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Robust Numerical Solutions of Higher-Order Stiff and Non-Stiff Third-Order Oscillatory Differential Equations Using a Direct Block Approach

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Keywords

Block Algorithm, Third-order Differential Equations, Direct Method, Numerical Methods, Numerical Solution, Stiff ODEs, Oscillatory Systems, Non-stiff Problems.

Abstract

This work introduces a block algorithm formulated for directly solving third-order ordinary differential equations (ODEs), without the intermediate step of converting them into systems of first-order equations. The method is applied to three distinct types of problems non-stiff, oscillatory and stiff previously investigated by other researchers. Numerical results obtained using the proposed method are compared with those from existing methods in literature. The results reveal that the block algorithm consistently produces numerical solutions that closely align with the analytical solutions across all problem types. Graphical comparisons further demonstrate the superiority of the new method in terms of accuracy and stability. This confirms the method's effectiveness and versatility in solving higher-order ODEs directly, making it a valuable tool in computational mathematics.

1. Introduction

This research is centered on the formulation and application of block algorithms specifically designed to address third-order initial value problems of the form:

$$y'''(t) = f(t, y, y', y''), y(t_0) = y_0, y'(t_0) = y_0', \; y''(t_0) = y_0'', \; t \in [t_0, t_n] \tag{1}$$

It is assumed that equation (1) satisfies the necessary conditions of the existence and uniqueness theorem for differential equations. Furthermore, the solutions associated with problems of this form are regarded as bounded. Such equations are widely applied in disciplines including engineering, thermodynamics, and several other practical domains [1].

Equations of the form (1) are encountered in numerous scientific and engineering contexts, including thermodynamics, fluid mechanics, elasticity, and quantum mechanics [2, 3]. In these areas, third-order differential equations often arise as a consequence of modeling complex natural processes. These equations often lack closed-form analytical solutions [4], making it necessary to use numerical approaches. Among the numerical techniques developed to approximate solutions, linear multistep methods stand out for their zero-stability and their ability to operate without the need for external predictors or initial values from other schemes [5, 6]. Traditional methods for solving (1) involve reducing them to a system of first-order equations. However, this traditional method presents several limitations, such as increased formulation time, longer computational runtime,

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and higher implementation costs [6, 7]. To address these issues, the two-step hybrid block method has been introduced. This method is self-starting, straightforward to implement, and capable of producing numerical results that closely match the exact solutions of third-order ODEs [8].

Several researchers have proposed a method for solving (1) directly, avoiding the computational inefficiencies associated with converting them into systems of first-order equations. Author [9] developed a novel two-step hybrid block method that interpolates power series at both initial and hybrid points while collocating the third derivative at all step and off-step nodes. This self-starting approach showed high accuracy and consistency in comparison with existing methods and is suitable for software implementation. Author [10] introduced a continuous hybrid scheme using a new class of orthogonal polynomials as basis functions, applying interpolation and collocation techniques to ensure convergence and accuracy. These studies exemplify the move toward direct, self-sufficient numerical methods that do not require predictor values or auxiliary computations.

Earlier efforts contributed significantly to this evolution in methodology. Author [11] proposed a hybrid method with block extension that improves accuracy and removes the need for predictor-corrector structures, thus offering a more efficient computational framework. In [12], a fifth-order hybrid block technique was developed using power series interpolation combined with third-derivative collocation, and the results showed improved accuracy compared to earlier methods. Another author [5] advanced the field further by formulating a ninth-order, three off-step hybrid method based on Legendre polynomials, addressing limitations in linear multistep and predictor-corrector methods while providing a cost-effective, self-starting numerical tool.

2. Derivation of the Method

A numerical block algorithm will be developed to solve third-order oscillatory problems represented by equation (1). This approach will utilize a power series approximation for the solution, expressed as follows:

$$y(t) = \sum_{j=0}^{r+s-1} a_j t^j \tag{2}$$

For the derivative within the block algorithm, which is given by

$$A^{(0)} Y_m^{(i)} = \sum_{j=0}^{1} \frac{(jh)^{(i)}}{i!} e_i y_n^{(i)} + h^{(3-i)} \left[d_i f(y_n) + b_i F(Y_m) \right] \tag{3}$$

Performing three successive differentiations on equation (2) and inserting the result into equation (1) gives

$$f(t, y, y', y'') = \sum_{j=0}^{r+s-1} j(j-1)(j-2) a_j t^{j-3} \tag{4}$$

This study employs a grid of equal intervals, where the step size remains constant throughout, h given by $h = t_{n+i} - t_n,\ i = 0,1$ and off-step points at $t_{n+\frac{1}{6}}, t_{n+\frac{1}{3}}, t_{n+\frac{1}{2}}$.

Interpolating (2) at a points $t_{n+s},\ s = \frac{1}{6}\left(\frac{1}{3}\right)\frac{1}{2}$ and collocating (4) at $t_{n+r},\ r = 0\left(\frac{1}{6}\right)1$, give a system of nonlinear equation of the form

$$TA = U \tag{5}$$

Where,

$$A=\begin{bmatrix}a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9\end{bmatrix}$$

$$U=\begin{bmatrix}y_{n+\frac{1}{6}} & y_{n+\frac{1}{3}} & y_{n+\frac{1}{2}} & f_n & f_{n+\frac{1}{6}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5}{6}} & f_{n+1}\end{bmatrix}$$

$$T=\begin{bmatrix}
1 & t_{n+\frac{1}{6}} & t^2_{n+\frac{1}{6}} & t^3_{n+\frac{1}{6}} & t^4_{n+\frac{1}{6}} & t^5_{n+\frac{1}{6}} & t^6_{n+\frac{1}{6}} & t^7_{n+\frac{1}{6}} & t^8_{n+\frac{1}{6}} & t^9_{n+\frac{1}{6}} \\
1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & t^3_{n+\frac{1}{3}} & t^4_{n+\frac{1}{3}} & t^5_{n+\frac{1}{3}} & t^6_{n+\frac{1}{3}} & t^7_{n+\frac{1}{3}} & t^8_{n+\frac{1}{3}} & t^9_{n+\frac{1}{3}} \\
1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} & t^8_{n+\frac{1}{2}} & t^9_{n+\frac{1}{2}} \\
0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 & 336t_n^5 & 504t_n^6 \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{6}} & 60t^2_{n+\frac{1}{6}} & 120t^3_{n+\frac{1}{6}} & 210t^4_{n+\frac{1}{6}} & 336t^5_{n+\frac{1}{6}} & 504t^6_{n+\frac{1}{6}} \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{3}} & 60t^2_{n+\frac{1}{3}} & 120t^3_{n+\frac{1}{3}} & 210t^4_{n+\frac{1}{3}} & 336t^5_{n+\frac{1}{3}} & 504t^6_{n+\frac{1}{3}} \\
0 & 0 & 0 & 6 & 24t_{n+\frac{1}{2}} & 60t^2_{n+\frac{1}{2}} & 120t^3_{n+\frac{1}{2}} & 210t^4_{n+\frac{1}{2}} & 336t^5_{n+\frac{1}{2}} & 504t^6_{n+\frac{1}{2}} \\
0 & 0 & 0 & 6 & 24t_{n+\frac{2}{3}} & 60t^2_{n+\frac{2}{3}} & 120t^3_{n+\frac{2}{3}} & 210t^4_{n+\frac{2}{3}} & 336t^5_{n+\frac{2}{3}} & 504t^6_{n+\frac{2}{3}} \\
0 & 0 & 0 & 6 & 24t_{n+\frac{5}{6}} & 60t^2_{n+\frac{5}{6}} & 120t^3_{n+\frac{5}{6}} & 210t^4_{n+\frac{5}{6}} & 336t^5_{n+\frac{5}{6}} & 504t^6_{n+\frac{5}{6}} \\
0 & 0 & 0 & 6 & 24t_{n+1} & 60t_{n+1}^2 & 120t_{n+1}^3 & 210t_{n+1}^4 & 336t_{n+1}^5 & 504t_{n+1}^6
\end{bmatrix}$$

Equation (5) is solved to determine the required constants, which, when substituted back into equation (2), produce a continuous block algorithm expressed as

$$y(t)=a_{\frac{1}{6}}(t)y_{n+\frac{1}{6}}+a_{\frac{1}{3}}(t)y_{n+\frac{1}{3}}+a_{\frac{1}{2}}(t)y_{n+\frac{1}{2}}+h^3\left[\sum_{j=0}^{1}\beta_j(t)f_{n+j}+\beta_s(t)f_{n+s}\right],\ s=\frac{1}{6}\left(\frac{1}{6}\right)\frac{5}{6} \tag{6}$$

where $a_r(t)$, $\beta_j(t)$ and $\beta_s(t)$ are express as a function of x with

$$x=\frac{t-t_n}{h} \tag{7}$$

to obtained the continuous form as

$$\left.\begin{aligned}
&a_{\frac{1}{6}}(t)=3-15x+18x^2,\ a_{\frac{1}{3}}(t)=-324x-36x^2,\ a_{\frac{1}{2}}(t)=1-9x+18x^2\\
&\beta_0(t)=-\frac{221}{6531840}+\frac{116857}{65318400}x-\frac{36733}{145120}x^2+\frac{1}{6}x^3-\frac{49}{80}x^4+\frac{203}{150}x^5-\frac{147}{80}x^6+\frac{3}{2}x^7-\frac{27}{40}x^8+\frac{9}{70}x^9\\
&\beta_{\frac{1}{6}}(t)=-\frac{4703}{2177280}+\frac{319073}{10886400}x-\frac{367}{3024}x^2+\frac{3}{2}x^4-\frac{261}{50}x^5+\frac{87}{10}x^6-\frac{279}{35}x^7+\frac{27}{7}x^8-\frac{27}{35}x^9\\
&\beta_{\frac{1}{3}}(t)=-\frac{2831}{1088640}+\frac{2363}{124416}x+\frac{1153}{483840}x^2-\frac{15}{8}x^4+\frac{351}{40}x^5-\frac{1383}{80}x^6+\frac{1233}{70}x^7-\frac{513}{56}x^8+\frac{27}{14}x^9\\
&\beta_{\frac{1}{2}}(t)=\frac{887}{3265920}+\frac{4589}{3265920}x-\frac{193}{5184}x^2+\frac{5}{3}x^4-\frac{127}{15}x^5+\frac{93}{5}x^6-\frac{726}{35}x^7+\frac{81}{7}x^8-\frac{18}{7}x^9\\
&\beta_{\frac{2}{3}}(t)=-\frac{307}{2177280}+\frac{3623}{4354560}x+\frac{1153}{483840}x^2-\frac{15}{16}x^4+\frac{99}{20}x^5-\frac{921}{80}x^6+\frac{963}{70}x^7-\frac{459}{56}x^8+\frac{27}{14}x^9\\
&\beta_{\frac{5}{6}}(t)=\frac{17}{435456}+\frac{479}{1555200}x-\frac{79}{12096}x^2+\frac{3}{10}x^4-\frac{81}{10}x^5+\frac{39}{10}x^6-\frac{171}{35}x^7+\frac{108}{35}x^8-\frac{27}{35}x^9\\
&\beta_1(t)=-\frac{1}{204120}-\frac{3023}{65318400}x-\frac{1313}{1451520}x^2+\frac{1}{24}x^4+\frac{137}{600}x^5-\frac{9}{16}x^6+\frac{51}{70}x^7-\frac{27}{56}x^8+\frac{9}{70}x^9
\end{aligned}\right\} \tag{8}$$

From equation (6), the independent solution is determined, which when substituted yields a new block algorithm of the form

$$y(t)=\sum_{i=0}^{1}\frac{(jh)^i}{i!}y_n^{(i)}+h^3\left[\sum_{j=0}^{1}\sigma_j(t)f_{n+j}+\sigma_s(t)f_{n+s}\right],\ s=\frac{1}{6}\left(\frac{1}{6}\right)\frac{5}{6} \tag{9}$$

When equation (7) is evaluated at $t=\frac{1}{6}\left(\frac{1}{6}\right)1$, the resulting formulation produces a discrete block method of the form

$$Y_m^{(i)}=\begin{bmatrix} y_{n+\frac{1}{6}}^{(i)} & y_{n+\frac{1}{3}}^{(i)} & y_{n+\frac{1}{2}}^{(i)} & y_{n+\frac{2}{3}}^{(i)} & y_{n+\frac{5}{6}}^{(i)} & y_{n+1}^{(i)} \end{bmatrix},\ F(Y_m)=\begin{bmatrix} f_{n+\frac{1}{6}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5}{6}} & f_{n+1} \end{bmatrix},$$

$$y_m^{(i)}=\begin{bmatrix} y_{n-\frac{1}{6}}^{(i)} & y_{n-\frac{1}{3}}^{(i)} & y_{n-\frac{1}{2}}^{(i)} & y_{n-\frac{2}{3}}^{(i)} & y_{n-\frac{5}{6}}^{(i)} & y_{n}^{(i)} \end{bmatrix}$$

$A^{(0)}=n\times n$.

For $i=0$

$$e_0=\begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix},\ e_1=\begin{bmatrix} 0&0&0&0&0&\frac{1}{6}\\ 0&0&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&0&\frac{2}{3}\\ 0&0&0&0&0&\frac{5}{6}\\ 0&0&0&0&0&1 \end{bmatrix},\ e_2=\begin{bmatrix} 0&0&0&0&0&\frac{1}{72}\\ 0&0&0&0&0&\frac{1}{18}\\ 0&0&0&0&0&\frac{1}{8}\\ 0&0&0&0&0&\frac{4}{18}\\ 0&0&0&0&0&\frac{25}{72}\\ 0&0&0&0&0&\frac{1}{2} \end{bmatrix}$$

$$d_0=\begin{bmatrix} 0&0&0&0&0&\frac{343801}{783820800}\\ 0&0&0&0&0&\frac{6887}{3061800}\\ 0&0&0&0&0&\frac{1959}{358400}\\ 0&0&0&0&0&\frac{3863}{382725}\\ 0&0&0&0&0&\frac{505625}{31352832}\\ 0&0&0&0&0&\frac{33}{1400} \end{bmatrix},\ b_0=\begin{bmatrix} \frac{6031}{9331200} & -\frac{32981}{52254720} & \frac{5177}{9797760} & -\frac{15107}{52254720} & \frac{5947}{65318400} & -\frac{9809}{783820800}\\ \frac{1499}{255150} & -\frac{233}{58320} & \frac{52}{15309} & -\frac{379}{204120} & \frac{149}{255150} & -\frac{491}{6123600}\\ \frac{1599}{89600} & -\frac{537}{71680} & \frac{1}{120} & -\frac{327}{71680} & \frac{129}{89600} & -\frac{71}{358400}\\ \frac{4664}{127575} & -\frac{226}{25515} & \frac{272}{15309} & -\frac{31}{3645} & \frac{344}{127575} & -\frac{142}{382725}\\ \frac{162125}{2612736} & -\frac{85625}{10450944} & \frac{66875}{1959552} & -\frac{119375}{10450944} & \frac{1625}{373248} & -\frac{18625}{31352832}\\ \frac{33}{350} & -\frac{3}{560} & \frac{2}{35} & -\frac{3}{280} & \frac{3}{350} & -\frac{1}{1200} \end{bmatrix}$$

For $i=1$

$$e_1=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\ e_2=\begin{bmatrix}0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&\frac{1}{3}\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&\frac{2}{3}\\0&0&0&0&0&\frac{5}{6}\\0&0&0&0&0&1\end{bmatrix},\ d_1=\begin{bmatrix}0&0&0&0&0&\frac{28549}{4354560}\\0&0&0&0&0&\frac{1027}{68040}\\0&0&0&0&0&\frac{253}{10752}\\0&0&0&0&0&\frac{272}{8505}\\0&0&0&0&0&\frac{35225}{870912}\\0&0&0&0&0&\frac{41}{840}\end{bmatrix}$$

$$b_1=\begin{bmatrix}\frac{275}{20736}&-\frac{5717}{483840}&\frac{10621}{1088640}&-\frac{7703}{1451520}&\frac{403}{241920}&-\frac{199}{870912}\\\frac{97}{1890}&-\frac{2}{81}&\frac{197}{8505}&-\frac{97}{7560}&\frac{23}{5670}&-\frac{19}{34020}\\\frac{163}{1792}&-\frac{267}{17920}&\frac{5}{128}&-\frac{363}{17920}&\frac{57}{8960}&-\frac{47}{53760}\\\frac{376}{2835}&-\frac{2}{945}&\frac{656}{8505}&-\frac{2}{81}&\frac{8}{945}&-\frac{2}{1701}\\\frac{8375}{48384}&-\frac{3125}{290304}&\frac{25625}{217728}&-\frac{625}{96768}&\frac{275}{20736}&-\frac{1375}{870912}\\\frac{3}{14}&-\frac{3}{140}&\frac{17}{105}&-\frac{3}{280}&\frac{3}{70}&0\end{bmatrix}$$

For $i=2$

$$e_2=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\ d_2=\begin{bmatrix}0&0&0&0&0&\frac{19087}{362880}\\0&0&0&0&0&\frac{1139}{22680}\\0&0&0&0&0&\frac{137}{2688}\\0&0&0&0&0&\frac{143}{2835}\\0&0&0&0&0&\frac{3715}{72576}\\0&0&0&0&0&\frac{41}{840}\end{bmatrix},\ b_1=\begin{bmatrix}\frac{2713}{15120}&-\frac{15487}{120960}&\frac{293}{2835}&-\frac{6737}{120960}&\frac{263}{15120}&-\frac{863}{362880}\\\frac{47}{189}&\frac{11}{7560}&\frac{166}{2835}&-\frac{269}{7560}&\frac{11}{945}&-\frac{37}{22680}\\\frac{27}{112}&\frac{387}{4480}&\frac{17}{105}&-\frac{243}{4480}&\frac{9}{560}&-\frac{29}{13440}\\\frac{232}{945}&\frac{64}{945}&\frac{752}{2835}&\frac{29}{945}&\frac{8}{945}&-\frac{4}{2835}\\\frac{725}{3024}&\frac{2125}{24192}&\frac{125}{567}&\frac{3875}{24192}&\frac{235}{3024}&-\frac{275}{72576}\\\frac{9}{35}&\frac{9}{280}&\frac{34}{105}&\frac{9}{280}&\frac{9}{35}&\frac{41}{840}\end{bmatrix}$$

3. Analysis of Basic Properties of the Block Algorithm

In this part of the work, attention is given to assessing the basic characteristics of the new block scheme formulated for third-order oscillatory problems described by equation (1). The analysis considers order, error constant, consistency, and zero stability [1, 13].

3.1.Order and Error Constant

Definition 1: Order of an Algorithm

The discrete formulation in equation (2) has an associated linear operator, which is defined as

$$\ell\{y(t):h\} = A^{(0)}Y_m^{(i)} - \sum_{i=0}^{1}\frac{(jh)^{(i)}}{i!}e_i y_n^{(i)} + h^{(3-i)}\left[d_i f(y_n) + b_i F(Y_m)\right] \quad (10)$$

Under the assumption that $y(t)$, is adequately differentiable, the components of equation (8) are expanded in a Taylor series about the point t, leading to the expression:

$$\ell\{y(t):h\} = C_0 y(t) + C_1 y'(t) + \ldots + C_p h^p y^p(t) + C_{p+1}h^{p+1}y^{p+1}(t) + C_{p+2}h^{p+2}y^{p+2}(t) + \ldots \quad (11)$$

The block algorithm defined in (3) along with the related linear difference operators is considered to have order p when $C_0 = C_1 = \ldots = C_p = C_{p+1} = C_{p+2} = 0,\ C_{p+3} \neq 0.$ In numerical analysis, the order is described as the maximum positive real number p that measures how rapidly the approximation produced by a method converges to the exact solution of the differential problem.

Definition 2: Error Constant
This coefficient C_{p+3} is defined as the error constant and establishes that the local truncation error corresponding to (3) takes the form

$$T_{n+k} = C_{p+3}h^{p+3}y^{p+3}(t) + O(h^{p+4}) \quad (12)$$

Once the order of a numerical algorithm is determined, the remaining accumulated error is termed the error constant. Referring to Definitions 1 and 2, the derived block algorithm achieves order eight, with its error constant expressed as

$$C_{p+3} = \begin{bmatrix} \frac{4001}{109709829734400} & \frac{199}{857108044800} & \frac{29}{50164531200} & \frac{29}{26784626400} & \frac{7625}{4388393189376} & \frac{1}{391910400} \end{bmatrix} \quad (13)$$

3.2.Consistency

Definition 3: Consistency
The consistency of the continuous block algorithm (3) is established when it the order $p \geq 1$,

3.3. Zero-Stability

Definition 4: Zero-Stability
A block algorithm (3) is said to be zero-stable if the roots $z_s,\ s = 1,2,\ldots,n$ of the first characteristic polynomial $\bar{\rho}(z)$, defined by

$$\bar{\rho}(z) = \det\left[zA^{(0)} - E\right] \quad (14)$$

satisfies $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ *and* E. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

Applying Definition 4 on the one-step algorithm with three partitions, the first characteristic polynomial is given by,

$$\rho(z)=\left|\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-z\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right|=\begin{bmatrix}z&0&0&0&0&-1\\0&z&0&0&0&-1\\0&0&z&0&0&-1\\0&0&0&z&0&-1\\0&0&0&0&z&-1\\0&0&0&0&0&z-1\end{bmatrix}=z^5(z-1)$$

Thus, solving for z in

$$z^5(z-1)=0 \tag{15}$$

gives $z_1=z_2=z_3=z_4=z_5=0,\ z_6=1$. Hence, the block algorithm is zero-stable.

3.4.Convergence

Definition 5: Convergence

The block algorithm (3) is said to convergent if, for all problems of the form (1) subject to the hypothesis of Lipschitz condition, we have that

$$\lim_{\substack{h\to 0\\ jh=x-a}} y_j = y(x_j) \tag{16}$$

holds for $x\in[a,b]$ and for all solutions $\{y_j\}$.

The convergence of the continuous linear multistep method (3) is examined based on the fundamental properties previously outlined, alongside Dahlquist's fundamental theorem for linear multistep methods (LMMs). Below, we present the Dahlquist theorem without proof.

Theorem 1:

The convergence of any linear multistep method requires that it fulfills the dual conditions of consistency and zero-stability. Accordingly, the proposed block algorithm is convergent.

3.5. Region of Absolute Stability (RAS)

Definition 6: The continuous block scheme given in (3) is said to be absolutely stable in a specified region of the complex plane provided that, for every $h(\lambda h)\in\Re$, all the roots of the corresponding stability polynomial $\pi(z,h)$ satisfy $|z_s|<1,\ s=1,2,\cdots,k\ and\ |z_s|<|z_1|, s=2,3,\cdots,k$.

Following Sunday (2018), the application of Definition 6 leads to the derivation of the stability polynomial through the boundary locus technique as

$$\bar{h}(w)=\left(-\frac{8757}{3433432}w^6+\frac{113}{34836480}w^5\right)h^{12}+\left(-\frac{3503}{37623}w^5-\frac{61}{829440}w^6\right)h^{10}+\left(-\frac{123}{5225}w^5+\frac{481}{414720}w^6\right)h^8$$
$$+\left(-\frac{1288}{696729}w^6-\frac{67}{5184}w^5\right)h^6+\left(-\frac{817}{5040}w^5+\frac{113}{1152}w^6\right)h^4+\left(-\frac{17}{30}w^5-\frac{11}{24}w^6\right)h^2-2w^5+w^6 \tag{17}$$

The equation (17) was used to get the region of absolute stability as

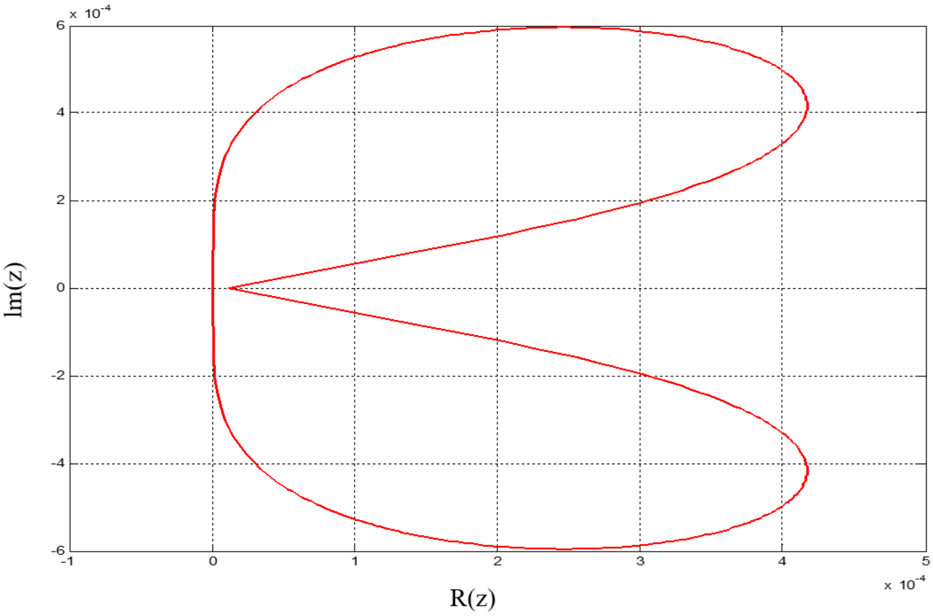

Figure 1. RAS of the block algorithm (3)

4. Numerical Experiments

The block algorithm developed in this study enables the direct solution of higher-order differential equations of the form (1), eliminating the necessity to transform them into equivalent first-order systems. The acronyms listed below will be used throughout the tables and figures.

T	: Point of evaluation
AS	: Analytical Solution
NS	: Numerical Solution
ENBM	: Error in new Block Method
ES19	: Error in [15]
EF18	: Error in [14]
ES18	: Error in [1]
ER23	: Error in [16]
EKO17	: Error in [4]
EAO19	: Error in [13]

Problem 1: As presented in [14, 15], a highly non-stiff third-order linear problem can be written in the form:

$$y'''(t)-3\cos(t)=0, \quad y(0)=1, y'(0)=0, y''(0)=2 \tag{18}$$

The exact analytical solution for the problem is formulated as

$$y(t) = t^2 - 3\,sin(t) + 3t + 1 \tag{19}$$

Table 1. The Numerical results for problem 1

t	AS	NS	ENBM	ES19	EF18
0.1	1.01049975005951554310	1.0104997500595155431	1.0000E-19	1.9700E-16	0.0000E-00
0.2	1.04399200761481635360	1.0439920076148163536	3.0000E-19	1.2639E-15	2.2205E-16
0.3	1.10343938001598127470	1.1034393800159812747	7.0000E-19	4.0627E-15	8.8818E-16
0.4	1.19174497307404852500	1.1917449730740485250	1.4000E-18	9.4370E-15	1.5543E-15
0.5	1.31172338418739099920	1.3117233841873909992	2.4000E-18	1.8205E-14	2.8866E-15
0.6	1.46607257981489392840	1.4660725798148939284	3.5000E-18	3.1152E-14	5.3291E-15
0.7	1.65734693828692683900	1.6573469382869268390	4.9000E-18	4.9021E-14	7.5495E-15
0.8	1.88793172730143171510	1.8879317273014317151	6.5000E-18	7.2504E-14	1.0436E-14
0.9	2.16001927111754983460	2.1600192711175498346	8.4000E-18	1.0224E-13	1.4211E-14
1.0	2.47558704557631048000	2.4755870455763104798	9.8000E-18	1.3880E-13	1.8208E-14

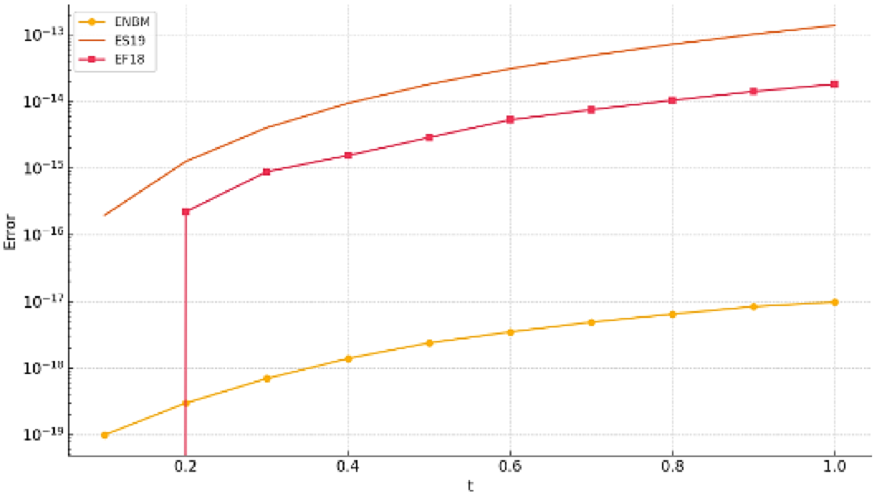

Figure 2. Graphical curve of table 1 showing the comparison for ENBM, ES19 and EF18

Problem 2: As reported in [1, 16], the third-order oscillatory differential equation can be represented in the form:

$$y'''(t)+4y'(t)-t=0,\ y(0)=y'(0)=0,\ y''(0)=-1,\ h=0.1 \quad (20)$$

The exact analytical solution for the problem is formulated as

$$y(t)=\frac{3}{16}(1-\cos 2t)+\frac{t^2}{8} \quad (21)$$

Table 2. The Numerical results for problem 2

t	AS	NS	ENBM	ES18	ER23
0.1	0.00498751665476719416	0.00498751665476714555	4.8610E-17	8.3209E-13	2.5521E-12
0.2	0.01980106362445904698	0.01980106362445885599	1.9099E-16	3.4752E-12	3.6421E-12
0.3	0.04399957220443531927	0.04399957220443490276	4.1651E-16	7.8178E-12	4.5313E-12
0.4	0.07686749199740648358	0.07686749199740577542	7.0816E-16	1.3681E-11	1.3406E-12
0.5	0.11744331764972380299	0.11744331764972275952	1.0435E-15	2.0825E-11	3.2855E-12
0.6	0.16455792103562370419	0.16455792103562230833	1.3959E-15	2.8962E-11	4.5913E-12
0.7	0.21688116070620482401	0.21688116070620308777	1.7362E-15	3.7764E-11	5.4732E-12
0.8	0.27297491043149163616	0.27297491043148960136	2.0348E-15	4.6879E-11	1.9652E-12
0.9	0.33135039275495382287	0.33135039275495156010	2.2628E-15	5.5941E-11	2.3453E-12
1.0	0.39052753185258919756	0.39052753185258680323	2.3943E-15	6.4592E-11	2.5559E-12

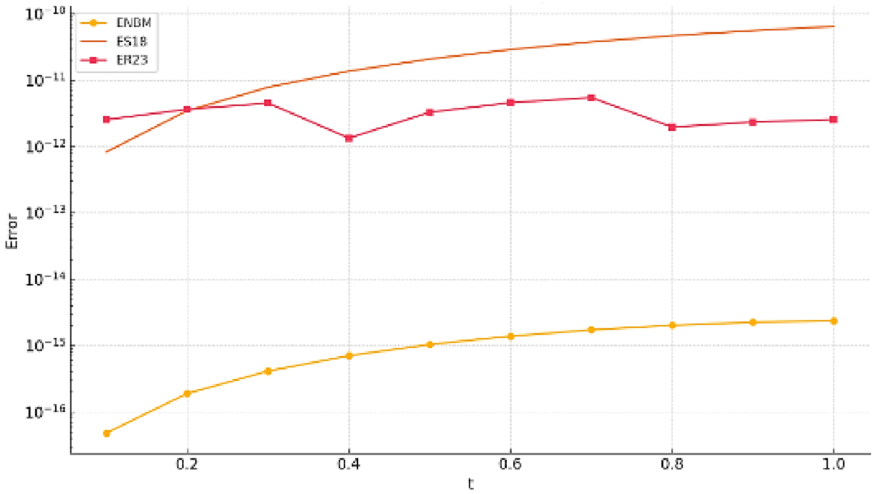

Figure 3. Graphical curve of table 2 showing the comparison for ENBM, ES18 and EF23

Problem 3: According to [4, 13], a highly stiff third-order linear problem may be formulated in the form:

$$y'''(t)+\exp(t)=0,\ y(0)=1,\ y'(0)=-1,\ y''(0)=3,\ h=0.1 \tag{22}$$

The exact analytical solution for the problem is formulated as

$$y(t)=2t^2-\exp(t)+2 \tag{23}$$

Table 3. The Numerical results for problem 3

t	AS	NS	ENBM	EKO17	EAO19
0.1	0.9148290819243523752	0.9148290819243523 7543	2.3000E-19	1.8241E-13	1.6209E-14
0.2	0.8585972418398301661	0.8585972418398301 6723	1.1300E-18	1.6708E-12	6.6058E-14
0.3	0.8301411924239968960	0.8301411924239968 9893	2.9300E-18	6.0014E-12	1.5277E-13
0.4	0.8281753023587296822	0.8281753023587296 8786	5.6600E-18	1.4860E-11	2.7955E-13
0.5	0.8512787292998718532	0.8512787292998718 6285	9.6500E-18	3.0121E-11	4.5020E-13
0.6	0.8978811996094910251	0.8978811996094910 4030	1.5200E-17	5.3842E-11	6.6835E-13
0.7	0.9662472925295234784	0.9662472925295235 0071	2.2310E-17	8.8316E-11	9.3925E-13
0.8	1.0544590715075323954	1.0544590715075324 2690	3.1500E-17	1.3606E-10	1.2670E-12
0.9	1.1603968888430503362	1.1603968888430503 7930	4.3100E-17	1.9987E-10	1.6578E-12
1.0	1.2817181715409547646	1.2817181715409548 2180	5.7200E-17	2.8281E-10	2.1174E-12

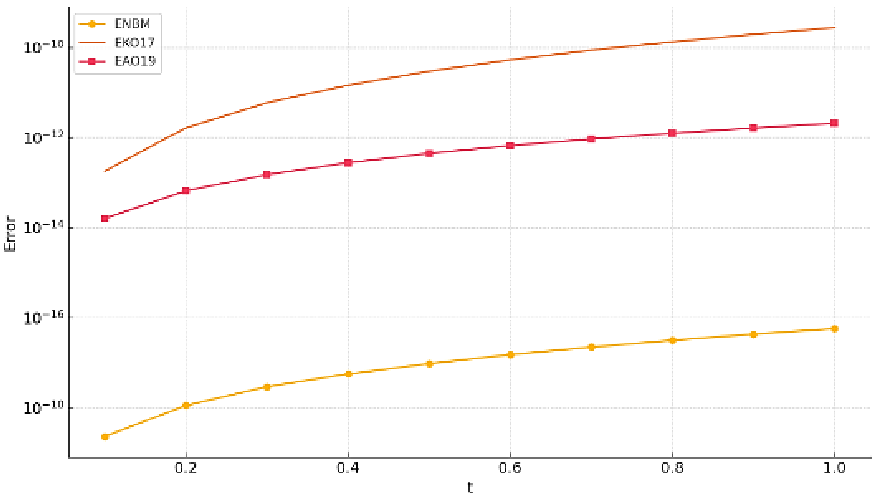

Figure 4. Graphical curve of table 3 showing the comparison for ENBM, EKO17 and EAO19

5. Discussion

This section examines the fundamental numerical properties of the proposed block method for solving third-order oscillatory problems. The order of the method indicates the accuracy with which it approximates the exact solution, whereas the error constant quantifies the magnitude of the local truncation error. The method is considered consistent if it satisfies specific mathematical criteria, including possessing a well-defined order. Zero-stability is assessed by studying the roots of the characteristic polynomial, ensuring that errors do not amplify as the computation progresses. According to Dahlquist’s theorem, a method that is both consistent and zero-stable is convergent, implying that its numerical solutions converge to the true solution as the step size is reduced. Finally, the region of absolute stability is established using the boundary locus method, identifying the domain in the complex plane where the method remains stable.

The results in Table 1 and the corresponding Figure 2 relate to Problem 1, a non-stiff third-order linear differential equation. The block algorithm developed in this study produces numerical results that match the analytical solution almost perfectly. Throughout the entire interval, the numerical solution curve closely tracks the analytical solution, demonstrating the method’s strong consistency and reliability. When compared visually in Figure 2, the solution generated by the new method remains closely aligned with the benchmark, clearly outperforming the previously established methods by [14, 15], which show wider deviations as the value of t increases.

In Table 2 and Figure 3, Problem 2 presents a third-order oscillatory equation. Despite the oscillatory nature of the solution, which can be more challenging for numerical solvers, the new block method maintains excellent agreement with the analytical solution. The figure highlights this consistency by showing that the curve for the new method remains smooth and in close proximity to the expected analytical trend. In contrast, the older approaches used by [1, 16] show more noticeable divergence, particularly at higher points of evaluation.

The third scenario, presented in Table 3 and Figure 4, examines a stiff third-order differential equation, which typically poses difficulty for conventional methods due to its rapidly changing solution behavior. However, the block method continues to perform robustly, matching the analytical solution across all points with minimal deviation. The figure illustrates how closely the new method’s curve adheres to the reference trend, while the solutions from [4, 13] tend to diverge more prominently as the interval progresses. This further emphasizes the stability and adaptability of the proposed method in handling stiff problems.

Across all three problems, the figures consistently show that the block algorithm's numerical solutions trace the analytical curves more faithfully than other referenced methods. Whether the equation is non-stiff, oscillatory, or stiff, the method maintains a high level of accuracy and stability. This consistency across varying types of differential equations confirms the versatility and precision of the block approach developed in this study, making it a reliable tool for solving higher-order differential equations directly without transformation.

6. Conclusion

This study presents a high-order, A-stable block hybrid method for the direct solution of third-order ordinary differential equations (ODEs), avoiding the need to convert them into systems of first-order equations. Constructed using power series interpolation and collocation techniques, the method is self-starting, consistent, zero-stable, and convergent. It was applied to non-stiff, oscillatory, and stiff ODEs, with its performance evaluated both analytically and through numerical experiments. The results show that the proposed method closely reproduces analytical solutions while yielding significantly smaller errors than existing approaches, highlighting its accuracy, robustness, and computational efficiency.

From the results, the proposed block algorithm consistently outperforms other known numerical techniques across different problem types. Graphical and tabular comparisons reveal minimal deviations between the numerical and analytical solutions, even for stiff and oscillatory equations, which are often challenging for many solvers. The method's strong performance across diverse problem settings validates its applicability in a wide range of scientific and engineering contexts. Overall, the study concludes that the new method is a versatile and reliable tool for solving higher-order ODEs directly and efficiently.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authorship Contribution Statement

Bello Kareem Akanbi: Conceptualization, Methodology, Supervision, Reviewing and Editing
Oyedepo Taiye: Writing – Original Draft Preparation, Reviewing and Editing, Data Interpretation
Ayinde Muhammed Abdullahi: Methodology, Formal Analysis, Validation
Raji Musiliu Tayo: Data Preparation, Computation, Visualization, Proofreading

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