} \\ &\vdots \\ &m_{i1}u_{p+2}^{p+1} + m_{i2}u_{p+2}^{p} + \cdots + m_{ip+2} = u_{p+2}^{n+p+2-i}. \end{aligned}$$

So, for each $i,\ j = 1, 2, \cdots, p+2$, we obtain m_{ij} as follows:

$$m_{ij} = \frac{\det\left(V_p^{(i,j)}\right)}{\det(V_p)}.$$

Theorem 2.1 gives immediately: $\square$

2.3. Corollary. $Pap\,(n) = \frac{\det\left(V_p^{(p+2,1)}\right)}{\det(V_p)} = \frac{\det\left(V_p^{(2,3)}\right)}{\det(V_p)} = \frac{\det\left(V_p^{(p+1,p+2)}\right)}{\det(V_p)}$.

Now we give the generating function of the Padovan p-numbers and an exponential representation for the Padovan p-numbers with the following Theorem.

2.4. Theorem. *The generating function $g(x)$ of the Padovan p-numbers is given by*

$$g(x) = \frac{1}{1 - x^2 - x^{p+2}}$$

for $0 \leq x^2 + x^{p+2} < 1$ *and it has exponential representation*

$$g(x) = \exp\left(\sum_{i=1}^{\infty} \frac{x^{2i}}{i}(1+x^p)^i\right).$$

Proof. Let $g(x)$ be a generating function for the Padovan p-numbers. Then

$$\begin{aligned} g(x) = {} & Pap\,(p+1) + Pap\,(p+2)\,x + Pap\,(p+3)\,x^2 + \cdots + \\ & + Pap\,(p+n+1)\,x^n + Pap\,(p+n+2)\,x^{n+1} + \cdots. \end{aligned} \tag{2.5}$$

By the definition of the Padovan p-numbers, we can write

$$g(x) - x^2 g(x) - x^{p+2} g(x) = Pap\,(p+1) = 1.$$

So we get

$$g(x) = \frac{1}{1 - x^2 - x^{p+2}}$$

for $0 \leq x^2 + x^{p+2} < 1$. Also by a simple calculation, we obtain

$$\ln g(x) = -\ln\left\{1 - x^2(1 + x^p)\right\} = \sum_{i=1}^{\infty} \frac{x^{2i}}{i}(1 + x^p)^i .$$

Thus the proof is complete. $\square$

Now we give a combinatorial representation for the Padovan p-numbers by the following Theorem.

2.5. Theorem.

$$Pap(n + p + 1) = \sum_{\frac{n}{p+2} \leq m \leq n} \binom{m}{j}$$

where $j = \frac{n-2m}{p}$.

Proof. From (2.5), it is clear that the coefficient of x^n in $g(x)$ is $Pap(p + n + 1)$. Since

$$g(x) = \frac{1}{1 - x^2 - x^{p+2}} = \frac{1}{1 - (x^2 + x^{p+2})} = 1 + \left(x^2 + x^{p+2}\right) +$$

$$+\left(x^2 + x^{p+2}\right)^2 + \cdots + \left(x^2 + x^{p+2}\right)^n + \cdots$$

$$= 1 + x^2(1 + x^p) + x^4 \sum_{j=0}^{2} \binom{2}{j} x^{pj} + \cdots + x^{2n} \sum_{j=0}^{n} \binom{n}{j} x^{pj} + \cdots ,$$

we only consider the first $n + 1$ terms on the right-side. By the binomial theorem, we can write

$$\left(x^2 + x^{p+2}\right)^m = \left(x^2(1 + x^p)\right)^m = x^{2m} \sum_{j=0}^{m} \binom{m}{j} x^{pj}.$$

Then by the above equation we see that the coefficient of x^n in $\left(x^2 + x^{p+2}\right)^m$ for positive m and n is

$$\binom{m}{j}$$

where $j = \frac{n-2m}{p}$. Thus the proof is complete. $\square$

Let $E(e_1, e_2, \ldots, e_l)$ be the $l \times l$companion matrix

$$E(e_1, e_2, \cdots, e_l) = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_l \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

2.6. Theorem. ***(Chen and Louck*** [4]***).*** *The* $(i,\ j)$ *entry* $e_{ij}^{(n)}(e_1, e_2, \ldots, e_l)$ *in the matrix* $E^n(e_1, e_2, \ldots, e_l)$ *is given by the following formula:*

(2.6)

$$e_{ij}^{(n)}(e_1, e_2, \ldots, e_l) = \sum_{(t_1, t_2, \cdots, t_l)} \frac{t_j + t_{j+1} + \cdots + t_l}{t_1 + t_2 + \cdots + t_l} \times \binom{t_1 + t_2 + \cdots + t_l}{t_1, t_2, \ldots, t_l} e_1^{t_1} \cdots e_l^{t_l}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + lt_l = n - i + j$, $\binom{t_1 + t_2 + \cdots + t_l}{t_1, t_2, \ldots, t_l} = \frac{(t_1 + t_2 + \cdots + t_l)!}{t_1! t_2! \cdots t_l!}$ *is a multinomial coefficient, and the coefficients in (2.6) are defined to be 1 if* $n = i - j$.

Now we give other combinatorial representations than the above for the Padovan p-numbers.

2.7. Corollary. ***i.***

$$Pap\,(n) = \sum_{(t_1,t_2,\ldots,t_{p+2})} \binom{t_1+t_2+\cdots+t_{p+2}}{t_1,t_2,\cdots,t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+2)\,t_{p+2} = n-p-1$.

ii.

$$Pap\,(n) = \sum_{(t_1,t_2,\ldots,t_{p+2})} \frac{t_{p+2}}{t_1+t_2+\cdots+t_{p+2}} \binom{t_1+t_2+\cdots+t_{p+2}}{t_1,t_2,\ldots,t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+2)\,t_{p+2} = n+1$.

iii.

$$Pap\,(n) = \sum_{(t_1,t_2,\cdots,t_{p+2})} \frac{t_3+t_4+\cdots+t_{p+2}}{t_1+t_2+\cdots+t_{p+2}} \binom{t_1+t_2+\cdots+t_{p+2}}{t_1,t_2,\ldots,t_{p+2}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+2)\,t_{p+2} = n+1$.

Proof. If we take $i=p+2$ and $j=1$ for case i., $i=p+1$ and $j=p+2$ for case ii. and $i=2$ and $j=3$ for case iii. in Theorem 2.4, then we can directly see the conclusions from equation (2.2). $\square$

Let the sums of the Padovan p-numbers from $p+1$ to $p+n$ be denoted by S_n, that is,

$$S_n = \sum_{i=1}^{n} Pap\,(p+i) \tag{2.7}$$

and let T and K_n be the $(p+3)\times(p+3)$ matrices

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \quad 0 \quad \cdots \quad 0 \quad 0 \\ 1 & \\ 0 & \\ 0 & M \\ \vdots & \\ 0 & \end{bmatrix}$$

and

$$K_n = \begin{bmatrix} 1 & 0 \quad 0 \quad \cdots \quad 0 \quad 0 \\ S_n & \\ S_{n-1} & \\ \vdots & M^n \\ \vdots & \\ S_{n-p-1} & \end{bmatrix}.$$

Then by induction on n, it is easy to see that $K_n = T^n$.

Now we can give the sums of the Padovan p-numbers by the following Theorem.

2.8. Theorem. *Let the sums of the Padovan p-numbers from $p+1$ to $p+n$, S_n be as in (2.7). Then*

$$S_n = \sum_{i=0}^{p+1} Pap\,(n+p+2-i) - 1.$$

Proof. Let U and D_1 be the $(p+3)\times(p+3)$ matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & u_1^{p+1} & u_2^{p+1} & \cdots & u_{p+2}^{p+1} \\ -1 & u_1^{p} & u_2^{p} & \cdots & u_{p+2}^{p} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & u_1 & u_2 & \cdots & u_{p+2} \\ -1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} 1 & & & \\ & u_1 & & \\ & & \ddots & \\ & & & u_{p+2} \end{bmatrix}$$

where $u_1, u_2, \ldots, u_{p+2}$ are the roots of the equation $u^{p+2} - u^p - 1 = 0$. Expanding $\det(U)$ by the Laplace expansion of the determinant with respect to the first row gives us $\det(U) = \det(V_p)$ where V_p is as in (2.4). It is easy to see that $(1-u)\left(u^{p+2} - u^p - 1\right) = 0$ is the characteristic equation of the matrix U and the eigenvalues of U are $1, u_1, u_2, \ldots, u_{p+2}$. By Lemma 2.1, it is known that $1, u_1, u_2, \cdots, u_{p+2}$ are distinct. Hence, the matrix U is diagonalizable. Also, we can write $TU = UD_1$. Since the matrix U is invertible, $U^{-1}TU = D_1$. Thus, the matrix T is similar to the matrix D_1. Then $T^nU = UD_1^n$, and hence $K_nU = UD_1^n$. Since $S_n = k_{2,1}$ where $K_n = [k_{ij}]$, by using matrix multiplication, we can write

$$S_n - \left(\sum_{i=0}^{p+1} Pap\,(n+p+2-i)\right) = -1.$$

So we get

$$S_n = \sum_{i=0}^{p+1} Pap\,(n+p+2-i) - 1.$$

□

Now we consider the relationship between the Padovan p-numbers and the permanent of a certain matrix which is obtained using the Padovan p-matrix M.

2.9. Definition. An $n \times m$ real matrix $C = [c_{ij}]$ is called a contractible matrix in the α^{th} column (resp. row.) if the α^{th} column (resp. row.) contains exactly two non-zero entries.

Let $u_1, u_2, \ldots, u_n$ be row vectors of the matrix C and let C be contractible in the α^{th} column with $c_{i\alpha} \neq 0$, $c_{j\alpha} \neq 0$ and $i \neq j$. Then the $(n-1)\times(m-1)$ matrix $C_{ij:\alpha}$ obtained from C by replacing the i^{th} row with $c_{i\alpha}u_j + c_{j\alpha}u_i$ and deleting the j^{th} row and the α^{th} column is called the contraction in the α^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson showed that $per\,(A) = per\,(B)$ if A is a real matrix of order $n > 1$ and B is a contraction of A.

Let p be a fixed integer such that $p \geq 2$ and let $A_p^n = [a_{ij}]$ be the $n \times n$ super-diagonal matrix with $a_{i,i+1} = a_{i+1,i} = a_{i,i+p+1} = 1$ and for all i and 0 otherwise, that is,

$$A_p^n = \begin{array}{c} \begin{array}{ccccccccc} & & & & & (p+2)\,\text{th} & & & \\ & & & & & \downarrow & & & \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \ddots & & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Note that if $n = 1$, $A_p^1 = 0$.

2.10. Theorem. *The permanent of A_p^n $(n \geq 1,\ p \geq 2)$ is $Pap\,(n+p+1)$.*

Proof. We prove this by induction. First, let us consider the case $n < p+2$. From the definitions of the matrix A_p^n and the Padovan p-numbers it is clear that $perA_p^1 = Pap\,(p+2) = 0$ and $perA_p^2 = Pap\,(p+3) = 1$. Also, we obtain the following matrix for $3 \leq \lambda \leq p+1$

$$A_p^\lambda = \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}.$$

Then

$$perA_p^\lambda = \begin{cases} 1 & \text{if } \lambda \text{ is even}, \\ 0, & \text{if } \lambda \text{ is odd}. \end{cases}$$

Furthermore, we know that

$$Pap\,(\lambda+p+1) = \begin{cases} 1 & \text{if } \lambda \text{ is even}, \\ 0, & \text{if } \lambda \text{ is odd}, \end{cases} \quad \text{for } 3 \leq \lambda \leq p+1.$$

So we get $perA_p^n = Pap\,(n+p+1)$ for $1 \leq n \leq p+1$.
Now, let us consider the case $n \geq p+2$. Suppose that the equation holds for $n \geq p+2$. Then we show that the equation holds for $n+1$. If we expand the $perA_p^n$ by the Laplace expansion of the permanent, we obtain

$$perA_p^{n+1} = perA_p^{n-1} + perA_p^{n-p-1}.$$

Since $perA_p^{n-1} = Pap\,(n+p)$ and $perA_p^{n-p-1} = Pap\,(n)$, we get
$perA_p^{n+1} = Pap\,(n+p+2)$.
So, the proof is complete. $\square$

3. The Padovan p-Sequences in Groups

We consider the Padovan p-numbers modulo m.
Reducing the Padovan p-sequence $\{Pap\,(n)\}$ by a modulus m, we can get the repeating sequence, denoted by

$$\{Pap_m\,(n)\} = \{\ Pap_m\,(1)\,,\ Pap_m\,(2)\,,\ldots, Pap_m\,(p+2)\,,\ldots,\ Pap_m\,(i)\,,\ \ldots\}$$

where $Pap_m\,(i) \equiv Pap\,(i)\,(\mathrm{mod}\ m)$. It has the same recurrence relation as in (2.1).
A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the *period* of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, ...$ is periodic after the initial element a and has period 3. *A sequence is simply periodic with period k* if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, ...$ is simply periodic with period 4.

3.1. Theorem. *The sequence $\{Pap_m\,(n)\}$ is simply periodic.*

Proof. Let $X_p = \{\,(x_1, x_2, \cdots, x_{p+2})|\ x_i{}'$s are integers such that $0 \le x_i \le m-1\}$, then $|X_p| = m^{p+2}$. Since there are m^{p+2} distinct $(p+2)$-tuples of elements of $\mathbb{Z}_m$, at least one of the $(p+2)$-tuples appears twice in the sequence $\{Pap_m\,(n)\}$. Therefore, the subsequence following this $(p+2)$-tuple repeats; hence, the sequence $\{Pap_m\,(n)\}$ is periodic. So if

$$Pap_m\,(u+1) = Pap_m\,(v+1)\,,\ Pap_m\,(u+2) = Pap_m\,(v+2)\,,\ldots,$$
$$Pap_m\,(u+p+2) = Pap_m\,(v+p+2)$$

and $v > u$, then $v \equiv u\,(\mathrm{mod}\ p+2)$. From the definition, we can easily derive that

$$Pap\,(n) = Pap\,(n+p+2) - Pap\,(n+p)\,.$$

Thus we obtain

$$\begin{aligned}&Pap_m\,(u) = Pap_m\,(v)\,,\ Pap_m\,(u-1) = Pap_m\,(v-1)\,,\ldots,\\ &Pap_m\,(2) = Pap_m\,(v-u+2)\,,\ Pap_m\,(1) = Pap_m\,(v-u+1)\,,\end{aligned}$$

which implies that the sequence is simply periodic. □

We denote the period of the sequence $\{Pap_m\,(n)\}$ by $hPap_m$.

3.2. Example. The sequence $\{Pa3_2\,(n)\}$ is

$$\{0,0,0,1,0,1,0,1,1,1,0,1,1,0,0,0,1,1,1,1,1,0,0,1,1,0,1,0,0,1,0,0,0,0,1,0,\ldots\}\,.$$

Since $Pa3_2\,(32) = Pa3_2\,(1) = 0$, $Pa3_2\,(33) = Pa3_2\,(2) = 0$, $Pa3_2\,(34) = Pa3_2\,(3) = 0$, $Pa3_2\,(35) = Pa3_2\,(4) = 1$ and $Pa3_2\,(36) = Pa3_2\,(5) = 0$, the sequence is simply periodic with period $hPa3_2 = 31$.

Given an integer matrix $A = [a_{ij}]$, $A\ \ (\mathrm{mod}\ m)$ means that all entries of A are reduced modulo m, that is, $A\ \ (mod\ m) = (a_{ij}\ \ (mod\ m))$. Let us consider the set $\langle A\rangle_m = \left\{\,A^i\ (\mathrm{mod}\ m)\middle|\, i \ge 0\right\}$. If $\gcd(m,\ \det A) = 1$, then the set $\langle A\rangle_m$ is a cyclic group. Let the notation $\left|\langle A\rangle_m\right|$ denote the order of the set $\langle A\rangle_m$. By equation (2.2), it is clear that the set $\langle M\rangle_m$ is a cyclic group for every positive integer m.
Now we can give the relationship between the Padovan p-matrix M and the period $hPap_m$ by the following Theorem.

3.3. Theorem. *If m has the prime factorization $m = t^{\alpha}$ where t is prime and α is a positive integer, then $hPap_{t^{\alpha}} = \left|\langle M\rangle_{t^{\alpha}}\right|$.*

Proof. Let $\left|\langle M\rangle_{t^{\alpha}}\right| = u$. Then by equation (2.2), it is easy to see that $Pap\,(u+1) \equiv Pap\,(u+2) \equiv \cdots \equiv Pap\,(u+p) \equiv 0\,(\bmod\, t^{\alpha})$, $Pap\,(u+p+1) \equiv 1\,(\bmod\, t^{\alpha})$ and $Pap\,(u+p+2) \equiv 0\,(\bmod\, t^{\alpha})$, that is, $Pap\,(u+1) \equiv Pap\,(1)\,(\bmod\, t^{\alpha})$, $Pap\,(u+2) \equiv Pap\,(2)\,(\bmod\, t^{\alpha}), \ldots, Pap\,(u+p) \equiv Pap\,(p)\,(\bmod\, t^{\alpha})$, $Pap\,(u+p+1) \equiv Pap\,(p+1)\,(\bmod\, t^{\alpha})$ and $Pap\,(u+p+2) \equiv Pap\,(p+2)\,(\bmod\, t^{\alpha})$. Since $hPap_{t^{\alpha}}$ is the period of the sequence $\{Pap_{t^{\alpha}}\,(n)\}$, we obtain $hPap_{t^{\alpha}} | u$. Now we need only to prove that $hPap_{t^{\alpha}}$ is divisible by $\left|\langle M\rangle_{t^{\alpha}}\right|$. From equation (2.2), we obtain $M^{hPap_{t^{\alpha}}}\ (\bmod\, t^{\alpha}) \equiv I$, where I is the $(p+2)\times(p+2)$ identity matrix. So we get $\left|\langle M\rangle_{t^{\alpha}}\right| \big| hPap_{t^{\alpha}}$. Thus the proof is complete. □

The auxiliary equation for the Padovan p-numbers can be written as

$$x^{p+2} = x^{p} + 1.$$

3.4. Lemma. *Let* $n \geq p+2$ and $p \geq 2$, *then*

$$x^{n} = Pap\,(n)\,x^{p+1} + Pap\,(n+1)\,x^{p} + \sum_{i=0}^{p-1} Pap\,(n-1-i)\,x^{i}. \tag{3.1}$$

Proof. This follows directly from induction on n. □

Let t be a prime and let $G_{p}^{t^{\alpha}} = \left\{x^{n}\,(mod t^{\alpha}) : n \in \mathbb{Z},\ x^{p+2} = x^{p}+1\right\}$ such that α is a positive integer. Then, it is clear that the set $G_{p}^{t^{\alpha}}$ is a cyclic group.
Now we can give a relationship between the characteristic equation of the Padovan p-numbers and the period $hPap_{m}$ by the following Theorem.

3.5. Theorem. *The cyclic group* $G_{p}^{t^{\alpha}}$ *is isomorphic to the cyclic group* $\langle M\rangle_{t^{\alpha}}$, *where* t *is prime and* α *is a positive integer.*

Proof. Let t be a prime and let α be a positive integer. It is clear that $hPap_{t^{\alpha}} > 2p+2$. Then by equation (3.1) we see that $x^{hPap_{t^{\alpha}}} \equiv 1\,(mod\, t^{\alpha})$. Thus we obtain $\left|G_{p}^{t^{\alpha}}\right| = hPap_{t^{\alpha}}$. So by Theorem 3.3 we have $G_{p}^{t^{\alpha}} \cong \langle M\rangle_{t^{\alpha}}$. □

Now we give some properties of the period $hPap_{m}$ by the following Theorem.

3.6. Theorem. ***i.*** *Let* t *be a prime and let* u *be the smallest positive integer where* $hPap_{t^{u+1}} \neq hPap_{t^{u}}$, *then* $hPap_{t^{\sigma}} = t^{\sigma-u}\cdot hPap_{t^{u}}$ *for every integer* $\sigma > u$. *In particular, if* $hPap_{t} \neq hPap_{t^{2}}$, *then* $hPap_{t^{\sigma}} = t^{\sigma-1}\cdot hPap_{t}$ *holds for every integer* $\sigma > 1$.
ii. *If* $m = \prod_{i=1}^{v} t_{i}^{e_{i}}$, $(v \geq 1)$ *where* t_{i}*'s are distinct primes, then*

$$hPap_{m} = lcm\left[hPap_{t_{1}^{e_{1}}}, hPap_{t_{2}^{e_{2}}}, \ldots,\ hPap_{t_{v}^{e_{v}}}\right].$$

Proof. **i.** By Theorem 3.3 we see that for each positive integer a, $M^{hPap_{t^{a+1}}} \equiv I\ \left(\bmod\, t^{a+1}\right)$, hence $M^{hPap_{t^{a+1}}} \equiv I\ \ (\bmod\, t^{a})$, which means that $hPap_{t^{a}}$ divides $hPap_{t^{a+1}}$. On the other hand, writing $M^{hPap_{t^{a}}} = I + \left(m_{ij}^{(a)}\cdot t^{a}\right)$, by the binomial theorem, we obtain

$$M^{hPap_{t^{a}}\cdot t} = \left(I + \left(m_{ij}^{(a)}\cdot t^{a}\right)\right)^{t} = \sum_{i=0}^{t}\begin{pmatrix} t \\ i \end{pmatrix}\left(m_{ij}^{(a)}\cdot t^{a}\right)^{i} \equiv I\ \left(\bmod\, t^{a+1}\right),$$

which implies that $hPap_{t^{a+1}}$ divides $hPap_{t^{a}}\cdot t$. Thus, $hPap_{t^{a+1}} = hPap_{t^{a}}$ or $hPap_{t^{a+1}} = hPap_{t^{a}}\cdot t$, and the latter holds if and only if there is a $m_{ij}^{(a)}$ which is not divisible by t. Due to fact that we assume u is the smallest positive integer such that $hPap_{t^{u+1}} \neq hPap_{t^{u}}$, there is an $m_{ij}^{(u)}$ which is not divisible by t. Since there is an $m_{ij}^{(u)}$ such that t does not divide $m_{ij}^{(u)}$, it is easy to see that there is an $m_{ij}^{(u+1)}$ which is not divisible by t. This shows that $hPap_{t^{u+2}} \neq hPap_{t^{u+1}}$. Then we see that $hPap_{t^{u+2}} = t \cdot hPap_{t^{u+1}} = t^2 \cdot hPap_{t^u}$. So by induction on u we obtain $hPap_{t^\sigma} = t^{\sigma-u} \cdot hPap_{t^u}$ for every integer $\sigma > u$. Also, if $u = 1$, then $hPap_{t^\sigma} = t^{\sigma-1} \cdot hPap_t$ for every integer $\sigma > 1$.

ii. Since $hPap_{t_i^{e_i}}$ is the length of the period of the sequence $\left\{Pap_{t_i^{e_i}}(n)\right\}$, the sequence $\left\{Pap_{t_i^{e_i}}(n)\right\}$ repeats only after blocks of length $\lambda \cdot hPap_{t_i^{e_i}}$, $(\lambda \in \mathbb{N})$. Also, $hPap_m$ is the length of the period $\{Pap_m(n)\}$, which implies that $\left\{Pap_{t_i^{e_i}}(n)\right\}$ repeats after $hPap_m$ terms for all values i. Thus, $hPap_m$ is the form $\lambda \cdot hPap_{t_i^{e_i}}$ for all values of i, and since any such number gives a period of $\{Pap_m(n)\}$. So we get $hPap_m = \mathrm{lcm}\left[hPap_{t_1^{e_1}}, hPap_{t_2^{e_2}}, \ldots, hPap_{t_v^{e_v}}\right]$. □

We consider the Padovan p-sequences in p-generated groups such that $p \geq 2$.
Let G be a finite p-generator group and let
$X = \{(x_1, x_2, \ldots, x_p) \in \underbrace{G \times G \times \cdots \times G}_{p} \mid <\{x_1, x_2, \ldots, x_p\}>= G\}$. We call $(x_1, x_2, \ldots, x_p)$ a generating p-tuple for G.

3.7. Definition. For a p-tuple $(x_1, x_2, \ldots, x_p) \in X$, we define the Padovan p-orbit $Pap\,(G; x_1, x_2, \ldots, x_p) = \{a_i\}$ by

$$a_0 = e,\ a_1 = x_1,\ a_2 = x_2, \ldots,\ a_p = x_p,\ a_{p+1} = e,\ a_{n+p+1} = a_{n-1} \cdot a_{n+p-1},$$

$n \geq 1$.

3.8. Theorem. *A Padovan p-orbit of a finite group is simply periodic.*

Proof. Let n be the order of G. Since there are n^{p+2} distinct $(p+2)$-tuples of elements of G, at least one of the $(p+2)$-tuples appears twice in a Padovan p-orbit of G. Therefore, the subsequence following this $(p+2)$-tuple repeats; hence, the Padovan p-orbit is periodic. □

Since the Padovan p-sequence is periodic, there exist natural numbers u and v, with $u > v$, such that

$$a_{u+1} = a_{v+1},\ a_{u+2} = a_{v+2},\ \ldots\ ,\ a_{u+p+2} = a_{v+p+2}.$$

By definition 3.7, we know that

$$a_u = (a_{u+p+2}) \cdot (a_{u+p})^{-1} \text{ and } a_v = (a_{v+p+2}) \cdot (a_{v+p})^{-1}.$$

Therefore, $a_u = a_v$, and hence,

$$a_{u-v} = a_{v-v} = a_0,\ a_{u-v+1} = a_{v-v+1} = a_1,\ \ldots\ ,\ a_{u-v+p+1} = a_{v-v+p+1} = a_{p+1},$$

which implies that the Padovan p-orbit is simply periodic.
We denote the length of the period of the Padovan p-orbit $Pap\,(G; x_1, x_2, \ldots, x_p)$ by $LPap\,(G; x_1, x_2, \ldots, x_p)$ and we call this length the Padovan p-length with respect to the generating p-tuple $(x_1, x_2, \ldots, x_p)$. From the definition it is clear that the Padovan p-length of a finite group depends on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_p$ are made.
The classic Padovan p-sequence in a cyclic group $C = \langle x \rangle$ can be written as $Pap(C; \underbrace{x, x, \ldots, x}_{p+2})$.
It is clear that the period of $hPap_m$ is the period of the Padovan p-sequence in the cyclic group of order m.
We will now address the lengths of the Padovan p-orbits of the quaternion group Q_{2^n}.

The quaternion group Q_{2^n}, $(n \geq 3)$ is defined by the presentation

$$Q_{2^n} = \left\langle x, y : x^{2^{n-1}} = e,\ y^2 = x^{2^{n-2}},\ y^{-1}xy = x^{-1} \right\rangle.$$

Note that $|Q_{2^n}| = 2^n$, $|x| = 2^{n-1}$ and $|y| = 4$.

3.9. Theorem. *Consider the quaternion group Q_{2^n}, $(n \geq 3)$ with generators x, y. Then the lengths of the periods of the Padovan 2-orbits $LPa2\,(Q_{2^n}; x, y)$ and $LPa2\,(Q_{2^n}; y, x)$ are $2^{n-1} \cdot 3$.*

Proof. We prove the result by direct calculation. We first note that $xy = yx^{-1}$ and $yx = x^{-1}y$.

First, let us consider the Padovan 2-orbit $Pa2\,(Q_{2^n}; x, y)$. Then we have the sequence

$$\begin{array}{l} e,\ x,\ y,\ e,\ y,\ x,\ y^2,\ x,\ y^3,\ x^2,\ y,\ x^3,\ e,\ x^5,\ y,\ x^8, \\ y,\ x^{13},\ y^2,\ x^{21},\ y^3,\ x^{34},\ y,\ x^{55},\ e,\ x^{89},\ y,\ x^{144}, \ldots. \end{array}$$

Using the above, the sequence becomes:

$$\begin{array}{l} a_0 = e,\ a_1 = x,\ a_2 = y,\ e,\ \ldots, \\ a_{12} = e,\ a_{13} = x^5,\ a_{14} = y,\ a_{15} = x^8,\ \ldots, \\ a_{24} = e,\ a_{25} = x^{89},\ a_{26} = y,\ a_{27} = x^{144},\ \ldots, \\ a_{48} = e,\ a_{49} = x^{28657},\ a_{50} = y,\ a_{51} = x^{46368},\ \ldots, \\ a_{12 \cdot 2^i} = e,\ a_{12 \cdot 2^i+1} = x^{2^{i+2} \cdot \lambda_1+1},\ a_{12 \cdot 2^i+2} = y,\ a_{12 \cdot 2^i+3} = x^{2^{i+3} \cdot \lambda_2}, \ldots, \end{array}$$

where λ_1 and λ_2 are odd integers and i is an nonnegative integer. So we need the smallest integer i such that $2^{n-1} \big| 2^{i+2}$ for $n \geq 3$. If we choose $i = n - 3$, we obtain $x_{2^{n-1} \cdot 3} = e, x_{2^{n-1} \cdot 3+1} = x$ $x_{2^{n-1} \cdot 3+2} = y$, $x_{2^{n-1} \cdot 3+3} = e$. Since the elements succeeding $x_{2^{n-1} \cdot 3}$, $x_{2^{n-1} \cdot 3+1}$, $x_{2^{n-1} \cdot 3+2}$ and $x_{2^{n-1} \cdot 3+3}$ depend on e, x, y and e for their values, the cycle begins again with the $\left(2^{n-1} \cdot 3\right)^{\mathrm{nd}}$ element. Thus, $LPa2\,(Q_{2^n}; x, y) = 2^{n-1} \cdot 3$.

Now consider the Padovan 2-orbit $Pa2\,(Q_{2^n}; y, x)$. Then we have the sequence

$$\begin{array}{l} e, y,\ x,\ e,\ x,\ y,\ x^2,\ y,\ x^3,\ y^2,\ x^5,\ y^3,\ x^8,\ y,\ x^{13},\ e, \\ x^{21},\ y,\ x^{34},\ y,\ x^{55},\ y^2,\ x^{89},\ y^3,\ x^{144},\ y,\ x^{233},\ e,\ \ldots. \end{array}$$

Using the above, the sequence becomes:

$$\begin{array}{l} a_0 = e,\ a_1 = y,\ a_2 = x,\ e,\ \ldots, \\ a_{12} = x^8,\ a_{13} = y,\ a_{14} = x^{13},\ a_{15} = e,\ \ldots, \\ a_{24} = x^{144},\ a_{25} = y,\ a_{26} = x^{233},\ a_{27} = e,\ \ldots, \\ a_{48} = x^{46368},\ a_{49} = y,\ a_{50} = x^{75025},\ a_{51} = e,\ \ldots, \\ a_{12 \cdot 2^i} = x^{2^{i+3} \cdot \beta_1},\ a_{12 \cdot 2^i+1} = y,\ a_{12 \cdot 2^i+2} = x^{2^{i+2} \cdot \beta_2+1},\ a_{12 \cdot 2^i+3} = e, \ldots, \end{array}$$

where β_1 and β_2 are odd integers and i is an integer such that $i \geq 0$. Similar to the above, we obtain that $LPa2\,(Q_{2^n}; y, x) = 2^{n-1} \cdot 3$. □

4. Further Questions

There are many open problems in this area. Below are a few of them:

(1) Does there exist a relationship among the Padovan sequence and the considered sequences in this paper?
(2) Does there exist a formula for calculating the period $hPap_m$?
(3) As it is known that the quaternion group Q_{2^n} is a special class of the binary polyhedral group $\langle l, m, n \rangle$, the polyhedral group (l, m, n) is a factor group of $\langle l, m, n \rangle$ and $(n, 2, 2)$ is isomorphic to the dihedral group D_n. Furthermore, the quaternion group Q_{2^n} is known as dicyclic group and it is a metacyclic group. Due to these relations and goal for contributing further researches, we select the quaternion group Q_{2^n} for applications of the Padovan p-sequences in groups. In

terms of a further research, one can consider the question "What are the lengths of Padovan p-orbits of the groups which are related to the quaternion group Q_{2^n}".

(4) What general theories can be obtained regarding the length of the period of the Padovan p-orbit of a general group? For example does there exist a decision process to determine whether, or not, a given group has finite length?

(5) Let us consider infinite groups such that the lengths of the periods of the Padovan p-orbits of these groups are finite. To find these lengths it would be useful to have a program. This would possibly rely on using the Knuth-Bendix method, see [18].

Acknowledgement. The authors thank the referees for their valuable suggestions which improved the presentation of the paper. This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2014-FEF-34.

References

[1] Aydin H. and Dikici R., *General Fibonacci sequences in finite groups*, Fibonacci Quart., **36 (3)**, 216-221, 1998.

[2] Brualdi R. A. and Gibson P. M., *Convex polyhedra of doubly stochastic matrices I: applications of permanent function*, J. Combin. Theory, **22**, 194-230, 1977.

[3] Campbell C. M., Doostie H. and Robertson E. F., *Fibonacci length of generating pairs in groups* in Applications of Fibonacci Numbers, Vol. 3 Eds. G. E. Bergum et al. Kluwer Academic Publishers, 27-35, 1990.

[4] Chen W. Y. C. and Louck J. D., *The combinatorial power of the companion matrix*, Linear Algebra Appl., **232**, 261-278, 1996.

[5] Deveci O., *The polytopic-k-step Fibonacci sequences in finite groups*, Discrete Dyn. Nat. Soc., 431840-1-431840-13, 2011.

[6] Deveci O. and Karaduman E., *The Pell sequences in finite groups*, Util. Math., **96**, 263-276, 2015.

[7] Deveci O., *The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups*, Util. Math., **98**, 257-270, 2015.

[8] Doostie H. and Hashemi M., *Fibonacci lengths involving the Wall number $k(n)$*, J. Appl. Math. Comput., **20** (1-2), 171-180, 2006.

[9] Falcon S. and Plaza A., *k-Fibonacci sequences modulo m*, Chaos Solitons Fractals, **41** (1), 497-504, 2009.

[10] Frey D. D. and Sellers J. A., *Jacobsthal numbers and alternating sign matrices*, J. Integer Seq., **3**, Article 00.2.3, 2000.

[11] Gil J. B., Wiener M. D. and Zara C., *Complete Padovan sequences in finite fields*, http://arxiv.org/pdf/math/0605348.pdf.

[12] Gogin N. D. and Myllari A. A., *The Fibonacci-Padovan sequence and MacWilliams transform matrices*, Programing and Computer Software, published in Programmirovanie, **33** (2), 74-79, 2007.

[13] http://mathworld.wolfram.com/Padovan Sequence.html

[14] Kalman D., *Generalized Fibonacci numbers by matrix methods*, Fibonacci Quart., **20** (1), 73-76, 1982.

[15] Kilic E., *The generalized Pell (p,i)-numbers and their Binet formulas, combinatorial representations, sums*, Chaos Solitons Fractals, **40** (4), 2047-2063, 2009.

[16] Kilic E. and Tasci D., *The generalized Binet formula, representation and sums of the generalized order-k Pell numbers,* Taiwanese J. Math., **10** (6), 1661-1670, 2006.

[17] Knox S. W., *Fibonacci sequences in finite groups*, Fibonacci Quart., **30** (2), 116-120, 1992.

[18] Knuth D. E. and Bendix P. B, *Simple word problems in universal algebra, In: Computational problems in abstract algebra,* (edited by L. Leech), Pergamon Press, Oxford, 263-297, 1970.

[19] Koken F. and Bozkurt D., *On the Jacobsthal numbers by matrix methods*, Int. J. Contemp. Math. Sciences, **3** (13), 605-614, 2008

[20] Lee G-Y., *k-Lucas numbers and associated bipartite graphs*, Linear Algebra Appl., **320** (1-3), 51-61, 2000.

[21] Lü K. and Wang J., *k-step Fibonacci sequence modulo m*, Util. Math., **71**, 169-178, 2006.

[22] Ozgur N. Y, *On the sequences related to Fibonacci and Lucas numbers*, J. Korean Math. Soc., **42** (1), 135-151, 2005.

[23] Ozkan E., Aydin H. and Dikici R., *3-step Fibonacci series modulo m*, Appl. Math. Comput., **143**, 165-172, 2003.

[24] Shannon A. G., Horadam A. F. and Anderson P. G., *The auxiliary equation associated with plastic number*, Notes on Number Theory and Disc. Math., **12** (1), 1-12, 2006.

[25] Stakhov A. P., *A generalization of the Fibonacci Q-matrix*, Rep. Natl. Acad. Sci., Ukraine, **9**, 46-49, 1999.

[26] Tas S., Deveci O. and Karaduman E., *The Fibonacci-Padovan sequences in finite groups*, Maejo Int. J. Sci., **8** (03), 279-287, 2014.

[27] Tasci D. and Firengiz M. C., *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Modell., **52**, 1763-1770, 2010.

[28] Tasci D. and Kilic E., *On the order-k generalized Lucas numbers*, Appl. Math. Comput. **20**, 171-180, 2006.

[29] Tuglu N., Kocer E. G. and Stakhov A. P., *Bivarite Fibonacci like p-polinomials*, Appl. Math. Comput., **155**, 637-641, 2004.

[30] Wall D. D., *Fibonacci series modulo m*, Amer. Math. Monthly, **67**, 525-532, 1960.

[31] Yilmaz F. and Bozkurt D., *The generalized order-k Jacobsthal numbers*, Int. J. Contemp. Math. Sciences, **4** (34), 1685-1694, 2009.