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Perturbed Statistical Cluster Points

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Abstract— In recent years, generalizations of statistical convergence have appeared in the literature. The notion of perturbed statistical convergence has recently been established as one such generalization. This recent convergence approach aims to enhance the convergence behavior of a sequence by utilizing specific perturbation functions, thereby increasing its flexibility. This paper defines cluster points related to perturbed statistical convergence and investigates some of their fundamental properties.

Keywords — *Statistical convergence, statistical cluster points, perturbed metric spaces*

Mathematics Subject Classification (2020) 40A05, 40A35

1. Introduction

A wide range of convergence concepts has been developed in recent mathematical research, extending classical notions and introducing density-based approaches [1–4]. Being one of them, the concept of statistical convergence has become a crucial tool in density-based research, as it provides a more flexible framework than classical convergence. Recently, statistical convergence has been extended through the incorporation of perturbation functions, resulting in a new framework called perturbed statistical convergence [5]. This approach is more tolerant, compensating for the effects of local fluctuations in sequences and accounting for measurement errors that are frequently encountered in real-world problems. Understanding the limiting behavior of sequences implies identifying the limit and determining how frequently it occurs, similar to classical statistical convergence. Thus, to analyze convergence in a broader scope, the concept of statistical cluster points was introduced. It is necessary to develop new types of cluster point ideas, as this structure may not be sufficient in cases of minor perturbations or noisy data.

As convergence methods have developed, several forms of cluster and limit points have been extensively investigated to enhance the understanding of the geometric and statistical behavior of sequences [6–8]. However, most of these studies follow existing convergence methods and fail to adequately clarify the effect of perturbation functions on cluster points.

In this paper, the term cluster point is defined for the first time in the literature concerning perturbed statistical convergence, and its basic characteristics are investigated. The suggested definition proposes a more comprehensive framework that takes into account the effects of perturbation functions and is a

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logical extension of standard statistical cluster points. Furthermore, the relationship between these new cluster points and classical cluster points is demonstrated. This paper has been motivated by the potential application of such generalized cluster point notions in disciplines such as sequential data analysis, functional analysis, and fixed point theory.

The rest of the paper is structured as follows: Section 2 presents some basic definitions and supplementary results. Section 3 introduces the paper's main results and provides an explanation of the term perturbed statistical cluster point. The final section presents the results and offers recommendations for future research.

2. Preliminaries

This section provides an overview of the fundamental ideas that are applied in the present research, as well as the structures established in earlier articles.

Definition 2.1. [9] A real sequence (x_k) is said to be statistically convergent to a real number x if, for any $\varepsilon > 0$, the subset of indices where the deviation from x is at least ε becomes asymptotically sparse. In formal terms, define

$$A(\varepsilon) := \{m \in \mathbb{N} : |x_m - x| \geq \varepsilon\}$$

Then, the statistical convergence condition is expressed as

$$\lim_{n\to\infty} \frac{1}{n}\Big| \{m \leq n : |x_m - x| \geq \varepsilon\} \Big| = 0$$

If this condition holds for all $\varepsilon > 0$, then

$$st\text{-}\lim_{k\to\infty} x_k = x$$

The notions of statistical limit and cluster points for real sequences trace back to Fridy's framework involving nonthin subsequences [10]. In this approach, a subsequence whose index set possesses a positive natural density is called nonthin. Based on this, a point is a statistical limit (or cluster) point depending on its limiting relation to such nonthin subsequences. Specifically, a subsequence $x_{n(j)}$ of the sequence $x = (x_n)$ is said to be nonthin if the index set $K = \{n(j) : j \in \mathbb{N}\}$ has a positive density. Based on this framework, Fridy [10] has defined the following:

Definition 2.2. [10] A real number λ is called a statistical limit point of the sequence x if there exists a nonthin subsequence of x that converges to λ.

Definition 2.3. [10] A real number γ is called a statistical cluster point of the sequence x if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}$ has positive natural density.

Definition 2.4. [11] Let $D, P : X \times X \to [0, \infty)$ be two functions. Then, D is called a perturbed metric on X with respect to P if the function $d(x, y) := D(x, y) - P(x, y)$ is a metric on X, i.e., for all $x, y, z \in X$, the following conditions hold:

i. $D(x, y) - P(x, y) \geq 0$

ii. $D(x, y) - P(x, y) = 0$ if and only if $x = y$

iii. $D(x, y) - P(x, y) = D(y, x) - P(y, x)$

iv. $D(x, y) - P(x, y) \leq D(x, z) - P(x, z) + D(z, y) - P(z, y)$

In this context, P is called a perturbation, $d = D - P$ is called an exact metric, and the ordered triple (X, D, P) is called a perturbed metric space.

Definition 2.5. [5] Consider a sequence (x_k) in a perturbed metric space (X, D, P). It is said to be (x_k) converges to $x \in X$ in the perturbed statistical sense, denoted by $\text{pst-}\lim x_k = x$, if, for all $\varepsilon > 0$, the subset of natural numbers $A_\varepsilon := \{k \in \mathbb{N} \mid D(x_k, x) \geq \varepsilon + P(x_k, x)\}$ has zero natural density, i.e., $\delta(A_\varepsilon) = 0$.

Here, it must be noted that if the perturbation function is constant, i.e., $P(x_k, x) = r$ for some fixed $r \geq 0$, then the concept of perturbed statistical convergence reduces to the notion of r-rough statistical convergence [12].

The use of a perturbation function P allows for a refined generalization of the classical statistical convergence structure. Instead of a fixed tolerance ε, the tolerance dynamically adapts to each pair (x, y), making the approach more robust against local oscillations or measurement noise. This enables the analysis of convergence properties for sequences that exhibit irregular behavior under traditional metrics. This is where perturbation comes into play: The function P defines a dynamic tolerance based on the distance of each term from the target point. This allows us to evaluate the sequence's behavior within a more flexible framework. This approach offers a much more sensitive and comprehensive analysis, particularly for

i. Noisy sequences

ii. Slowly oscillating structures

iii. Sequences that do not converge in the classical sense but exhibit significant limiting behavior

Moreover, perturbed structures provide a new perspective for sequences that fall outside the classical theory.

3. Main Results

This section investigates cluster points established within the context of perturbed statistical convergence concepts. Initially, it explores some basic properties of perturbed statistical cluster points; thereafter, it theoretically demonstrates the existence of these points and their connection to classical convergence. Additionally, this section provides several examples and counterexamples to clarify the results obtained. Thus, it clearly illustrates the features and limitations of the newly established cluster point concept. Definition 3.1 introduces the concept of perturbed statistical limit points, which generalizes the notion of statistical limit points in the context of perturbed metric spaces.

Definition 3.1. Let (X, D, P) be a perturbed metric space and (x_k) be a sequence in X. Then, a point $x \in X$ is said to be a perturbed statistical limit point of the sequence (x_k) if there exists a subsequence (x_{k_j}) of (x_k) such that $x_{k_j} \to x$ in the classical sense, and the set

$$\left\{j \in \mathbb{N} : D(x_{k_j}, x) < \varepsilon + P(x_{k_j}, x)\right\}$$

has positive natural density for all $\varepsilon > 0$.

The set of all perturbed statistical limit points of a sequence (x_k) is denoted by $\Lambda_{pst}(x_k)$.

Definition 3.2. Let (X, D, P) be a perturbed metric space and (x_k) be a sequence in X. Then, a point $\mu \in X$ is called a perturbed statistical cluster point of the sequence (x_k) if, for all $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : D(x_k, \mu) < \varepsilon + P(x_k, \mu)\}$$

has positive natural density, i.e.,

$$\delta\left(\{k \in \mathbb{N} : D(x_k, \mu) < \varepsilon + P(x_k, \mu)\}\right) > 0$$

The set of all perturbed statistical cluster points of a sequence (x_k) is called the perturbed statistical cluster point set of the sequence (x_k) and denoted by $\Gamma_{pst}(x_k)$.

It must be noted that if P is a constant function, then the concept of rough cluster points [13] is obtained.

The following example demonstrates that if a sequence converges in the classical sense, then its limit point serves as a perturbed statistical cluster point. This confirms that our newly introduced concept is consistent with classical convergence.

Example 3.3. Let $X = \mathbb{R}$ and consider the sequence (x_k) defined by

$$x_k = 1 + \frac{(-1)^k}{k}$$

Define the perturbation function $P(x, y) = \frac{1}{1+|x-y|}$ and the corresponding perturbed metric $D(x, y) = |x - y| + P(x, y)$ on X. Since $x_k \to 1$, the following inequality holds for all $\varepsilon > 0$ and for sufficiently large values of k

$$D(x_k, 1) < \varepsilon + P(x_k, 1)$$

Moreover, the set

$$\{k \in \mathbb{N} : D(x_k, 1) < \varepsilon + P(x_k, 1)\}$$

has natural density 1. Hence, 1 is a perturbed statistical cluster point of the sequence (x_k) (see Figure 1).

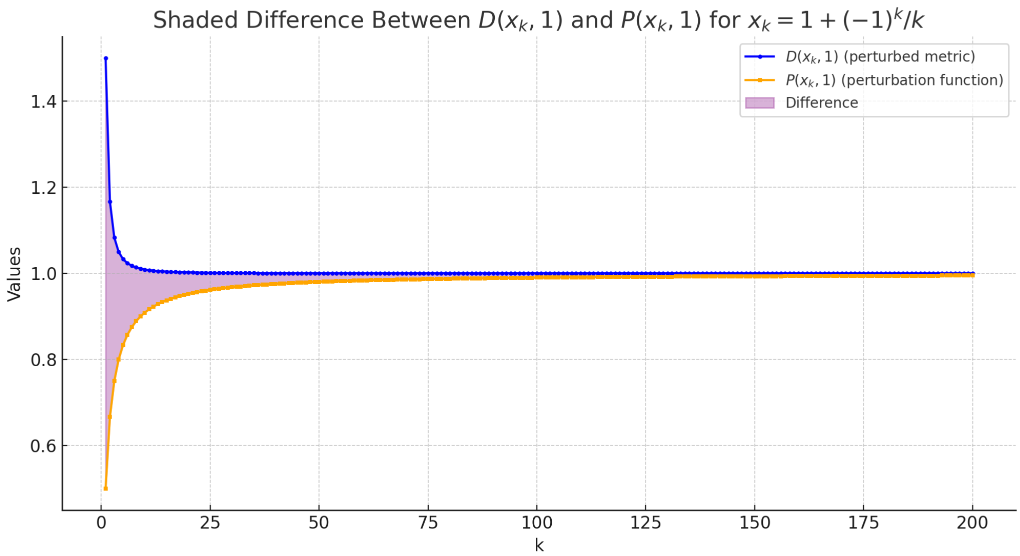

Figure 1. Comparison of the perturbed metric $D(x_k, 1)$ (blue) and the perturbation function $P(x_k, 1)$ (orange) for the sequence $x_k = 1 + (-1)^k/k$. The shaded region represents their difference, which is always positive and tends to zero, confirming that the classical limit 1 is also a perturbed statistical cluster point.

While the previous example shows that the concept of perturbed statistical cluster points is consistent with classical convergence, the following example demonstrates that sequences which are not classically convergent may still admit perturbed statistical cluster points. In fact, such sequences may even have more than one cluster point, illustrating the richness of the newly defined notion.

Example 3.4. Let $X = \mathbb{R}$ and consider the sequence (x_k) defined by

$$x_k = \begin{cases} 1 + \frac{1}{k}, & k \text{ is odd} \\ 2 - \frac{1}{k}, & k \text{ is even} \end{cases}$$

Define the perturbation function $P(x, y) = \frac{1}{1+|x-y|}$ and the perturbed metric $D(x, y) = |x-y| + P(x, y)$ on X. This sequence is not convergent, as it oscillates between values tending toward 1 and 2 along subsequences. However, for both $\mu = 1$ and $\mu = 2$, the inequality

$$D(x_k, \mu) < \varepsilon + P(x_k, \mu)$$

holds on a subset of $\mathbb{N}$ with the natural density $\frac{1}{2}$, for all $\varepsilon > 0$. Therefore, both points are perturbed statistical cluster points, i.e.,

$$\{1, 2\} \subseteq \Gamma_{pst}(x_k)$$

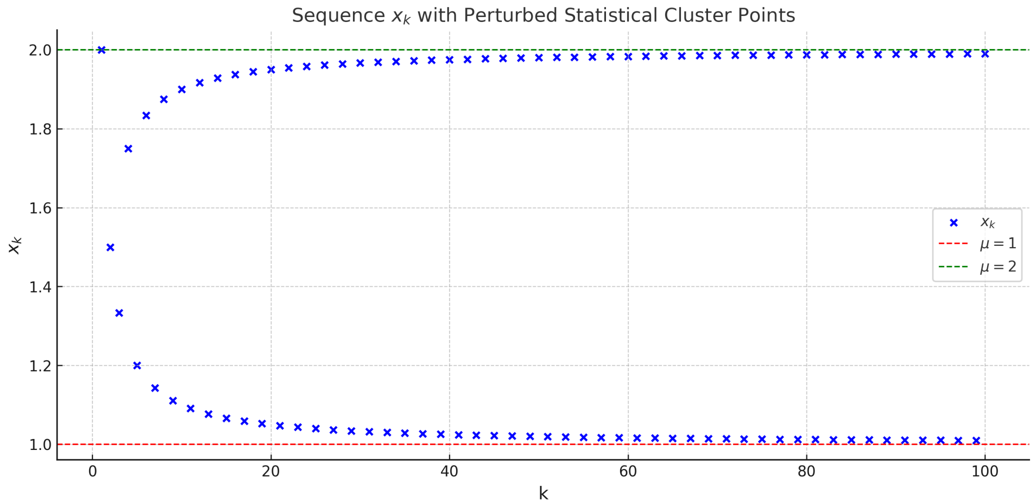

Figure 2. Two statistical cluster points $\mu = 1$ and $\mu = 2$ of the sequence (x_k) in Example 3.4

As illustrated in Figure 2, the sequence in Example 3.4 possesses two distinct statistical cluster points, namely $\mu = 1$ and $\mu = 2$.

The following example illustrates the influence of the perturbation function itself. While the previous examples showed that cluster points can exist consistently with classical convergence or in oscillating sequences, here we emphasize that the choice of P directly affects whether a point is detected as a perturbed statistical cluster point.

Example 3.5. Let $X = \mathbb{R}$ and define the perturbed metric

$$D(x, y) = |x - y| + P(x, y)$$

where P is a perturbation on $X \times X$. Consider the sequence

$$x_k = 1 + \frac{(-1)^k}{k}$$

which oscillates around 1 without being statistically convergent in the classical sense. Moreover, consider the following perturbation functions:

$$P_1(x, y) = 0.05 \quad \text{and} \quad P_2(x, y) = 0.2$$

Let $\varepsilon = 0.1$. Then, the perturbed tolerance bands are as follows:

$$\varepsilon + P_1 = 0.15 \quad \text{and} \quad \varepsilon + P_2 = 0.3$$

Since the band defined by P_2 is wider, more terms of the sequence satisfy the condition

$$D(x_k, 1) < \varepsilon + P(x_k, 1)$$

whereas for P_1, the condition is more restrictive. Therefore, the point $\mu = 1$ is a perturbed statistical cluster point of (x_k) under P_2 but may not be under P_1.

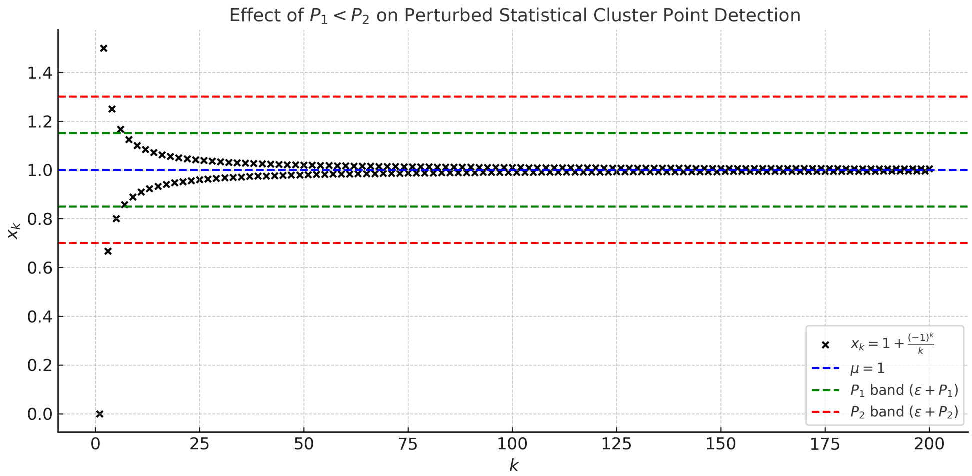

Figure 3. Effect of different perturbation functions on the perturbed statistical cluster point detection for the sequence (x_k) in Example 3.5

Figure 3 demonstrates how different perturbation functions affect the detection of perturbed statistical cluster points for the sequence considered in Example 3.5. This example illustrates that larger perturbation functions increase the likelihood of a point being a cluster point.

Remark 3.6. Example 3.5 illustrates that the choice of perturbation function significantly affects the set of perturbed statistical cluster points. A larger perturbation function expands the admissible band around a candidate point, increasing the likelihood that the sequence will satisfy the perturbed statistical clustering condition. In particular, if $P_1(x, y) \leq P_2(x, y)$, for all $x, y \in X$, then

$$\Gamma_{pst}^{(P_1)}(x_k) \subseteq \Gamma_{pst}^{(P_2)}(x_k)$$

This inclusion indicates that using a more permissive perturbation function yields a larger set of cluster points.

Remark 3.7. At first sight, it may seem that D is always written in terms of P. This is not a restriction but a structural feature of the framework: By definition, $D = d + P$, where $d = D - P$ is a genuine metric capturing the intrinsic geometry, and P represents bounded measurement error or noise. Hence, the role of P is to enlarge the admissible band around candidate points, as illustrated in Example 3.5. Larger P yields a larger cluster set, but the underlying metric properties are always governed by d. Therefore, this dependence does not reduce the generality of the framework but rather clarifies its intended interpretation as "true distance plus perturbation". In fact, as the following finite example shows, even if D is not explicitly written in terms of P, there still remains an implicit relation between them through the requirement that $d = D - P$ must be a genuine metric. The essential point is not to avoid such a relation, but the fact that d preserves the metric structure of the space.

Example 3.8. Let $X = \{a, b, c\}$. Define the perturbation function $P : X \times X \to [0, \infty)$ by

$$P(x,y) = \begin{cases} 0, & x = y \\ 0.5, & x \neq y \end{cases}$$

and the perturbed distance $D : X \times X \to [0, \infty)$ by

$$D(x,y) = \begin{cases} 0, & x = y \\ 1.5, & x \neq y \end{cases}$$

Then the exact metric $d = D - P$ is

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

which is precisely the discrete metric on X. This shows that D and P can be chosen independently on a finite set, while $d = D - P$ retains the required metric structure (see Table 1).

Table 1. Values of D, P, and $d = D - P$ for the finite set example $X = \{a, b, c\}$. The exact metric d coincides with the discrete metric

Pair (x, y)	$D(x, y)$	$P(x, y)$	$d(x, y) = D - P$
(a, a)	0	0	0
(b, b)	0	0	0
(c, c)	0	0	0
(a, b)	1.5	0.5	1
(a, c)	1.5	0.5	1
(b, c)	1.5	0.5	1

Consider the sequence

$$x_k = \begin{cases} a, & k \text{ odd} \\ b, & k \text{ even} \end{cases}$$

This sequence is not convergent in the classical sense, as it oscillates between a and b. For $\mu = a$, all odd terms satisfy $D(x_k, a) = 0 < \varepsilon$ for every $\varepsilon > 0$, and the even terms satisfy

$$D(b, a) = 1.5 < \varepsilon + P(b, a) = \varepsilon + 0.5$$

whenever $\varepsilon > 1$. Since both the set of odd indices and the set of even indices have natural density $\frac{1}{2}$, the point a is a perturbed statistical cluster point. By symmetry, the same argument shows that b is a perturbed statistical cluster point. However, c is not, because the sets

$$\{k : D(x_k, c) < \varepsilon + P(x_k, c)\}$$

have density zero for all $\varepsilon > 0$.

Consequently,

$$\Gamma_{pst}(x_k) = \{a, b\}$$

Example 3.9. Let $X = \mathbb{R}^2$ and consider the sequence

$$x_k = (\cos k,\ \tfrac{1}{k}), \qquad k \in \mathbb{N}$$

This sequence does not converge in the classical sense, since its first coordinate oscillates as $\cos k$, while the second coordinate tends to 0 (see Figure 4).

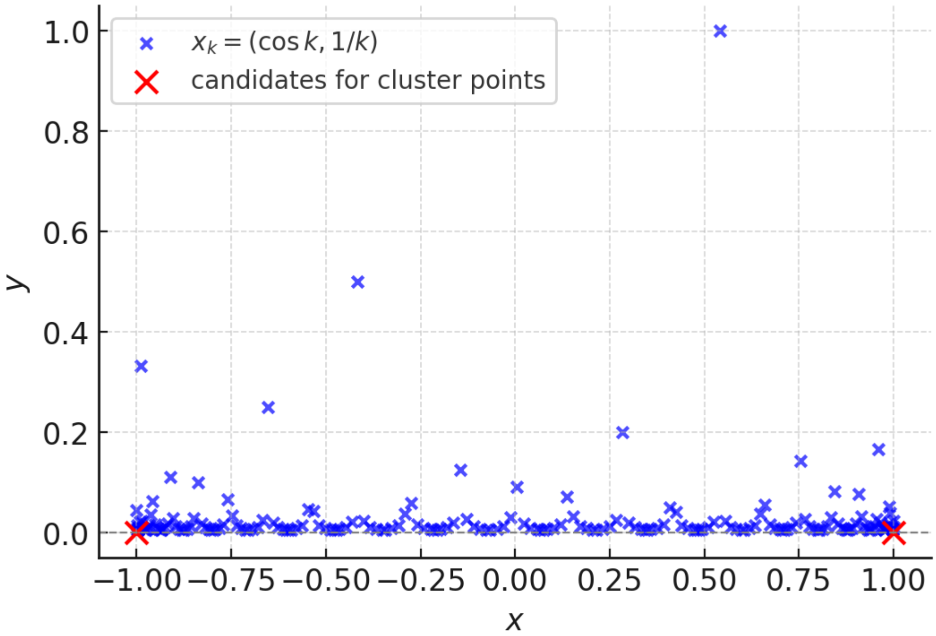

Figure 4. Scatter plot of the sequence $x_k = (\cos k, 1/k)$ for the first 200 terms

Define

$$P(x,y) = \frac{\sin^2(\|x-y\|_2)}{1+\|x-y\|_2}, \qquad D(x,y) = \|x-y\|_1 + P(x,y)$$

Here, P is a sinusoidal perturbation depending on the Euclidean norm, while D is based on the Manhattan norm plus the perturbation. The exact metric is

$$d(x,y) = D(x,y) - P(x,y) = \|x-y\|_1$$

which is a genuine metric.

For $\mu_1 = (1,0)$, there exists a subsequence of (x_k) with $\cos k \to 1$ and $1/k \to 0$, so that $x_{k_j} \to (1,0)$. Along this subsequence, the inequality

$$D(x_{k_j}, \mu_1) < \varepsilon + P(x_{k_j}, \mu_1)$$

holds for sufficiently large j. The set of such indices has positive natural density, so $\mu_1 = (1,0)$ is a perturbed statistical cluster point. Similarly, for $\mu_2 = (-1,0)$ another cluster point is obtained. Consequently,

$$\Gamma_{pst}(x_k) \supseteq \{(1,0),\ (-1,0)\}$$

This example shows that even when the perturbation P has a sinusoidal form and the sequence oscillates in a nontrivial way, the framework still yields meaningful perturbed statistical cluster points.

Theorem 3.10. Let (x_k) be a sequence in a perturbed metric space (X,D,P). Then, $\Gamma_{\mathrm{pst}}(x_k)$ is a closed subset of X.

Proof. Let (x_k) be a sequence in a perturbed metric space (X,D,P). If $\Gamma_{pst}(x_k) = \emptyset$, then $\Gamma_{\mathrm{pst}}(x_k)$ is closed. Then, assume that $\Gamma_{pst}(x_k) \neq \emptyset$ and let (y_i) be a sequence in $\Gamma_{pst}(x_k)$ such that $\lim\limits_{i\to\infty} y_i = y^*$.

Let $\varepsilon > 0$. Since $y_i \to y^*$, there exists an $i_0 \in \mathbb{N}$ such that for all $i > i_0$,

$$D(y_i, y^*) - P(y_i, y^*) < \frac{\varepsilon}{2}$$

Fix such an index $j_0 > i_0$, and consider the set

$$A := \left\{ k \in \mathbb{N} : D(x_k, y_{j_0}) < P(x_k, y_{j_0}) + \frac{\varepsilon}{2} \right\}$$

Since $y_{j_0} \in \Gamma_{pst}(x_k)$, the set A has positive natural density. Thus, for all $k \in A$,

$$D(x_k, y_{j_0}) < P(x_k, y_{j_0}) + \frac{\varepsilon}{2}$$

Moreover, since $D(y_{j_0}, y^*) - P(y_{j_0}, y^*) < \frac{\varepsilon}{2}$, by the triangle inequality on perturbed metric spaces,

$$D(x_k, y^*) - P(x_k, y^*) \leq D(x_k, y_{j_0}) - P(x_k, y_{j_0}) + D(y_{j_0}, y^*) - P(y_{j_0}, y^*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies that

$$\left\{ k \in \mathbb{N} : D(x_k, y_{j_0}) < P(x_k, y_{j_0}) + \frac{\varepsilon}{2} \right\} \subseteq \{ k \in \mathbb{N} : D(x_k, y^*) < P(x_k, y^*) + \varepsilon \}$$

Since the left-hand side has positive natural density, the right-hand side does as well. Therefore, $y^* \in \Gamma_{pst}(x_k)$. Consequently, $\Gamma_{pst}(x_k)$ is closed. □

Theorem 3.11. Let (x_k) be a sequence in a perturbed metric space (X, D, P). If (x_k) is perturbed statistically convergent to a point $x \in X$, then x is also a perturbed statistical cluster point of (x_k). That is,

$$pst\text{-}\lim x_k = x \Rightarrow x \in \Gamma_{pst}(x_k)$$

Proof. Let (x_k) be a sequence in a perturbed metric space (X, D, P) and $pst\text{-}\lim x_k = x$. Then, for all $\varepsilon > 0$, the set

$$A_\varepsilon := \{ k \in \mathbb{N} : D(x_k, x) \geq \varepsilon + P(x_k, x) \}$$

has natural density zero. Therefore, its complement

$$A_\varepsilon^c = \{ k \in \mathbb{N} : D(x_k, x) < \varepsilon + P(x_k, x) \}$$

has density 1 and hence is positive. Thus, for all $\varepsilon > 0$, x satisfies the condition of being a perturbed statistical cluster point. Therefore, $x \in \Gamma_{pst}(x_k)$. □

Remark 3.12. The converse of Theorem 3.1 is not valid in general. That is, a point may be a perturbed statistical cluster point of a sequence without being its perturbed statistical limit. To illustrate this, let $X = \mathbb{R}$ and consider the perturbed metric space (X, D, P), where $D(x, y) = |x - y| + P(x, y)$ and $P(x, y) = \frac{1}{1+|x-y|}$ and the sequence

$$x_k = \begin{cases} 1 + \frac{1}{k}, & k \text{ is odd} \\ 2 - \frac{1}{k}, & k \text{ is even} \end{cases}$$

defined on a perturbed metric space (X, D, P) with $D(x, y) = |x - y| + P(x, y)$ and $P(x, y) = \frac{1}{1+|x-y|}$. It can be observed that the natural densities of the following sets are $\frac{1}{2} > 0$:

$$\{ k \in \mathbb{N} : D(x_k, 1) < \varepsilon + P(x_k, 1) \} \quad \text{and} \quad \{ k \in \mathbb{N} : D(x_k, 2) < \varepsilon + P(x_k, 2) \}$$

Hence, 1 and 2 are perturbed statistical cluster points of the sequence (x_k). However, the sequence (x_k) does not perturbed statistically converge to any point, as it keeps oscillating between neighborhoods of 1 and 2, and no single point captures a set of indices with density 1. Therefore, in general, the set of *pst*-limits of a sequence is properly contained in $\Gamma_{pst}(x_k)$.

4. Conclusion

This paper investigates perturbed statistical cluster points and perturbed statistical limit points of a sequence, based on the previously established idea of perturbed statistical convergence. It begins by providing basic definitions before investigating deeply into the characteristics of these new point types. Moreover, the paper discusses the relationship between perturbed statistical cluster points and classical statistical cluster points, illustrating the separate features of the new definition through several examples and counterexamples. It is demonstrated that any perturbed statistical limit point is a perturbed statistical cluster point, highlighting that the converse is not always true. Our notion of perturbed statistical cluster points naturally extends several well-known concepts. If the perturbation function P is taken as a constant, our definition reduces to the rough cluster points introduced in [13]. When $P = 0$, it coincides with the classical statistical cluster points studied by Fridy [10]. Moreover, unlike lacunary statistical or λ-statistical points, which modify the density notion on the index set, our approach preserves the usual natural density but modifies the distance structure by adding perturbations. Thus, the proposed framework is orthogonal to these generalizations: it enlarges the admissible neighborhood of candidate points through P, while remaining consistent with the existing notions when P takes special forms. In the future, these ideas could find applications in areas, such as fixed-point theorems, the refinement of convergence methods, and the study of alternative perturbation structures, thereby opening up new and productive directions in nonlinear analysis research.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] R. Abazari, *Statistical convergence in g-metric spaces*, Filomat 36 (5) (2022) 1461–1468.

[2] M. İlkhan, E. E. Kara, *On statistical convergence in quasi-metric spaces*, Demonstratio Mathematica 52 (1) (2019) 225–236.

[3] F. Nuray, *Statistical convergence in partial metric spaces*, Korean Journal of Mathematics 30 (1) (2022) 155–160.

[4] C. Yalçın, *Statistical convergence on b-metric spaces*, Turkish Journal of Mathematics and Computer Science 17 (1) (2025) 275–281.

[5] C. Yalçın, *Perturbed statistical convergence*, Results in Nonlinear Analysis 8 (2) 305–313.

[6] I. S. Ibrahim, I. Brevik, R. P. Agarwal, M. A. Yousif, N. Chorfi, P. O. Mohammed, *Weighted statistical convergence and cluster points: The Fibonacci sequence-based approach using modulus functions*, Mathematics 12 (23) (2024) 3764.

[7] Ö. Kişi, V. Gürdal, M. B. Huban, *Ideal statistically limit points and ideal statistically cluster points of triple sequences of fuzzy numbers*, Journal of Classical Analysis 19 (2) (2022) 127–137.

[8] P. Malik, S. Das, *$\mathcal{A}^{\mathcal{I}}$-statistical limit points and $\mathcal{A}^{\mathcal{I}}$-statistical cluster points*, Filomat 36 (5) (2022) 1573–1585.

[9] H. Fast, *Sur la convergence statistique*, Colloquium Mathematicae 2 (3-4) (1951) 241–244.

[10] J. A. Fridy, *Statistical limit points*, Proceedings of the American Mathematical Society 118 (4) (1993) 1187–1192.

[11] M. Jleli, B. Samet, *On Banach's fixed point theorem in perturbed metric spaces*, Journal of Applied Analysis & Computation 15 (2) (2025) 993–1001.

[12] S. Aytar, *Rough statistical convergence*, Numerical Functional Analysis and Optimization 29 (3-4) (2008) 291–303.

[13] S. Aytar, *Rough statistical cluster points*, Filomat 31 (16) (2017) 5295–5304.